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PROBABILISTIC ANALYSIS OF A SYSTEM CONSISTING OF TWO SUBSYSTEMS IN THE SERIES CONFIGURATION UNDER COPULA REPAIR APPROACH

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Abstract. Redundancy is commonly employed to improve system reliability. In most situations, components in the standby configurations are assumed statistically similar but independent. In many realistic models, all parts in standby are not treated as identical as they have different failure possibilities. The operational structure of the system has subsystem-1 with five identical components working under 2-out-of-5: G; policy, and the subsystem-2 has two units and functioning under 1-out-of-2: G; policy. Failure rates of units of subsystems are constant and assumed to follow an exponential distribution. Computed results give a new aspect to the scientific community to adopt multi-dimension repair in the form of the copula.

1. Introduction

Many deteriorating repairable systems, such as aircraft, space shuttles, hydraulic control systems, nuclear plants, satellite projection systems, electric power generating systems, and communication system suffer from unavoidable failures due to complex degradation processes and environmental conditions. In practice, we come across many complex systems where an unpredicted failure of any of the parts results in the reduction of efficiency of the whole system or the complete failure of the system, and because of it, the reliability of the system reduces. Redundancy is a technique widely used to improve system reliability and availability. It is used in the form of identical components connected in such a way that when one component fails the others will keep the system functioning. Moreover, redundancy is highly

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cost-effective in achieving a certain reliability level of the system. Therefore, to enhance reliability and adequate performances, a k-out-of-n system structure in which at least k components out of n must be functioning for the system to be operational, play a vital role. For example, a communications system with three transmitters, where the average message load may be such that at least two transmitters must be operational at all times can be modeled as a 2-out-of-3: G system. Furthermore, it may be possible to drive a car with an eight-cylinder V configuration engine if only four cylinders are firing. However, if less than four cylinders fire, then the car cannot be driven. Thus, the functioning of the engine can be demonstrated by a 4-out-of-8: G system. Conclusively a k-out-of-n: G/F systems play a very crucial role in system reliability theory to the proper operation of the system. In the past decade, k-out-of-n redundant systems have been studied extensively in the context of computing the reliability and availability, optimization of the system, common cause failures, and repair facility for fixing failed components. Several articles have been published on reliability and availability analysis of complex systems by researchers and presented a highly significant work to improve the reliability of real-life industrial systems. More than a few authors, including Moustafa [3], Kullstam [4], and Liang et al. [5] examined the reliability characteristics using k-out-of-n repairable systems with different failure modes. Park and Pham [6] studied the block replacement policy for the$k$-out-of-n system on threshold numbers of fail components and the risk cost of the system. Among them, Kumar and Gupta [7], Vanderperre [8], Mokaddis et al. [9], Kumar [10], Dhillon [11] have studied the reliability measures of considered systems by taking different types of failure and general repair employing the k-out-of-n: G operation policy. The type of repair approach predicts the performance of repairable systems. The cited literature from [1] to [11] has considered general repair between two transition states, while in many real situations, more than one repair is possible. Such types of possibility insist the researchers to repair a completely failed state by employing Goumbel-Hougard copula repair distribution developed by Nelson [12]. To cite a few of them, Alka and Singh [13] analyzed the reliability characteristics of a complex repairable system composed of two subsystems in a parallel configuration using copula repair strategy. The authors' considered first subsystem L working under 2-out-of-3: G policy, while the second subsystem M is working on 5-out-of-5: G policy. Goyal et al. [14] studied a three-unit series system under k-out-of-n redundancy and done the sensitivity analysis for the system. Singh et al. ([15], [16]) have studied the performance analysis of the complex system in the series configuration under different failure and repair disciplines using copula concept. Lado et al. [17] evaluated the reliability measures (Availability, reliability, and MTTF, sensitivity, and profit analysis) of the repairable complex system with two subsystems connected in a series configuration using the supplementary variable and Laplace transforms. Kumar et al. [18] presented a novel method for availability analysis of an engineering system involving subsystems in series configuration incorporating waiting time to repair. Sharma and Kumar [19] analyzed availability improvement for the success of the k-out-of-n machining system with multiple working vacations. The repairman can choose multiple working vacations of random length during its dormant time. El-Damcese et al. [20] analyzed availability and reliability for the r-out-of-m: G system with three types of failures using the Markov model. Ram and Singh [21] studied availability and cost analysis of a parallel redundant complex system with two types of failure under preemptive-resume repair discipline using copula repair. Singh et al. [22] investigated cost analysis of an engineering system involving two subsystems in a series configuration with controllers and human failure under the concept of k-out-of-n: G policy using copula repair approach. Xinzhuo Bao [23] examined reliability characteristics for the series Markovian repairable system by considering the repair time of the system failure as too short and long and tried to delay the failure effect. Zheng et al. [24] studied a single unit Markov repairable system with neglecting repair time. Kumar and Ram [25] mugged upon sensitivity analysis of coal handling thermal power plant with two subsystems (Wagon Tripler and Conveyor) in a series configuration with one standby unit in both subsystems with different failure rates and general repair concept. Sensitivity assessment of air and refrigeration systems with four equipment (Compressor, condenser, expansion device, and evaporator) have premediated by Goyal et al. [26] El-Damcese et al. [27] have illustrated reliability and MTTF for the three-element system in series and parallel configuration utilizing Fuzzy failure rates. El- Damcese and El- Sodany [28] have studied reliability and sensitivity analysis of the k-out-of-n: G warm standby parallel repairable system with replacement at common cause employing Markov model.

