GENERIC LIGHTLIKE SUBMANIFOLDS OF AN INDEFINITE KAHLER MANIFOLD WITH A SEMI-SYMMETRIC METRIC CONNECTION

JAE WON LEE AND CHUL WOO LEE*

Abstract. Depending on the characteristic vector filed $\zeta$, a generic lightlike submanifold $M$ in an indefinite Kaehler manifold $\bar{M}$ with a semi-symmetric metric connection has various characterizations. In this paper, when the characteristic vector filed $\zeta$ belongs to the screen distribution $S(TM)$ of $M$, we provide some characterizations of (Lie-) recurrent generic lightlike submanifold $M$ in an indefinite Kaehler manifold $\bar{M}$ with a semi-symmetric metric connection. Moreover, we characterize various generic lightlike submanifolds in an indefinite complex space form $\bar{M}(c)$ with a semi-symmetric metric connection.

1. Introduction

A lightlike submanifold $M$ of an indefinite almost complex manifold $\bar{M}$, with an indefinite almost complex structure $J$, is called generic if there exists a screen distribution $S(TM)$ of $M$, which is a complementary non-degenerate distribution of $Rad(TM) = TM \cap TM^\perp$ in $TM$, such that

\[(1.1) \quad J(S(TM)^\perp) \subset S(TM),\]

where $S(TM)^\perp$ is the orthogonal complement of $S(TM)$ in the tangent bundle $TM$ of $\bar{M}$ such that $TM = S(TM) \oplus_{\text{orth}} S(TM)^\perp$. The notion of generic lightlike submanifolds was introduced by Jin-Lee [9] and later, studied by several authors [2, 5, 6, 10]. Moreover, Jin [8]...
studied generic lightlike submanifolds of an indefinite Kaehler manifold with a semi-symmetric non-metric connection. Lightlike hypersurfaces of an indefinite almost complex manifold are important examples of the generic lightlike submanifold. Much of the theory of generic submanifolds will be immediately generalized in a formal way to general lightlike submanifolds.

In 1924, Friedmann-Schouten [4] introduced the idea of a semi-symmetric connection as follow: A linear connection \( \bar{\nabla} \) on a semi-Riemannian manifold \((\bar{M}, \bar{g})\) is called a semi-symmetric connection if its torsion tensor \( \bar{T} \) satisfies
\[
\bar{T}(\bar{X}, \bar{Y}) = \theta(\bar{Y})\bar{X} - \theta(\bar{X})\bar{Y},
\]
where \( \theta \) is a 1-form associated with a smooth unit vector field \( \zeta \), which is called the characteristic vector field of \( \bar{M} \), by \( \theta(\bar{X}) = \bar{g}(\bar{X}, \zeta) \). In the followings, we denote by \( \bar{X}, \bar{Y} \) and \( \bar{Z} \) the smooth vector fields on \( \bar{M} \). Moreover, if this connection is a metric one, \( \bar{\nabla} \bar{g} = 0 \), then \( \bar{\nabla} \) is called a semi-symmetric metric connection on \( \bar{M} \). The notion of a semi-symmetric metric connection on a Riemannian manifold was introduced by Yano [12].

Remark 1.1. Denote \( \bar{\nabla} \) by the Levi-Civita connection of a semi-Riemannian manifold \((\bar{M}, \bar{g})\) with respect to \( \bar{g} \). It is well known that a linear connection \( \bar{\nabla} \) on \( \bar{M} \) is a semi-symmetric metric connection if and only if it satisfies
\[
\bar{\nabla}_X \bar{Y} = \bar{\nabla}_X \bar{Y} + \theta(\bar{Y})\bar{X} - \bar{g}(\bar{X}, \bar{Y})\zeta.
\]

The object of this paper is to study generic lightlike submanifolds \( M \) of an indefinite Kaehler manifold \( \bar{M} \) with a semi-symmetric metric connection \( \bar{\nabla} \) subject to the condition that the characteristic vector field \( \zeta \) of \( \bar{M} \) belongs to our screen distribution \( S(TM) \) of \( M \). In Section 3, we provide several new results on such a generic lightlike submanifold. In Section 4, we characterize generic lightlike submanifolds of an indefinite complex space form \( \bar{M}(c) \) with a semi-symmetric metric connection subject such that \( \zeta \) belongs to \( S(TM) \).

2. Semi-symmetric metric connections

Let \( \bar{M} = (\bar{M}, \bar{g}, J) \) be an indefinite Kaehler manifold, where \( \bar{g} \) is a semi-Riemannian metric and \( J \) is an indefinite almost complex structure;
\[
J^2 \bar{X} = -\bar{X}, \quad \bar{g}(J\bar{X}, J\bar{Y}) = \bar{g}(\bar{X}, \bar{Y}), \quad (\bar{\nabla}_X J)\bar{Y} = 0.
\]
Replacing the Levi-Civita connection $\tilde{\nabla}$ by the semi-symmetric metric connection $\nabla$, the third equation of three equations in (2.1) is reduced to

\[(2.2) \quad (\bar{\nabla}_X J)\bar{Y} = \theta(J\bar{Y})\bar{X} - \theta(\bar{Y})J\bar{X} - \bar{g}(\bar{X}, J\bar{Y})\zeta + \bar{g}(\bar{X}, \bar{Y})J\zeta.\]

Let $(M, g)$ be an $m$-dimensional lightlike submanifold of an indefinite Kaehler manifold $(\bar{M}, \bar{g})$ of dimension $(m + n)$. Then the radical distribution $\text{Rad}(TM) = TM \cap TM^\perp$ of $M$ is a subbundle of the tangent bundle $TM$ and the normal bundle $TM^\perp$, of rank $r$ $(1 \leq r \leq \min\{m, n\})$. In general, there exist two complementary non-degenerate distributions $S(TM)$ and $S(TM^\perp)$ of $\text{Rad}(TM)$ in $TM$ and $TM^\perp$, respectively, which are called the screen distribution and the co-screen distribution of $M$ [1], such that

\[TM = \text{Rad}(TM) \oplus_{\text{orth}} S(TM), \quad TM^\perp = \text{Rad}(TM) \oplus_{\text{orth}} S(TM^\perp),\]

where $\oplus_{\text{orth}}$ denotes the orthogonal direct sum. Denote by $F(M)$ the algebra of smooth functions on $M$ and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle $E$ over $M$. Let $X, Y, Z$ and $W$ be the vector fields on $M$, unless otherwise specified. We use the following range of indices:

\[i, j, k, \ldots \in \{1, \ldots, r\}, \quad a, b, c, \ldots \in \{r + 1, \ldots, n\}.\]

Let $\text{tr}(TM)$ and $\text{ltr}(TM)$ be complementary vector bundles to $TM$ in $TM|_M$ and $TM^\perp$ in $S(TM)^\perp$, respectively, and let $\{N_1, \ldots, N_r\}$ be a null basis of $\text{ltr}(TM)|_{U}$, where $U$ is a coordinate neighborhood of $M$. Then we have

\[\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0,\]

where $\{\xi_1, \ldots, \xi_r\}$ is a null basis of $\text{Rad}(TM)|_U$. Then we have

\[TM = TM \oplus \text{tr}(TM) = \{\text{Rad}(TM) \oplus \text{tr}(TM)\} \oplus_{\text{orth}} S(TM)
\]

\[= \{\text{Rad}(TM) \oplus \text{ltr}(TM)\} \oplus_{\text{orth}} S(TM) \oplus_{\text{orth}} S(TM^\perp).\]

A lightlike submanifold $M = (M, g, S(TM), S(TM^\perp))$ of $\bar{M}$ is called an $r$-lightlike submanifold [1, 3] if $1 \leq r < \min\{m, n\}$. For an $r$-lightlike $M$, we see that $S(TM) \neq \{0\}$ and $S(TM^\perp) \neq \{0\}$. In the sequel, by saying that $M$ is a lightlike submanifold we shall mean that it is an $r$-lightlike submanifold, with following local quasi-orthonormal field of frames of $M$:

\[\{\xi_1, \ldots, \xi_r, N_1, \ldots, N_r, F_{r+1}, \ldots, F_m, E_{r+1}, \ldots, E_n\},\]
where \( \{F_{r+1}, \ldots, F_m\} \) and \( \{E_{r+1}, \ldots, E_n\} \) are orthonormal bases of \( S(TM) \) and \( S(TM^\perp) \), respectively. Denote \( \epsilon_a = \bar{g}(E_a, E_a) \). Then \( \epsilon_a \delta_{ab} = \bar{g}(E_a, E_b) \).