2. Model Description and Notations

2.1. System Description Numerous models carrying the standby unit (s) have been widely studied in the explored literature above. Moreover, the configuration of k-out-of-n: G/F has also been studied by various researchers but the structure k-out-of-n: G type together with series and parallel configurations have not much attended by investigator due to the complexity of configuration. Treating the above realities in this paper, we have developed a system with two subsystems in a series configuration that includes a switching device. The subsystem-1 has five units that are working under 2-out-of-5: G; policy, and the subsystem-2 has two units that are working under 1-out-of-2: G; policy. Moreover, the switching device in the system is unreliable, and as long as the switch fails, the whole system fails immediately. The failure rates of units of both the subsystems are constant and assumed to follow an exponential distribution, but their repair supports two types of distribution namely general distribution and Gumbel-Hougaard family copula distribution. Furthermore, the failure rate of all the units in subsystem-1 is the same whereas in subsystem-2 it is different. We used the supplementary variable technique (Cox, 1; Oliveira et al., 2) and Laplace transformations to evaluate various characteristics like transition state probabilities, availability, reliability, MTTF, and profit analysis. Explicit expressions are obtained with help of MAPLE (software). Some particular cases have also been discussed for different values of failure rates. The results are demonstrated by graphs, and conclusions have been drawn. The paper is organized as follows: Section 2 introduces about system description with assumptions and notations, Sections 3 discusses states description of the model, Section 4 presents the mathematical formulation of the model and Section 5 gives the analytical study of the model that includes availability, reliability, MTTF and profit analysis. Conclusions of the proposed analysis are given in Section 6. At last, some special cases are taken to highlight the reliability characteristics of the system. These are as follows:

(i): Both the subsystems have a switching device.
(ii): Only subsystem-2 has the switching device.
(iii): Only subsystem-1 has the switching device.
(iv): No subsystems have switching devices.

The state description of the considered system is given in Table 1, and the transition state diagram of the investigated system is shown in Figure 2.

2.2. Assumptions Following assumptions have been considered for the study of model:

(1) Initially, the system is in state S_0, and all the units are in good working conditions.

(2) The subsystem-1 has five identical units and works successfully under the policy 2-out-of-5: G. The subsystem-1 has five identical units and works successfully under the policy 2-out-of-5: G .

(3) The subsystem-2 has two non-identical units and works successfully under the policy 1-out-of-2: G.

(4) Both the subsystems are connected via a switching device, which in the system may be unreliable at the time of need. Moreover, if the switch fails, the whole system fails immediately.

(5) The units in both the subsystems are in parallel mode and warm standby and ready to start within a negligible time after the failure of any unit in the subsystems.

(6) The repairman is available full time with the system and may be called as soon as the system reaches to wholly or partially failed state.

(7) The failure rate of all the units in subsystem-1 is the same, while the failure rates of both the units in subsystem-2 are different.

(8) Both the subsystems, including switching devices, have constant failure rates and follow an exponential distribution

(9) The complete failed system needs repair immediately. For this, copula repair can be used to restore the system. No damage has been reported due to the repair of the system.

(10) As soon as the failed unit repaired, it is ready to perform the task as good as new.

2.3. Notations

- t Time scale
- s Laplace transform variable
- λ_1 The failure rate of each unit in subsystem-1
- λ_A/λ_B The failure rate of both the units A and B in subsystem-2.
- $\lambda_{s_1}/\lambda_{s_1}$ The failure rate of switching devices between units for subsystem-1/subsystem-2.
- ϕ_1 Repair rate of each unit in subsystem-1.
- ϕ_A/ϕ_A Repair rate of each unit A and B in subsystem-2.
- $P_0(t)$ The state transition probability that the system is in S_i state at an instant $i = 0$
- $\bar{P}(s)$ Laplace transformation of the state transition probability $P(t)$.

- $P_i(x,t)$ The probability that the system is in the state S_i for $i = 1 \sim 8, s_1, s_2$ and the system is under repair with elapsed repair time is x, t. x is repaired variable and t is time variable.
- $E_p(t)$ Expected profit in the interval $[0,t)$.
- K_1, K_2 Revenue generated and service cost per unit time, respectively.
- μ_0 An expression of the joint probability from failed state S_i to good state S_0 according to the Gumbel-Hougaard family copula, is given as

$$\mu_0 = C_\theta\left\{u_1, u_2\right\} = exp\left[x^\theta + \{\log\phi(x)\}^\theta\right]^{\frac{1}{\theta}}$$

where $u_1 = \phi(x)$, $u_2 = e^x$. Here is the parameter $1 < \theta < \infty$

3. SYSTEM CONFIGURATION AND STATE TRANSITION DIAGRAM

The state description of the model is shown in Table 1 that highlights S is a perfect state where both the subsystems are in good working condition. S_1, S_2, S_3, S_5, and S_6 are the states where the system is in degraded mode and general repair is employed, while S_4, S_7, S_8 and S_9 are the states where the system is in the total failure mode and repair is being applied using Gumbel-Hougaard family copula distribution. System configuration is shown in Table 1 while the state transition diagram in Figure 2.

Table 1. State Description

State	Description	State	Description
S_0	Perfect state All Units good.	S_6	Degraded state Unit B of subsystem-2 failed General Repair.
S_1	Degraded state One unit of subsystem-1 failed General Repair	S_4	Totally failed state More than 3 units failed in subsystem-1 Copula Repair.
S_2	Degraded state Two units of subsystem-1 failed General Repair	S_7	Totally failed state Both the units failed in subsystem-2 Copula Repair.
S_3	Degraded state Three units of subsystem-1 failed General Repair	S_8	Totally failed state Switching device failed in subsystem-1Copula Repair.
S_5	Degraded state Unit A of subsystem-2 failed General Repair	S_9	Totally failed state Switching device failed in subsystem-2 Copula Repair.