Let \( P \) be the projection morphism of \( TM \) on \( S(TM) \). Then the local Gauss and Weingarten formulae of \( M \) and \( S(TM) \) are given respectively by

\[
\begin{align*}
\bar{\nabla}_X Y &= \nabla_X Y + \sum_{i=1}^{r} h^\ell_i(X,Y) N_i + \sum_{a=r+1}^{n} h^s_a(X,Y) E_a, \\
\bar{\nabla}_X N_i &= -A_{N_i} X + \sum_{j=1}^{r} \tau_{ij}(X) N_j + \sum_{a=r+1}^{n} \rho_{ia}(X) E_a, \\
\bar{\nabla}_X E_a &= -A_{E_a} X + \sum_{i=1}^{r} \lambda_{ai}(X) N_i + \sum_{b=r+1}^{n} \mu_{ab}(X) E_b, \\
\bar{\nabla}_X P Y &= \nabla^*_X P Y + \sum_{i=1}^{r} h^*_i(X, PY) \xi_i, \\
\bar{\nabla}_X \xi_i &= -A^*_\xi_i X - \sum_{j=1}^{r} \tau_{ji}(X) \xi_j,
\end{align*}
\]

where \( \nabla \) and \( \nabla^* \) are induced linear connections induced from \( \bar{\nabla} \) on \( M \) and \( S(TM) \), respectively, \( h^\ell_i \) and \( h^s_a \) are called the local second fundamental forms on \( M \), \( h^*_i \) are called the local second fundamental forms on \( S(TM) \). \( A_{N_i}, A_{E_a} \) and \( A^*_\xi_i \) are linear operators on \( M \), which are called the shape operators, and \( \tau_{ij}, \rho_{ia}, \lambda_{ai} \) and \( \mu_{ab} \) are 1-forms on \( M \). Using (1.2), (1.3) and (2.3), we see that

\[
\begin{align*}
(\nabla_X g)(Y, Z) &= \sum_{i=1}^{r} \left\{ h^\ell_i(X,Y) \eta_i(Z) + h^\ell_i(X,Z) \eta_i(Y) \right\}, \\
T(X, Y) &= \theta(Y) X - \theta(X) Y,
\end{align*}
\]

where \( \eta_i \)'s are 1-forms such that \( \eta_i(X) = \bar{g}(X, N_i) \).

From the facts that \( h^\ell_i(X,Y) = \bar{g}(\bar{\nabla}_X Y, \xi_i) \) and \( \epsilon_a h^s_a(X,Y) = \bar{g}(\bar{\nabla}_X Y, E_a) \), we know that \( h^\ell_i \) and \( h^s_a \) are symmetric and independent of the choice of \( S(TM) \). The above local second fundamental forms are
related to their shape operators by

\[(2.10) \quad h^i_\ell(X, Y) = g(A^*_\xi X, Y) - \sum_{k=1}^{r} h^k_\ell(X, \xi_k)\eta_k(Y),\]

\[(2.11) \quad \epsilon_a h^a_\sigma(X, Y) = g(A^e_a X, Y) - \sum_{k=1}^{r} \lambda_{ak}(X)\eta_k(Y),\]

\[(2.12) \quad h^i_\sigma(X, PY) = g(A^N_i X, PY).\]

Applying \(\overline{\nabla}X\) to \(\overline{g}(E_a, E_b) = \epsilon\delta_{ab}\), \(\overline{g}(\xi_i, \xi_j) = 0\), \(\overline{g}(\xi_i, E_a) = 0\), \(\overline{g}(N_i, N_j) = 0\) and \(\overline{g}(N_i, E_a) = 0\) by turns, we obtain \(\epsilon_b\mu_{ab} + \epsilon_a\mu_{ba} = 0\) and

\[(2.13) \quad \begin{align*}
\eta_j(A^N_i X) + \eta_i(A^N_j X) &= 0, \\
\eta_j(A^\sigma_{Ni} X) + \eta_i(A^\sigma_{Nj} X) &= 0, \\
\overline{g}(A^e_a X, N_i) &= \epsilon_{a\rho}(X).
\end{align*}\]

Furthermore, using (2.13), we see that

\[(2.14) \quad h^i_\ell(X, \xi_i) = 0, \quad h^j_\ell(\xi_j, \xi_k) = 0, \quad A^*_\xi \xi_i = 0.\]

Here, \((2.13)_i\) denotes the \(i\)-th equation of (2.13). We use the same notations for any others.

**Definition 2.1.** We say that a lightlike submanifold \(M\) of a semi-Riemannian manifold \((\overline{M}, \overline{g})\) is irrotational [11] if \(\overline{\nabla}X\xi_i \in \Gamma(TM)\) for all \(i \in \{1, \cdots, r\}\).

**Remark 2.2.** From (2.3) and (2.13), the above definition is equivalent to

\[(2.15) \quad h^j_\ell(X, \xi_i) = 0, \quad h^a_\sigma(X, \xi_i) = \lambda_{ai}(X) = 0.\]

**3. Structure equations**

Let \(M\) be a generic lightlike submanifold of \(\overline{M}\). From (1.1) we show that \(J(Rad(TM)), J(ltr(TM))\) and \(J(S(TM^\bot))\) are subbundles of \(S(TM)\). Thus there exist two non-degenerate almost complex distributions \(H_o\) and \(H\) with respect to \(J\), i.e., \(J(H_o) = H_o\) and \(J(H) = H\), such that

\[
S(TM) = \{J(Rad(TM)) \oplus J(ltr(TM))\} \oplus_{orth} J(S(TM^\bot)) \oplus_{orth} H_o, \\
H = Rad(TM) \oplus_{orth} J(Rad(TM)) \oplus_{orth} H_o.
\]

In this case, the tangent bundle \(TM\) of \(M\) is decomposed as follow:

\[(3.1) \quad TM = H \oplus J(ltr(TM)) \oplus_{orth} J(S(TM^\bot)).\]
Consider $r$-th local null vector fields $U_i$ and $V_i$, $(n - r)$-th local non-null unit vector fields $W_a$, and their 1-forms $u_i$, $v_i$ and $w_a$ defined by

\begin{align*}
(3.2) \quad & U_i = -JN_i, \quad V_i = -J\xi_i, \quad W_a = -JE_a, \\
(3.3) \quad & u_i(X) = g(X, V_i), \quad v_i(X) = g(X, U_i), \quad w_a(X) = \epsilon_a g(X, W_a).
\end{align*}

Denote by $S$ the projection morphism of $TM$ on $H$ and by $F$ the tensor field of type $(1, 1)$ globally defined on $M$ by $F = J \circ S$. Then $JX$ is expressed as

\begin{equation}
(3.4) \quad JX = FX + \sum_{i=1}^r u_i(X)N_i + \sum_{a=r+1}^n w_a(X)E_a.
\end{equation}

Applying $J$ to (3.4) and using (2.1)$_1$, (3.2) and (3.4), we have

\begin{equation}
(3.5) \quad F^2X = -X + \sum_{i=1}^r u_i(X)U_i + \sum_{a=r+1}^n w_a(X)W_a.
\end{equation}

By using (2.3)$_2$ and (3.4), we obtain

\begin{equation}
(3.6) \quad g(FX, FY) = g(X, Y) - \sum_{i=1}^r \{u_i(X)v_i(Y) + u_i(Y)v_i(X)\} \quad - \sum_{a=r+1}^n \epsilon_a w_a(X)w_a(Y).
\end{equation}

In the sequel, we say that $F$ is the \textit{structure tensor field} of $M$.