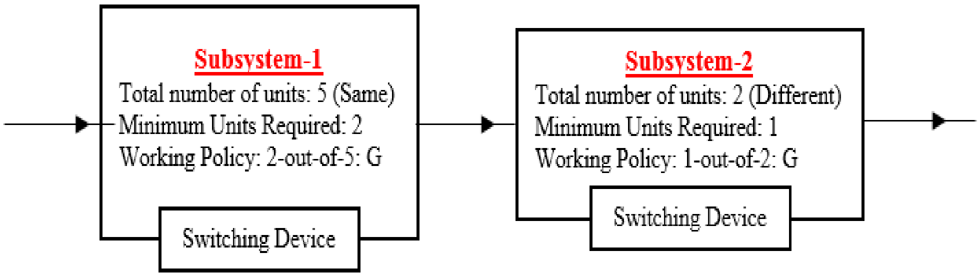

Figure 1. System Configuration

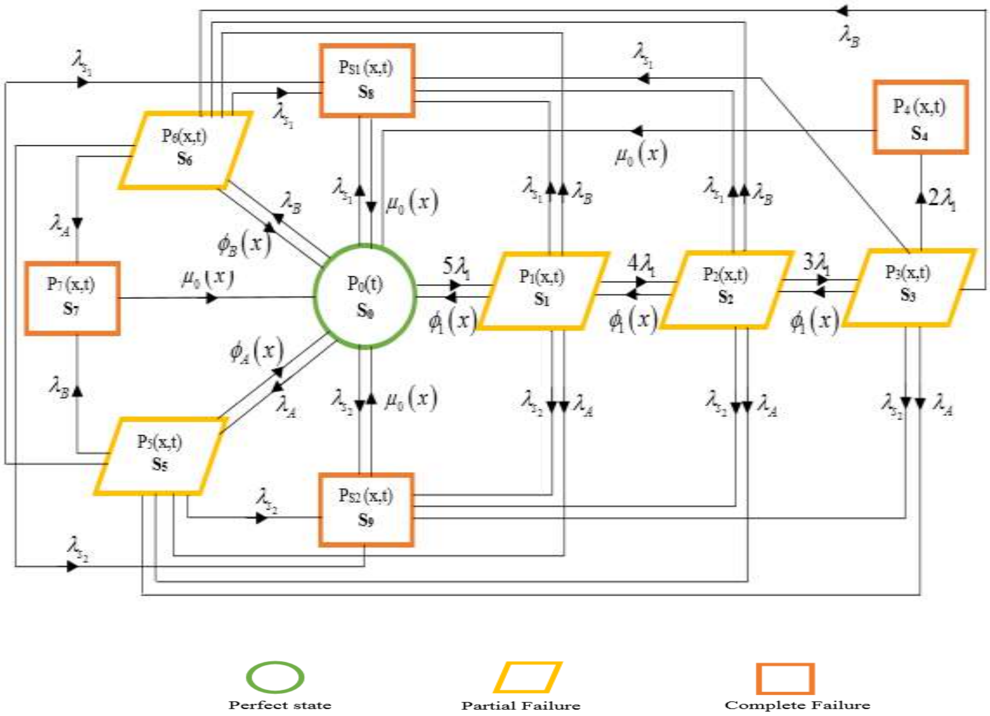

Figure 2. State Transition Diagram of the Model.

4. Formulation of the Mathematical Model

By a probability of considerations and continuity arguments, we can obtain the following set of difference-differential equations:

$$\left[\frac{\partial}{\partial x}+5\lambda_1+\lambda_A+\lambda_B+\lambda_{s_1}+\lambda_{s_2}\right]P_0(t)=\int_0^\infty \phi_1(x)P_1(x,t)dx \tag{4.1}$$
$$+\int_0^\infty \phi_A(x)P_5(x,t)dx+\int_0^\infty \phi_B(x)P_6(x,t)dx+\int_0^\infty \mu_0(x)P_i(x,t)dx,\ \ i=4,7,s_1,s_2$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+4\lambda_1+\lambda_A+\lambda_B+\lambda_{s_1}+\lambda_{s_2}+\phi_1\right]P_1(x,t) = 0 \tag{4.2}$$
$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+3\lambda_1+\lambda_A+\lambda_B+\lambda_{s_1}+\lambda_{s_2}+\phi_1\right]P_2(x,t) = 0 \tag{4.3}$$
$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+2\lambda_1+\lambda_A+\lambda_B+\lambda_{s_1}+\lambda_{s_2}+\phi_1\right]P_3(x,t) = 0 \tag{4.4}$$
$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\lambda_B+\lambda_{s_1}+\lambda_{s_2}+\phi_A\right]P_5(x,t) = 0 \tag{4.5}$$
$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\lambda_A+\lambda_{s_1}+\lambda_{s_2}+\phi_B\right]P_6(x,t) = 0 \tag{4.6}$$
$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\mu_0(x)\right]P_j(x,t) = 0;\ j=4,7,s_1,s_2 \tag{4.7}$$

Boundary conditions

$$P_1(0,t) = 5\lambda_1 P_0(t) \tag{4.8}$$
$$P_2(0,t) = 20\lambda_1^2 P_0(t) \tag{4.9}$$
$$P_3(0,t) = 60\lambda_1^3 P_0(t) \tag{4.10}$$
$$P_4(0,t) = 120\lambda_1^4 P_0(t) \tag{4.11}$$
$$P_5(0,t) = \lambda_A[P_0(t)+P_1(t)+P_2(t)+P_3(t)] \tag{4.12}$$
$$P_6(0,t) = \lambda_B[P_0(t)+P_1(t)+P_2(t)+P_3(t)] \tag{4.13}$$
$$P_7(0,t) = \lambda_A P_6(t)+\lambda_B P_5(t) \tag{4.14}$$
$$P_{s_1}(0,t) = \lambda_{s_1}[P_0(t)+P_1(t)+P_2(t)+P_3(t)+P_5(t)+P_6(t)] \tag{4.15}$$
$$P_{s_2}(0,t) = \lambda_{s_2}[P_0(t)+P_1(t)+P_2(t)+P_3(t)+P_5(t)+P_6(t)] \tag{4.16}$$

Initial conditions

$$P_0(0)=1 \tag{4.17}$$

and other state probabilities are zero at $t = 0$. Taking Laplace transformation of equations (4.1) to (4.16) and using equation (4.17), we obtain

$$(4.18)\quad [s+5\lambda_1+\lambda_A+\lambda_B+\lambda_{s_1}+\lambda_{s_2}]\,\bar{P}_0(s) = 1+\int_0^\infty \phi_1(x)\bar{P}_1(x,s)dx$$
$$+\int_0^\infty \phi_A(x)\bar{P}_5(x,s)dx+\int_0^\infty \phi_B(x)\bar{P}_6(x,s)dx+\int_0^\infty \mu_0(x)\bar{P}_i(x,s)dx;\ i=4,7,s_1,s_2$$

$$(4.19)\quad \left[s+\frac{\partial}{\partial x}+4\lambda_1+\lambda_A+\lambda_B+\lambda_{s_1}+\lambda_{s_2}+\phi_1\right]\bar{P}_1(x,s) = 0$$
$$(4.20)\quad \left[\frac{\partial}{\partial x}+3\lambda_1+\lambda_A+\lambda_B+\lambda_{s_1}+\lambda_{s_2}+\phi_1\right]\bar{P}_2(x,s) = 0$$
$$(4.21)\quad \left[s+\frac{\partial}{\partial x}+2\lambda_1+\lambda_A+\lambda_B+\lambda_{s_1}+\lambda_{s_2}+\phi_1\right]\bar{P}_3(x,s) = 0$$
$$(4.22)\quad \left[s+\frac{\partial}{\partial x}+\lambda_B+\lambda_{s_1}+\lambda_{s_2}+\phi_A\right]\bar{P}_5(x,s) = 0$$
$$(4.23)\quad \left[s+\frac{\partial}{\partial x}+\lambda_A+\lambda_{s_1}+\lambda_{s_2}+\phi_B\right]\bar{P}_6(x,s) = 0$$
$$(4.24)\quad \left[s+\frac{\partial}{\partial x}+\mu_0(x)\right]\bar{P}_j(x,s) = 0;\ j=4,7,s_1,s_2$$

Boundary conditions

$$(4.25)\quad \bar{P}_1(0,s) = 5\lambda_1\bar{P}_0(s)$$
$$(4.26)\quad \bar{P}_2(0,s) = 20\lambda_1^2\bar{P}_0(s)$$
$$(4.27)\quad \bar{P}_3(0,s) = 60\lambda_1^3\bar{P}_0(s)$$
$$(4.28)\quad \bar{P}_4(0,s) = 120\lambda_1^4\bar{P}_0(s)$$
$$(4.29)\quad \bar{P}_5(0,s) = \lambda_A\left(1+5\lambda_1+20\lambda_1^2+60\lambda_1^3\right)\bar{P}_0(s)$$
$$(4.30)\quad \bar{P}_6(0,s) = \lambda_B\left(1+5\lambda_1+20\lambda_1^2+60\lambda_1^3\right)\bar{P}_0(s)$$
$$(4.31)\quad \bar{P}_7(0,s) = 2\lambda_A\lambda_B\left(1+5\lambda_1+20\lambda_1^2+60\lambda_1^3\right)\bar{P}_0(s)$$
$$(4.32)\quad \bar{P_{s_1}}(0,s) = \lambda_{s_1}(1+\lambda_A+\lambda_B)\left(1+5\lambda_1+20\lambda_1^2+60\lambda_1^3\right)\bar{P}_0(s)$$
$$(4.33)\quad \bar{P_{s_2}}(0,s) = \lambda_{s_2}(1+\lambda_A+\lambda_B)\left(1+5\lambda_1+20\lambda_1^2+60\lambda_1^3\right)\bar{P}_0(s)$$