Now we shall assume that the characteristic vector field $\zeta$ belongs to the screen distribution $S(TM)$ of $M$. Applying $\nabla_X$ to (3.2) and (3.4) by turns and using (2.2)$\sim$ (2.7), (2.10)$\sim$ (2.12) and (3.2)$\sim$ (3.4), we get

\begin{equation}
(3.7) \quad \begin{cases}
h^l_j(X, U_i) = h^*_i(X, V_j) - \theta(V_j)\eta_i(X), \\
\epsilon_a h^*_a(X, U_i) = h^*_i(X, W_a) - \theta(W_a)\eta_i(X), \\
h^l_j(X, V_i) = h^*_l(X, V_j), \\
\epsilon_b h^*_b(X, W_a) = \epsilon_a h^*_a(X, W_b),
\end{cases}
\end{equation}
\[ \nabla_X U_i = F(A_{N_i} X) + \sum_{j=1}^{r} \tau_{ij}(X) U_j + \sum_{a=r+1}^{n} \rho_{ia}(X) W_a + \theta(U_i) X - v_i(X) \zeta - \eta_i(X) F \zeta, \]

\[ \nabla_X V_i = F(A_{\xi_i} X) - \sum_{j=1}^{r} \tau_{ji}(X) V_j + \sum_{a=r+1}^{n} h_{ij}^a(X, \xi_i) U_j \]

\[ \quad - \sum_{a=r+1}^{n} \epsilon_a \lambda_{ai}(X) W_a + \theta(V_i) X - u_i(X) \zeta, \]

\[ \nabla_X W_a = F(A_{E_a} X) + \sum_{i=1}^{r} \lambda_{ai}(X) U_i + \sum_{b=r+1}^{n} \mu_{ab}(X) W_b, \]

\[ \quad + \theta(W_a) X - \epsilon_a w_a(X) \zeta, \]

\[ (\nabla_X F) Y = \sum_{i=1}^{r} u_i(Y) A_{N_i} X + \sum_{a=r+1}^{n} w_a(Y) A_{E_a} X \]

\[ \quad - \sum_{i=1}^{r} h_{ij}^a(X, Y) U_i - \sum_{a=r+1}^{n} h_{ij}^a(X, Y) W_a \]

\[ \quad + \theta(FY) X - \theta(Y) FX - \bar{g}(X, JY) \zeta + g(X, Y) F \zeta. \]

4. Recurrent and Lie recurrent generic submanifolds

**Theorem 4.1.** There exist no generic lightlike submanifolds of an indefinite Kaehler manifold \( \bar{M} \) with a semi-symmetric metric connection such that \( \zeta \) belongs to \( S(TM) \) and \( F \) is parallel with respect to the connection \( \nabla \).

**Proof.** Assume that \( F \) is parallel with respect to the connection \( \nabla \). Replacing \( Y \) by \( \xi_j \) to (3.11) and using the fact that \( F \xi_j = -V_j \), we obtain

\[ \sum_{k=1}^{r} h_{ik}^j(X, \xi_j) U_k + \sum_{a=r+1}^{n} h_{aj}^a(X, \xi_j) W_a + \theta(V_j) X - u_j(X) \zeta = 0. \]

Taking the scalar product with \( N_i \) to (4.1) and then, taking \( X = \xi_j \), we get \( \theta(V_i) = 0 \). Also taking the scalar product with \( U_i \) to (4.1) and then, taking \( X = U_j \) and using \( \theta(V_j) = 0 \), we get \( \theta(U_i) = 0 \). Therefore, we obtain

\[ \theta(V_i) = 0, \quad \theta(U_i) = 0. \]
Taking the scalar product with $W_b$ to (4.1) and using $\theta(V_i) = 0$, we have
\begin{equation}
(4.2) \quad h_a^s(X, \xi_i) = \epsilon_a \theta(W_a) u_i(X).
\end{equation}
Replacing $Y$ by $W_a$ to (3.11) such that $\nabla X F = 0$, we have
\begin{equation}
A_{E_a} X = \sum_{i=1}^r h_i^\ell(X, W_a) U_i + \sum_{b=r+1}^n h_b^s(X, W_a) W_b \\
+ \theta(W_a) F X - \epsilon_a w_a(X) F \zeta.
\end{equation}
Taking the scalar product with $U_i$ to this equation, we obtain
\begin{equation}
(4.3) \quad h_a^s(X, U_i) = -\epsilon_a \theta(W_a) \eta_i(X).
\end{equation}
Taking $X = U_i$ to (4.2) and also, taking $X = \xi_i$ to (4.3) and then, comparing these two resulting equations, we obtain $\theta(W_a) = 0$. Taking the scalar product with $\zeta$ to (4.1) and using the facts that $\theta(V_i) = \theta(U_i) = \theta(W_a) = 0$, we have $u_j(X) = 0$ for all $X \in \Gamma(TM)$. It is a contradiction to $u_j(U_j) = 1$. Thus there exist no generic lightlike submanifolds of an indefinite Kaehler manifold $\bar{M}$ with a semi-symmetric metric connection subject such that $\zeta$ belongs to $S(TM)$ and $F$ is parallel with respect to the connection $\nabla$.

**Definition 4.2.** The structure tensor field $F$ of $M$ is said to be recurrent [6] if there exists a 1-form $\varpi$ on $TM$ such that
\[(\nabla_X F)Y = \varpi(X) FY.\]
A generic lightlike submanifold $M$ of an indefinite Kaehler manifold $\bar{M}$ is called recurrent if it admits a recurrent structure tensor field $F$.

**Theorem 4.3.** There exist no recurrent generic lightlike submanifolds of an indefinite Kaehler manifold $\bar{M}$ with a semi-symmetric metric connection such that the characteristic vector field $\zeta$ of $\bar{M}$ belongs to $S(TM)$.

**Proof.** From the above definition and (3.11), we obtain
\begin{equation}
(4.4) \quad \varpi(X) FY = \sum_{i=1}^r u_i(Y) A_{N_i} X + \sum_{a=r+1}^n w_a(Y) A_{E_a} X \\
- \sum_{i=1}^r h_i^\ell(X, Y) U_i - \sum_{a=r+1}^n h_a^s(X, Y) W_a \\
+ \theta(FY) X - \theta(Y) F X - \bar{g}(X, JY) \zeta + g(X, Y) F \zeta.
\end{equation}
Replacing $Y$ by $\xi_j$ to this and using the fact that $F\xi_j = -V_j$, we get (4.5)
\[ \varpi(X)V_j = \sum_{k=1}^{r} h^k(X, \xi_j)U_k + \sum_{b=r+1}^{n} h^b(X, \xi_j)W_b + \theta(V_j)X - u_j(X)\zeta. \]

Taking the scalar product with $N_i$ to this, we obtain $\theta(V_j)\eta_i(X) = 0$. Taking $X = \xi_i$ to this equation, we have $\theta(V_i) = 0$ for all $i$. Taking the scalar product with $V_i$ and $W_a$ to (4.5) and using $\theta(V_i) = 0$, we obtain
\[ h^i_j(X, \xi_i) = 0, \quad h^a_i(X, \xi_i) = \epsilon_a \theta(W_a)u_i(X). \]