Solving all the above equations with the implications of boundary conditions and and

$$\bar{P}_i(s) = \int_0^\infty \bar{P}_i(x,s)dx$$

we may get Laplace transform of state transition probabilities as:

$$\bar{P}_0(s) = \frac{1}{D(s)} \tag{4.34}$$

$$\bar{P}_1(s) = \frac{5\lambda_1}{D(s)}\frac{1}{(s+4\lambda_1+\lambda_A+\lambda_B+\lambda_{s_1}+\lambda_{s_2}+\phi_1)} \tag{4.35}$$

$$\bar{P}_2(s) = \frac{20\lambda_1^2}{D(s)}\frac{1}{(s+3\lambda_1+\lambda_A+\lambda_B+\lambda_{s_1}+\lambda_{s_2}+\phi_1)} \tag{4.36}$$

$$\bar{P}_3(s) = \frac{60\lambda_1^3}{D(s)}\frac{1}{(s+2\lambda_1+\lambda_A+\lambda_B+\lambda_{s_1}+\lambda_2+\phi_1)} \tag{4.37}$$

$$\bar{P}_4(s) = \frac{120\lambda_1^4}{D(s)}\frac{1}{(s+\mu_0)} \tag{4.38}$$

$$\bar{P}_5(s) = \frac{\lambda_A}{D(s)}\frac{1}{(s+\lambda_B+\lambda_{s_1}+\lambda_{s_2}+\phi_A)} \tag{4.39}$$

$$\bar{P}_6(s) = \frac{\lambda_B}{D(s)}\frac{1}{(s+\lambda_A+\lambda_{s_1}+\lambda_{s_2}+\phi_B)} \tag{4.40}$$

$$\bar{P}_7(s) = \frac{2\lambda_A\lambda_B}{D(s)}\frac{1}{(s+\mu_0)} \tag{4.41}$$

$$\bar{P_{s_1}}(s) = \frac{\lambda_{s_1}(1+\lambda_A+\lambda_B)}{D(s)}\frac{1}{(s+\mu_0)} \tag{4.42}$$

$$\bar{P_{s_2}}(s) = \frac{\lambda_{s_2}(1+\lambda_A+\lambda_B)}{D(s)}\frac{1}{(s+\mu_0)} \tag{4.43}$$

Where

$$\begin{aligned}D(s) &= (s+5\lambda_1+\lambda_A+\lambda_B+\lambda_{s_1}+\lambda_{s_2}) - 5\lambda_1\bar{S}_{\phi_1}\big(s+4\lambda_1+\lambda_A+\lambda_B+\lambda_{s_1}+\lambda_{s_2}\big)\\ &\quad - 120\lambda_1^4\bar{S}_{\mu_0}(s) - \big(\lambda_A\bar{S}_{\phi_A}(s+\lambda_B+\lambda_{s_1}+\lambda_{s_2}) + \lambda_B\bar{S}_{\phi_B}(s+\lambda_A+\lambda_{s_1}+\lambda_{s_2})\big)\\ &\quad - \bar{S}_{\mu_0}(s)\big[(1+\lambda_A+\lambda_B)(\lambda_{s_1}+\lambda_{s_2}) + 2\lambda_A\lambda_B\big]\end{aligned}$$

$$\begin{aligned}\bar{S}_{\phi_1}(s+4\lambda_1+\lambda_A+\lambda_B+\lambda_{s_1}+\lambda_{s_2}) &= \frac{\phi_1}{s+4\lambda_1+\lambda_A+\lambda_B+\lambda_{s_1}+\lambda_{s_2}+\phi_1}\\ \bar{S}_{\phi_A}(s+\lambda_A+\lambda_{s_1}+\lambda_{s_2}) &= \frac{\phi_A}{s+\lambda_A+\lambda_{s_1}+\lambda_{s_2}+\phi_A}\\ \bar{S}_{\phi_B}(s+\lambda_A+\lambda_{s_1}+\lambda_{s_2}) &= \frac{\phi_B}{s+\lambda_A+\lambda_{s_1}+\lambda_{s_2}+\phi_B}\end{aligned}$$