Replacing $Y$ by $W_a$ to (4.4) and using the fact that $FW_a = 0$, we have
\[ A_{\epsilon_a}X = \sum_{i=1}^{r} h^i_i(X, W_a)U_i + \sum_{b=r+1}^{n} h^b_a(X, W_a)W_b + \theta(W_a)FX - \epsilon_a w_a(X)F\zeta. \]

Taking the scalar product with $U_i$ to this equation, we obtain
\[ h^a_i(X, U_i) = -\epsilon_a \theta(W_a)\eta_i(X). \]

Taking $X = \xi_i$ to (4.7) and also, taking $X = U_i$ to (4.6) and then, comparing two resulting equations, we get $\theta(W_a) = 0$. As $\theta(W_a) = 0$, we get
\[ h^i_j(X, \xi_i) = 0, \quad h^a_i(X, \xi_i) = 0. \]

Using these equations and the fact that $\theta(V_i) = 0$, Eq. (4.5) is reduced to
\[ \varpi(X)V_j = -u_j(X)\zeta. \]

Taking the scalar product with $\zeta$ to this, we have $u_j(X) = 0$ for all $X \in \Gamma(TM)$. It is a contradiction to $u_j(U_j) = 1$. Thus there exist no recurrent generic lightlike submanifolds of an indefinite Kaehler manifold $\bar{M}$ with a semi-symmetric metric connection such that $\zeta$ belongs to $S(TM)$.

**Definition 4.4.** The structure tensor field $F$ of $M$ is said to be Lie recurrent [7] if there exists a 1-form $\vartheta$ on $M$ such that
\[ (\mathcal{L}_X F)Y = \vartheta(X)FY, \]
where $\mathcal{L}_X$ denotes the Lie derivative on $M$ with respect to $X$, that is,
\[ (\mathcal{L}_X F)Y = [X, FY] - F[X, Y]. \]

In case $\mathcal{L}_X F = 0$, we say that $F$ is Lie parallel. A generic lightlike submanifold $M$ of an indefinite Kaehler manifold $\bar{M}$ is called Lie recurrent if it admits a Lie recurrent structure tensor field $F$. 

**Theorem 4.5.** Let $M$ be a Lie recurrent lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ with a semi-symmetric metric connection such that the characteristic vector field $\xi$ of $\bar{M}$ belongs to $S(TM)$. Then $F$ is Lie parallel.

**Proof.** Using the above definition, (2.9) and (3.11), we obtain

\[
\vartheta(X)FY = -\nabla_{FY}X + F\nabla_{Y}X - \bar{g}(X, JY)\xi + g(X, Y)F\xi + \sum_{i=1}^{r} u_i(Y)A_{N_i}X + \sum_{a=r+1}^{n} w_a(Y)A_{E_a}X - \sum_{i=1}^{r} h_i^\xi(X, Y)U_i - \sum_{a=r+1}^{n} h_a^\xi(X, Y)W_a.
\]

Replacing $Y$ by $\xi$ and also, $Y$ by $V_j$ to (4.8), respectively, we have

\[
\vartheta(X)V_j = -\nabla_{V_j}X + F\nabla_{\xi_j}X + u_j(X)\xi - \sum_{i=1}^{r} h_i^\xi(X, \xi_j)U_i - \sum_{a=r+1}^{n} h_a^\xi(X, \xi_j)W_a,
\]

\[
\vartheta(X)\xi_j = -\nabla_{\xi_j}X + F\nabla_{V_j}X + u_j(X)F\xi - \sum_{i=1}^{r} h_i^\xi(X, V_j)U_i - \sum_{a=r+1}^{n} h_a^\xi(X, V_j)W_a.
\]

Taking the scalar product with $U_i$ to (4.9) and $N_i$ to (4.10), we get

\[
-\delta_{ij}\vartheta = g(\nabla_{V_j}X, U_i) - \bar{g}(\nabla_{\xi_j}X, N_i) + \vartheta u_j(X),
\]

\[
\delta_{ij}\vartheta = g(\nabla_{V_j}X, U_i) - \bar{g}(\nabla_{\xi_j}X, N_i) + \vartheta u_j(X),
\]

respectively. From these two equations, we get $\vartheta = 0$. Thus $F$ is Lie parallel.

**Proposition 4.6.** Let $M$ be a Lie recurrent lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ with a semi-symmetric metric connection such that the characteristic vector field $\xi$ of $\bar{M}$ belongs to $S(TM)$. Then $\tau_{ij}$ and $\rho_{ia}$ satisfy $\tau_{ij} \circ F = 0$ and $\rho_{ia} \circ F = 0$. Moreover,

\[
\tau_{ij}(X) = \sum_{k=1}^{r} u_k(X)g(A_{N_k}V_j, N_i).
\]

**Proof.** Taking the scalar product with $N_i$ to (4.9) such that $X = W_a$ and using (2.11), (2.13) and (3.10), we get $h_a^\xi(U_i, V_j) = \rho_{ia}(\xi_j)$. Also, taking the scalar product with $W_a$ to (4.10) such that $X = U_i$ and using
(3.8), we have \( h^s_a(U_i, V_j) = -\rho_{ia}(\xi_j) \). Thus \( \rho_{ia}(\xi_j) = 0 \) and \( h^s_a(U_i, V_j) = 0 \).

Taking the scalar product with \( U_i \) to (4.9) such that \( X = W_a \) and using (2.11), (2.13)\(_{2,4}\) and (3.10), we get \( \epsilon_a \rho_{ia}(V_j) = \lambda_{aj}(U_i) \). Also, taking the scalar product with \( W_a \) to (4.9) such that \( X = U_i \) and using (2.13)\(_2\) and (3.8), we get \( \epsilon_a \rho_{ia}(V_j) = -\lambda_{aj}(U_i) \). Thus \( \rho_{ia}(V_j) = 0 \) and \( \lambda_{aj}(U_i) = 0 \).

Taking the scalar product with \( V_i \) to (4.9) such that \( X = W_a \) and using (2.13)\(_2\), (3.7)\(_4\) and (3.10), we obtain \( \lambda_{ai}(V_j) = -\lambda_{aj}(V_i) \). Also, taking the scalar product with \( W_a \) to (4.9) such that \( X = V_i \) and using (2.13)\(_2\) and (3.9), we have \( \lambda_{ai}(V_j) = \lambda_{aj}(V_i) \). Thus we obtain \( \lambda_{ai}(V_j) = 0 \) and \( h^s_i(V_j, W_a) = 0 \).

Summarizing the above results, we obtain

\[
(4.11) \rho_{ia}(\xi_j) = 0, \quad \rho_{ia}(V_j) = 0, \quad \lambda_{ai}(U_j) = 0, \quad \lambda_{ai}(V_j) = 0, \quad \lambda_{ai}(\xi_j) = 0, \\
h^s_a(U_i, V_j) = h^s_j(U_i, W_a) = 0, \quad h^s_i(V_j, W_a) = h^s_a(V_j, V_i) = 0.
\]

Taking the scalar product with \( N_i \) to (4.8) and using (2.13)\(_4\), we have

\[
(4.12) -\bar{g}(\nabla_F Y X, N_i) + g(\nabla_Y X, U_i) + \theta(U_i)g(X, Y) \\
+ \sum_{k=1}^r u_k(Y)\bar{g}(A_{N_k} X, N_i) + \sum_{a=r+1}^n \epsilon_a w_a(Y)\rho_{ia}(X) = 0.
\]

Taking \( X = \xi_j \) and \( Y = U_k \) to (4.12) and using (2.7) and (2.10), we have

\[
h^s_j(U_k, U_i) = \eta_i(A_{N_k} \xi_j).
\]

As \( h^s_j \) are symmetric, from the last equation, we see that \( \eta_i(A_{N_k} \xi_j) \) is symmetric with respect to \( i \) and \( k \). From this result and (2.13)\(_4\), we obtain

\[
(4.13) g(A_{N_k} \xi_j, N_i) = 0, \quad h^s_i(U_k, V_j) = 0.
\]

Taking \( X = \xi_j \) to (4.12) and using (2.7), (2.10), (4.11)\(_1\) and (4.13)\(_1\), we get

\[
(4.14) h^s_j(X, U_i) = \tau_{ij}(FX).
\]
Taking $X = U_i$ to (4.8) and using (2.12), (3.5), (3.7) and (3.8), we get

$$\sum_{k=1}^{r} u_k(Y)A_{N_k} U_i + \sum_{a=r+1}^{n} w_a(Y)A_{k_a} U_i$$

(4.15)

$$- A_{N_i} Y + \eta(Y)\zeta + v_i(Y)F \zeta - F(A_{N_i} FY) - \sum_{j=1}^{r} \tau_{ij}(FY)U_j - \sum_{a=r+1}^{n} \rho_{ia}(FY)W_a = 0.$$  

Taking the scalar product with $V_j$ to (4.15) and using (2.11), (2.12), (3.7) and (4.13), we obtain

$$h^{'\ell}_{ij}(X, U_i) = -\tau_{ij}(FX).$$

Comparing this equation with (4.14), we obtain

(4.16)  

$$\tau_{ij}(FX) = 0, \quad h^{'\ell}_{ij}(X, U_i) = 0.$$  

Taking $X = V_j$ to (4.12) and using (2.10), (3.9), (4.11) and (4.16), we get

$$\tau_{ij}(X) = \sum_{k=1}^{r} u_k(X)\bar{g}(A_{N_k} V_j, N_i).$$

Taking the scalar product with $W_a$ to (4.15), we have

$$\epsilon_a \rho_{ia}(FY) = -h^*_i(Y, W_a) + \theta(W_a)\eta(Y)$$

(4.17)

$$+ \sum_{k=1}^{r} u_k(Y)h^*_k(U_i, W_a) + \sum_{b=r+1}^{n} \epsilon_b w_b(Y)h^*_b(U_i, W_a).$$

Taking the scalar product with $W_a$ to (4.15), we have

$$\epsilon_a \rho_{ia}(FY) = -h^*_i(Y, W_a) + \theta(W_a)\eta(Y)$$

(4.18)

$$+ \sum_{k=1}^{r} u_k(Y)h^*_k(U_i, W_a) + \sum_{b=r+1}^{n} \epsilon_b w_b(Y)h^*_b(U_i, W_a).$$

Comparing the last two equations, we obtain $\rho_{ia}(FY) = 0$.  

□
Theorem 4.7. There exist no generic lightlike submanifolds of an indefinite Kaehler manifold $\bar{M}$ with a semi-symmetric metric connection such that $\zeta$ belongs to $S(TM)$, $V_i (i = 1, \cdots, r)$ are parallel with respect to $\nabla$ and $h^s_a (X, \xi_i) = 0$ for any vector field $X$ on $M$.

Proof. Assume that $V_i (i = 1, \cdots, r)$ are parallel with respect to the connection $\nabla$ and $h^s_a (X, \xi_i) = 0$ for any vector field $X$ on $M$. Taking the scalar product with $W_a$ to (3.9) and using $\lambda_{ai} (X) = h^s_a (X, \xi_i) = 0$, we get

$$\epsilon_a \theta(V_i) w_a (X) = \theta(W_a) u_i (X).$$

Taking $X = W_a$ and $X = U_i$ to this equation by turns, we obtain

$$\theta(V_i) = 0, \quad \theta(W_a) = 0.$$

Taking the scalar product with $V_j$ to (3.9) and using $\theta(V_i) = 0$, we have

$$h^s_i (X, \xi_j) = 0.$$

Taking the scalar product with $\zeta$ and $N_j$ to (3.9) by turns and using the last two equations, we obtain

$$h^s_i (X, F\zeta) = -u_i (X), \quad h^s_i (X, U_j) = 0.$$

From these two equations, we have the following impossible result:

$$-\delta_{ij} = -u_i (U_j) = h^s_i (U_j, F\zeta) = h^s_i (F\zeta, U_j) = 0.$$

Thus we have our theorem \hfill \Box

5. Indefinite complex space forms

Denote by $\bar{R}, R$ and $R^*$ the curvature tensor of the semi-symmetric metric connection $\bar{\nabla}$ on $\bar{M}$ and the induced linear connections $\nabla$ and $\nabla^*$ on $M$ and $S(TM)$, respectively. Using the Gauss-Weingarten formulae,
we obtain Gauss equations for $M$ and $S(TM)$, respectively:

\begin{equation}
\bar{\mathcal{R}}(X,Y)Z = R(X,Y)Z \\
+ \sum_{i=1}^{r} \{ h_i^f(X,Z)A_{N_i} Y - h_i^f(Y,Z)A_{N_i} X \} \\
+ \sum_{a=r+1}^{n} \{ h_a^s(X,Z)A_{E_a} Y - h_a^s(Y,Z)A_{E_a} X \} \\
+ \sum_{i=1}^{r} \{ (\nabla_X h_i^f)(Y,Z) - (\nabla_Y h_i^f)(X,Z) \} \\
+ \sum_{j=1}^{r} \{ \tau_{ji}(X)h_j^f(Y,Z) - \tau_{ji}(Y)h_j^f(X,Z) \} \\
+ \sum_{a=r+1}^{n} \{ \lambda_{ai}(X)h_a^s(Y,Z) - \lambda_{ai}(Y)h_a^s(X,Z) \} \\
- \theta(X)h_i^f(Y,Z) + \theta(Y)h_i^f(X,Z) \} N_i \\
+ \sum_{a=r+1}^{n} \{ (\nabla_X h_a^s)(Y,Z) - (\nabla_Y h_a^s)(X,Z) \} \\
+ \sum_{i=1}^{r} \{ \rho_{ia}(X)h_i^f(Y,Z) - \rho_{ia}(Y)h_i^f(X,Z) \} \\
+ \sum_{b=r+1}^{n} \{ \mu_{ba}(X)h_a^s(Y,Z) - \mu_{ba}(Y)h_a^s(X,Z) \} \\
- \theta(X)h_a^s(Y,Z) + \theta(Y)h_a^s(X,Z) \} E_a,
\end{equation}

\begin{equation}
R(X,Y)PZ = R^*(X,Y)PZ \\
+ \sum_{i=1}^{r} \{ h_i^s(X,PZ)A_{\xi_i} Y - h_i^s(Y,PZ)A_{\xi_i} X \} \\
+ \sum_{i=1}^{r} \{ (\nabla_X h_i^s)(Y,PZ) - (\nabla_Y h_i^s)(X,PZ) \} \\
+ \sum_{k=1}^{r} \{ \tau_{ik}(Y)h_k^s(X,PZ) - \tau_{ik}(X)h_k^s(Y,PZ) \} \\
- \theta(X)h_i^s(Y,PZ) + \theta(Y)h_i^s(X,PZ) \} \xi_i.
\end{equation}
Definition. An indefinite complex space form $\tilde{M}(c)$ is a connected indefinite Kaehler manifold of constant holomorphic sectional curvature $c$:

(5.3) $\tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z} = \frac{c}{4}\{\tilde{g}(\tilde{Y}, \tilde{Z})\tilde{X} - \tilde{g}(\tilde{X}, \tilde{Z})\tilde{Y}
+ \tilde{g}(J\tilde{X}, \tilde{Z})J\tilde{X} - \tilde{g}(J\tilde{X}, \tilde{Z})J\tilde{Y} + 2\tilde{g}(\tilde{X}, J\tilde{Y})J\tilde{Z}\}$,

where $\tilde{R}$ is the curvature tensor of the Levi-Civita connection $\tilde{\nabla}$ on $\tilde{M}$.