The up and downstate probabilities of the system are given by

$$\begin{aligned}\bar{P}_{up} &= \frac{1}{D(s)}\Big[1+\frac{5\lambda_1}{(4\lambda_1+V)}+\frac{20\lambda_1^2}{(3\lambda_1+V)}+\frac{60\lambda_1^3}{(2\lambda_1+V)}+\frac{\lambda_A}{(\lambda_B+\phi_A+W)}\\ &\quad +\frac{\lambda_B}{(\lambda_A+\phi_B+W)}\Big]\end{aligned}$$

where $V = (s + \lambda_A + \lambda_B + \lambda_{s_1} + \lambda_{s_2} + \phi_1)$ and $W = (s + \lambda_{s_1} + \lambda_{s_2})$

$$\bar{P}_{down} = \frac{1}{D(s)}\left[\frac{120\lambda_1^4}{(s+\mu_0)} + \frac{2\lambda_A\lambda_B}{(s+\mu_0)} + \frac{(\lambda_{s_1}+\lambda_{s_2})(1+\lambda_A+\lambda_B)}{(s+\mu_0)}\right] \tag{4.44}$$

5. Analytical Study

5.1. Availability Analysis When repair follows two types of distributions as general and Gumbel-Hougaard family copula distribution, then we have

$$\bar{S}_\mu(s) = \bar{S}_{exp\left[x^\theta + \{\log\phi(x)\}^\theta\right]^{\frac{1}{\theta}}} = \frac{exp\left[x^\theta + \{\log\phi(x)\}^\theta\right]^{\frac{1}{\theta}}}{s + exp\left[x^\theta + \{\log\phi(x)\}^\theta\right]^{\frac{1}{\theta}}}$$

Let us choose the values of parameters as $\lambda_1 = 0.02$, $\lambda_A = 0.03$, $\lambda_B = 0.021$, $\lambda_{s_1} = 0.022$, $\lambda_{s_2} = 0.025$, $x = 1$, $\phi_i = 1$ for $= 1, A$ and B then taking inverse Laplace transform, and we obtain the availability of the system as per the following three cases on switching device:

(a): When both the subsystems have switching device, we get,

$$\begin{aligned} \bar{P}_{up}(t) \;=\; & -0.0002e^{-1.1170t} - 0.002738e^{-1.136006} + 0.018651e^{-2.7717t} \\ & -0.030599e^{-1.2752t} + 0.014734e^{-1.0847t} + 1.02123e^{-0.0057t} \\ & -0.021033e^{-1.0980t} \end{aligned} \tag{5.1}$$

(b): When subsystem-1 does not have a switching device, i.e $\lambda_{s_1} = 0$.
(c): When subsystem-2 does not have a switching device, i.e $\lambda_{s_2} = 0$.
(d): When both subsystems 1 and 2 do not have a switching device, i.e. $\lambda_{s_1} = \lambda_{s_2} = 0$.

We can write similar expressions for availability in case (b),(c) and (d) using Maple. For different values of time-variable units of time $t = 0, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100$, one may get different values of $\bar{P}_{up}(t)$ as shown in Table 2 and the corresponding Figure 3.

5.2. Reliability of the system Taking all repair rates equal to zero and obtain inverse Laplace transform, we get an expression for the reliability of the system after taking the failure rates as $\lambda_1 = 0.02$, $\lambda_A = 0.03$, $\lambda_B = 0.021$, $\lambda_{s_1} = 0.022$, $\lambda_{s_2} = 0.025$ in in (4.44). Now consider the same cases as availability, we have

Table 2. Variation of availability with respect to time in various cases

$Time(t)$	$P_{up}(t)(a)$	$P_{up}(t)(b)$	$P_{up}(t)(c)$	$P_{up}(t)(d)$
0	1.00	1.00	1.00	1.00
10	0.699	0.769	0.782	0.918
20	0.554	0.625	0.641	0.830
30	0.439	0.509	0.525	0.750
40	0.347	0.414	0.430	0.679
50	0.275	0.337	0.353	0.637
60	0.219	0.274	0.289	0.555
70	0.173	0.223	0.237	0.502
80	0.137	0.181	0.194	0.454
90	0.108	0.147	0.159	0.410
100	0.086	0.120	0.130	0.371