By directed calculations from (1.2) and (1.3), we see that

(5.4) $\tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z} = \tilde{g}(\tilde{X}, \tilde{Z})\nabla_{\tilde{Y}}\tilde{X} - \tilde{g}(\tilde{Y}, \tilde{Z})\nabla_{\tilde{X}}\tilde{Y}
+ \{((\nabla_{\tilde{X}}\theta)(\tilde{Z}) - \tilde{g}(\tilde{X}, \tilde{Z}))\tilde{Y} - (((\nabla_{\tilde{Y}}\theta)(\tilde{Z}) - \tilde{g}(\tilde{Y}, \tilde{Z}))\tilde{X}.$

Taking the scalar product with $\xi_i$ and $N_i$ to (5.4) by turns and then, substituting (5.1) and (5.3) into the resulting equation and using (5.2) and the facts that $g(\zeta, \xi_i) = \tilde{g}(\zeta, N_i) = \tilde{g}(\zeta, E_a) = 0$ and $\nabla$ is metric, we obtain

(5.5) $(\nabla_X h^\ell_i)(Y, Z) - (\nabla_Y h^\ell_i)(X, Z)$
+ $\sum_{k=1}^{r}\{\tau_{ki}(X)h^\ell_k(Y, Z) - \tau_{ki}(Y)h^\ell_k(X, Z)\}$
+ $\sum_{a=r+1}^{n}\{\lambda_{ai}(X)h^s_a(Y, Z) - \lambda_{ai}(Y)h^s_a(X, Z)\}$
- $\theta(X)h^\ell_i(Y, Z) + \theta(Y)h^\ell_i(X, Z)$
- $g(X, Z)h^\ell_i(Y, \zeta) + g(Y, Z)h^\ell_i(X, \zeta)$

- $\frac{c}{4}(u_i(X)\tilde{g}(JY, Z) - u_i(Y)\tilde{g}(JX, Z) + 2u_i(Z)\tilde{g}(X, JY))$,

(5.6) $(\nabla_X h^\ell_i)(Y, PZ) - (\nabla_Y h^\ell_i)(X, PZ)$
- $\sum_{k=1}^{r}\{\tau_{ik}(X)h^\ell_k(Y, PZ) - \tau_{ik}(Y)h^\ell_k(X, PZ)\}$
- $\sum_{k=1}^{r}\{h^\ell_k(Y, PZ)\eta_i(A_{N_k}X) - h^\ell_k(X, PZ)\eta_i(A_{N_k}Y)\}$
- $\sum_{a=r+1}^{n}\{h^s_a(Y, PZ)\eta_i(A_{E_a}X) - h^s_a(X, PZ)\eta_i(A_{E_a}Y)\}$.
Replacing $X$ is symmetric, we get

$$h(X, PZ) = g(X, PZ)h^*(X, PZ) + (\nabla_X h)(PZ)\eta(Y) + (\nabla_Y h)(PZ)\eta(X)$$

$$= \left(\frac{c}{4} + 1\right)\{g(Y, PZ)\eta(X) - g(X, PZ)\eta(Y)\}$$

$$+ \frac{c}{4}\{v_i(X)\bar{g}(JY, PZ) - v_i(Y)\bar{g}(JX, PZ) + 2v_i(PZ)\bar{g}(X, JY)\}.$$

**Theorem 5.1.** Let $M$ be a Lie recurrent generic lightlike submanifold of an indefinite complex space form $\tilde{M}(c)$ with a semi-symmetric metric connection such that $\zeta$ belongs to $S(TM)$. Then $c = 0$, i.e., $\tilde{M}(c)$ is flat.

**Proof.** In case $M$ is Lie recurrent. As $\tau_{ij}(FX) = 0$, from (4.14) we get

$$h(X, U_j) = 0. \quad (5.7)$$

Applying $\nabla_X$ to this equation and using (3.8) and (5.7), we have

$$\nabla_X h(X, U_j) = -h(X, F(A_{N_j}X)) - \sum_{a=r+1}^n \rho_{ja}(X)h_a(Y, W_a)$$

$$- \theta(U_j)h(Y, X) + h(Y, \zeta) + \eta(X)h(Y, F\zeta).$$

Substituting the last two equations into (5.5) such that $Z = U_j$, we have

$$h(X, F(A_{N_j}Y)) - h(X, F(A_{N_j}X))$$

$$- \sum_{a=r+1}^n \{\rho_{ja}(X)h_a(Y, W_a) - \rho_{ja}(Y)h_a(X, W_a)\}$$

$$+ \sum_{a=r+1}^n \{\lambda_{a}(X)h_a(Y, U_j) - \lambda_{a}(Y)h_a(X, U_j)\}$$

$$+ \eta_j(X)h_a(Y, F\zeta) - \eta_j(Y)h_a(X, F\zeta)$$

$$= \frac{c}{4}\{u_i(Y)\eta_j(X) - u_i(X)\eta_j(Y) + 2\delta_{ij}\bar{g}(X, JY)\}.$$

Taking $X = \xi_j$ and $Y = U_i$ to this and using (4.11)$_{3, 5}$ and (5.7), we get

$$h(\xi_j, F(A_{N_j}U_i)) + \sum_{a=r+1}^n \rho_{ja}(U_i)h_a(\xi_j, W_a) = \frac{3}{4}c. \quad (5.8)$$

Replacing $X$ by $\xi_j$ to (2.10) and using (2.14)$_2$ and the fact that $h^f$ is symmetric, we get $h(X, \xi_j) = g(A^*_\zeta\xi_j, X)$. From this result and
(2.13), we obtain \( g(A^*_\xi_j + A^*_\xi_i, X) = 0 \) for all \( X \). As \( S(TM) \) is non-degenerate, we get \( A^*_\xi_j = -A^*_\xi_i \). Thus \( A^*_\xi_j \) is skew-symmetric with respect to \( i \) and \( j \).

On the other hand, taking \( Y = U_j \) to (4.15), we have
\[
A_{N_j} U_i = A_{N_i} U_j.
\]
Applying \( F \) to this equation, we have
\[
F(A_{N_j} U_i) = F(A_{N_i} U_j).
\]
Thus \( F(A_{N_i} U_j) \) is symmetric with respect to \( i \) and \( j \). Therefore, we obtain
\[
(5.9) \quad h^i_j(X, W_a) = \epsilon_a h^a_i(X, V_i) = \epsilon_a h^a_i(V_i, \xi_j) = -\lambda_{aj}(V_i) = 0.
\]
From (5.8)~ (5.10), we obtain \( c = 0 \).

**Definition 5.2.** A lightlike submanifold \( M \) is said to be screen conformal [5] if there exist non-vanishing smooth functions \( \varphi_i \) on \( \mathcal{U} \) such that
\[
(5.11) \quad h^*_i(X, PY) = \varphi_i h^*_i(X, PY), \quad \forall i.
\]

**Theorem 5.3.** Let \( M \) be a screen conformal irrotational generic lightlike submanifold of an indefinite complex space form \( \bar{M}(c) \) with a semi-symmetric metric connection such that \( \zeta \) belongs to \( S(TM) \). Then \( c = 0 \), i.e., \( \bar{M}(c) \) is flat.

**Proof.** Using (3.7)\(_1,3\) and (5.11), we get
\[
h^j_i(X, U_i - \varphi_i V_i) = -\theta(V_i)\eta_i(X).
\]
Replacing \( X \) by \( \xi_j \) to this equation and using (2.14)\(_1\), we have
\[
(5.12) \quad \theta(V_i) = 0, \quad h^j_i(X, U_i - \varphi_i V_i) = 0.
\]
If \( M \) is irrotational, then we have (2.15). Using (3.7)\(_2,4\) and (5.11), we get
\[
h^a_i(X, U_i - \varphi_i V_i) = -\epsilon_a \theta(W_a)\eta_i(X).
\]
Replacing \( X \) by \( \xi_i \) to this equation and using (2.15)\(_2\), we obtain
\[
(5.13) \quad \theta(W_a) = 0, \quad h^a_i(X, U_i - \varphi_i V_i) = 0.
\]
Applying \( \bar{\nabla}_X \) to \( \theta(V_i) = 0 \) and using (2.15)\(_1,2\), (3.9) and (5.12)\(_1\), we obtain
\[
(5.14) \quad (\bar{\nabla}_X \theta)(V_i) = h^j_i(X, F\zeta) + u_i(X).
\]
Applying $\nabla_X$ to $h^e_i(Y, PZ) = \varphi_i h^e_i(Y, PZ)$, we have

$$(\nabla_X h^e_i)(Y, PZ) = (X \varphi_i) h^e_i(Y, PZ) + \varphi_i(\nabla_X h^e_i)(Y, PZ).$$

Substituting this equation into (4.6) and using (4.5), we have

$$(X \varphi_i) h^e_i(Y, PZ) - (Y \varphi_i) h^e_i(X, PZ)$$

$$- \sum_{j=1}^{r} \{ \varphi_j \tau_{ji}(X) + \varphi_j \tau_{ij}(X) + \eta(A_{x_j} X) \} h^e_j(Y, PZ)$$

$$+ \sum_{j=1}^{r} \{ \varphi_j \tau_{ji}(Y) + \varphi_j \tau_{ij}(Y) + \eta(A_{x_j} Y) \} h^e_j(X, PZ)$$

$$- \sum_{a=r+1}^{n} \epsilon_a \{ \rho_{ia}(X) h^e_a(Y, PZ) - \rho_{ia}(Y) h^e_a(X, PZ) \}$$

$$- (\nabla_X \theta)(PZ) \eta_i(Y) + (\nabla_Y \theta)(PZ) \eta_i(X)$$

$$= (c + 1) \{ \eta_i(X) g(Y, PZ) - \eta_i(Y) g(X, PZ) \}$$

$$+ \frac{c}{4} \{ [v_i(X) - \varphi_i u_i(X)] g(FY, PZ) - [v_i(Y) - \varphi_i u_i(Y)] g(FX, PZ)$$

$$+ 2[v_i(PZ) - \varphi_i u_i(PZ)] g(X, JY) \}.$$

Taking $Y = \xi_i$ and $PZ = V_j$ to this and using (2.15) and (5.14), we have

$$- (\xi_i \varphi_i) h^e_i(X, V_j) - h^e_j(X, F\zeta)$$

$$+ \sum_{j=1}^{r} \{ \varphi_j \tau_{ji}(\xi_i) + \varphi_j \tau_{ij}(\xi_i) + \eta(A_{x_j} \xi_i) \} h^e_j(X, V_j)$$

$$+ \sum_{a=r+1}^{n} \epsilon_a \rho_{ia}(\xi_i) h^e_a(X, V_j) = - \frac{3}{4} cu_j(X).$$

Taking $X = U_j + \varphi_j V_j$ to this and using (5.12) and (5.13), we get $c = 0$.

**Definition 5.4.** [1] We say that $S(TM)$ is totally umbilical in $M$ if there exist smooth functions $\gamma_i$ on a coordinate neighborhood $U$ such that

$$(5.15) \quad h^e_i(X, PY) = \gamma_i g(X, PY), \quad \forall i.$$ 

In case $\gamma_i = 0$ on $U$, we say that $S(TM)$ is totally geodesic in $M$.

**Theorem 5.5.** Let $M$ be an irrotational generic lightlike submanifold of an indefinite complex space form $\bar{M}(c)$ with a semi-symmetric metric...
connection such that \( \zeta \) belongs to \( S(TM) \). If \( S(TM) \) is totally umbilical in \( M \), then \( c = 0 \) and \( \gamma_i = 0 \), i.e., \( S(TM) \) is totally geodesic in \( M \).

Proof. If \( S(TM) \) is totally umbilical, then, from (3.7)\(_1\) and (5.15), we have

\[
h^\ell_j(X, U_i) = \gamma_i u_j(X) - \theta(V_j)\eta_i(X).
\]

Replacing \( X \) by \( \xi_j, V_k, U_k \) and \( \zeta \) to this by turns and using (2.14)\(_1\), we get

\[
(5.16) \quad \theta(V_i) = 0, \quad h^\ell_j(V_k, U_i) = 0, \quad h^\ell_j(U_k, U_i) = \gamma_i \delta_{kj}, \quad h^\ell_j(U_i, \zeta) = 0,
\]

\[
(5.17) \quad h^\ell_j(X, U_i) = \gamma_i u_j(X).
\]

If \( M \) is irrotational, then we have (2.15). From (3.7)\(_2\) and (5.15), we get

\[
h^a_i(X, U_i) = \gamma_i w_a(X) - \theta(W_a)\eta_i(X).
\]

Replacing \( X \) by \( \xi_i, V_k, U_k \) and \( \zeta \) to this by turns and using (2.15)\(_2\), we have

\[
(5.18) \quad \theta(W_a) = 0, \quad h^a_i(V_k, U_i) = 0, \quad h^a_i(U_k, U_i) = 0, \quad h^a_i(U_i, \zeta) = 0.
\]

Applying \( \nabla_X \) to \( \theta(V_i) = 0 \) and using (2.10), (2.15), (3.4) and (3.9), we obtain

\[
(\nabla_X \theta)(V_i) = h^\ell_i(X, F\zeta) + u_i(X).
\]

Taking \( X = F\zeta \) to (5.17), we get \( h^\ell_j(U_i, F\zeta) = 0 \). Replacing \( X \) by \( U_j \) to the last equation and using the fact that \( h^\ell_j(U_i, F\zeta) = 0 \), we obtain

\[
(5.19) \quad (\nabla_{U_j} \theta)(V_i) = \delta_{ij}.
\]

Applying \( \nabla_X \) to \( h^a_i(Y, PZ) = \gamma_i g(Y, PZ) \) and using (2.7), we obtain

\[
(\nabla_X h^a_i)(Y, PZ) = (X\gamma_i)g(Y, PZ) + \gamma_i \sum_{j=1}^{r} h^\ell_j(X, PZ)\eta_j(Y).
\]
Substituting this equation and (5.15) into (5.6), we have

\[
\begin{align*}
\{ X \gamma_i - \sum_{j=1}^{r} \gamma_j \tau_{ij}(X) - \left[ \frac{c}{4} + 1 \right] \eta_i(X) \} g(Y, PZ) \\
- \{ Y \gamma_i - \sum_{j=1}^{r} \gamma_j \tau_{ij}(Y) - \left[ \frac{c}{4} + 1 \right] \eta_i(Y) \} g(X, PZ) \\
+ \sum_{j=1}^{r} \{ \gamma_i \eta_j(Y) + \eta_i(A_{x_j} Y) \} h^\ell_j(Y, PZ) \\
- \sum_{j=1}^{r} \{ \gamma_i \eta_j(X) + \eta_i(A_{x_j} X) \} h^\ell_j(Y, PZ) \\
- \sum_{a=r+1}^{n} \{ h_a^s(Y, PZ) \eta_i(A_{E_a} X) - h_a^s(X, PZ) \eta_i(A_{E_a} Y) \} \\
- (\nabla X \theta)(PZ) \eta_i(Y) + (\nabla Y \theta)(PZ) \eta_i(X) \\
= \frac{c}{4} \{ v_i(X) g(FY, PZ) - v_i(Y) g(FX, PZ) + 2v_i(PZ) \bar{g}(X, Y) \}.
\end{align*}
\]

Replacing \( Y \) by \( \xi_k \) to this and using (2.15), (3.2) and (3.3), we have

\[
(5.20) \quad \{ \xi_k \gamma_i - \sum_{j=1}^{r} \gamma_j \tau_{ij}(\xi_k) - \left[ \frac{c}{4} + 1 \right] \delta_{ik} \} g(X, PZ) \\
- \sum_{j=1}^{r} \{ \gamma_i \delta_{jk} + \eta_i(A_{x_j} \xi_k) \} h^\ell_j(X, PZ) \\
- \sum_{a=r+1}^{n} \eta_i(A_{E_a} \xi_k) h_a^s(X, PZ) \\
+ (\nabla X \theta)(PZ) \delta_{ik} - (\nabla \xi_k \theta)(PZ) \eta_i(X) \\
= \frac{c}{4} \{ v_i(X) u_k(PZ) + 2v_i(PZ) \bar{u}_k(X) \}.
\]

Taking \( X = U_h \) and \( PZ = V_h \) and using (5.16)\(_2\), (5.18)\(_2\) and (5.19), we have

\[
(5.21) \quad \xi_k \gamma_i - \sum_{j=1}^{r} \gamma_j \tau_{ij}(\xi_k) = \frac{3}{4} c \delta_{ik}.
\]
Applying $\tilde{\nabla}_X$ to $g(\zeta, \zeta) = 1$ and using the fact that $\tilde{\nabla}$ is metric, we obtain
\begin{equation}
(\tilde{\nabla}_X \theta)(\zeta) = 0.
\end{equation}
Taking $X = U_k$ and $Z = \zeta$ to (5.20) and using (5.16), (5.21) and (5.22), we get $\theta(U_i) = 0$. As $\tilde{g}(J\zeta, \zeta) = 0$, we see that $g(F\zeta, \zeta) = 0$. Thus
\begin{equation}
\theta(U_i) = 0, \quad g(F\zeta, \zeta) = 0.
\end{equation}
As $\theta(V_j) = \theta(U_i) = \theta(W_a) = 0$, we get $J\zeta = F\zeta \in \Gamma(S(TM))$. Applying $\tilde{\nabla}_X$ to $\theta(U_i) = 0$ and using (3.8), (5.18) and (5.23), we obtain
\begin{equation}
(\tilde{\nabla}_X \theta)(U_i) = \gamma_i g(X, F\zeta) + v_i(X).
\end{equation}
Taking $X = V_j$ and $X = U_j$ to this equation by turns, we obtain
\begin{equation}
(\tilde{\nabla}_{V_j} \theta)(U_i) = \delta_{ij}, \quad (\tilde{\nabla}_{U_j} \theta)(U_i) = 0.
\end{equation}
Taking $X = V_h$ and $PZ = U_h$ to (5.20) and using (5.16), (5.21) and (5.24), we have $c = 0$. Thus $\tilde{M}(c)$ is flat.

Theorem 5.6. Let $M$ be a generic lightlike submanifold of an indefinite Kaehler manifold $\tilde{M}(c)$ with a semi-symmetric metric connection such that $\zeta$ belongs to $S(TM)$ and $U_i$s are parallel with respect to the connection $\nabla$. If either $\rho_{ia} = 0$ or $\tau_{ij} = 0$, then $c = 0$, i.e., $M(c)$ is flat.

Proof. (1) In case $\rho_{ia} = 0$. Taking the scalar product with $W_a$ to (3.8), we get $\epsilon_a \theta(U_i) w_a(X) - \theta(W_a)v_i(X) = 0$. Taking $X = W_a$ and $X = V_i$ to this result by turns, we have
\begin{equation}
\theta(U_i) = 0, \quad \theta(W_a) = 0.
\end{equation}
Taking the scalar product with $U_j, N_j, \zeta$ and $F\zeta$ to (3.8) by turns and using (3.6), (5.26) and the fact that $g(F\zeta, \zeta) = 0$, we obtain

\begin{align*}
(5.27) \quad \bar{g}(A_{N_i}X, N_j) &= 0, \quad h_i^*(X, U_j) = 0, \\
g(F(A_{N_i}X), \zeta) &= v_i(X), \quad h_i^*(X, \zeta) = \eta_i(X).
\end{align*}

Applying $\nabla_X$ to $\theta(U_i) = 0$ and using (3.8) and (5.27), we have

\begin{equation}
(5.28) \quad (\nabla_X \theta)(U_i) = 0.
\end{equation}

Applying $\nabla_Y$ to (5.27) and using the fact that $\nabla_Y U_j = 0$, we have

\begin{equation}
(\nabla_X h_i^*)(Y, U_j) = 0.
\end{equation}

Substituting this equation and (5.27) into (5.6) such that $PZ = U_j$ and using (2.13), (5.27) and (5.28) and the fact that $\rho_{ia} = 0$, we have

\begin{equation}
\frac{c}{4} \left( v_j(Y) \eta_i(X) - v_j(X) \eta_i(Y) + v_i(Y) \eta_j(X) - v_i(X) \eta_j(Y) \right) = 0.
\end{equation}

Taking $X = \xi_i$ and $Y = V_j$ to this equation, we obtain $c = 0$.

(2) In case $\tau_{ij} = 0$. Taking the scalar product with $V_j$ to (3.8), we get $\theta(U_i) u_j(X) - \theta(V_j) v_i(X) = 0$. Taking $X = U_j$ and $X = V_j$ to this equation by turns, we have

\begin{equation}
(5.29) \quad \theta(U_i) = 0, \quad \theta(V_i) = 0.
\end{equation}

Taking the scalar product with $U_j, N_j, F\zeta$ and $\zeta$ to (3.8) by turns and using (3.6), (5.29) and the fact that $g(F\zeta, \zeta) = 0$, we obtain

\begin{align*}
(5.30) \quad \bar{g}(A_{N_i}X, N_j) &= 0, \quad h_i^*(X, U_j) = 0, \quad h_i^*(X, \zeta) = \eta_i(X), \\
g(F(A_{N_i}X), \zeta) + \sum_{a=r+1}^n \theta(W_a) \rho_{ia}(X) &= v_i(X).
\end{align*}

Applying $\nabla_X$ to $\theta(U_i) = 0$ and using (3.8) and (5.30), we have

\begin{equation}
(5.31) \quad (\nabla_X \theta)(U_i) = 0.
\end{equation}

Applying $\nabla_Y$ to (5.30) and using the fact that $\nabla_Y U_j = 0$, we have

\begin{equation}
(\nabla_X h_i^*)(Y, U_j) = 0.
\end{equation}

Substituting this equation and (5.30) into (5.6) with $PZ = U_j$ and using (5.30) and (5.31), we have

\begin{equation}
\frac{c}{4} \left( v_j(Y) \eta_i(X) - v_j(X) \eta_i(Y) + v_i(Y) \eta_j(X) - v_i(X) \eta_j(Y) \right) = 0.
\end{equation}

Taking $X = \xi_i$ and $Y = V_j$ to this equation, we obtain $c = 0$. \qed
Generic lightlike submanifolds of an indefinite Kaehler manifold

References


Jae Won Lee
Department of Mathematics Education and RINS,
Gyeongsang National University, Jinju 52828, Republic of Korea.
E-mail: leeaew@gnu.ac.kr

Chul Woo Lee
Department of Mathematics, Kyungpook National University,
Daegu 41566, Republic of Korea.
E-mail: mathisu@knu.ac.kr