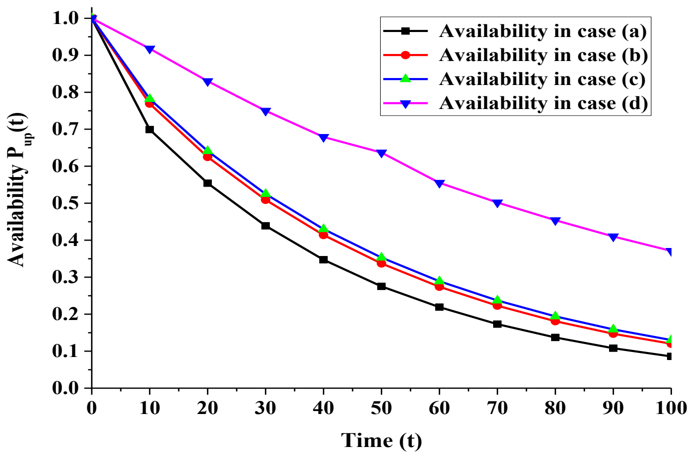

Figure 3. Availability as a function of Time

(a): When both the subsystems have switching device, we obtain,

$$R(t) = 0.31037e^{-0.0980t} - 20.0000e^{-0.1780t} + 0.21626e^{-0.1360t} - 0.008571e^{-0.1170t} + 20.14812e^{-0.1730t} + 0.136708e^{-0.0680t} \quad (5.2)$$

(b): When Subsystem-1 does not have switching devices, i.e. $\lambda_{s_1} = 0$.

(c): When subsystem-2 does not have a switching device, i.e $\lambda_{s_2} = 0$.

(d): When both subsystem-1 and 2 do not have a switching device, i.e $\lambda_{s_1} = \lambda_{s_1}$

We can write similar expressions for reliability in case (b), (c) and (d) using Maple. By taking different values of time-variable $t = 0, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100$ units of time, one may get reliability $R(t)$ with the help of as shown in Table 3 and the corresponding Figure 4.

Table 3. Computed values of reliability corresponding to the different cases

$Time(t)$	a	b	c	d
0	1.00	1.00	1.00	1.00
10	0.534	0.543	0.586	0.728
20	0.205	0.226	0.226	0.410
30	0.078	0.096	0.126	0.242
40	0.032	0.047	0.070	0.159
50	0.014	0.025	0.040	0.133
60	0.006	0.014	0.023	0.085
70	0.003	0.008	0.014	0.0.66
80	0.002	0.005	0.010	0.052
90	0.001	0.003	0.006	0.041
100	0.001	0.002	0.004	0.033

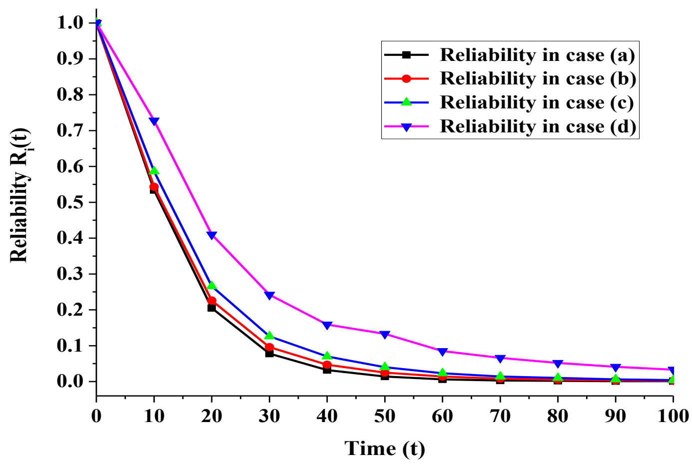

Figure 4. Availability as a function of Time

5.3. Mean Time to Failure (MTTF) Taking all repair rate and Laplace parameters to zero in (4.44) for the exponential distribution, we can obtain the meantime to failure as:

$$MTTF = \frac{1}{(5\lambda_1 + \lambda_A + 5\lambda_B + \lambda_{s_1} + \lambda_{s_2})}\Big[1 + \frac{5\lambda_1}{4\lambda_1+\mu} + \frac{20\lambda_1^2}{3\lambda_1+\mu} + \frac{60\lambda_1^3}{2\lambda_1+\mu} + \frac{\lambda_A\nu}{\mu-\lambda_A} + \frac{\lambda_B\nu}{\mu-\lambda_B}\Big] \tag{5.3}$$

where $\mu = \lambda_{s_1} + \lambda_{s_2}$ and $\nu = 1+5\lambda_1+20\lambda_1^2+60\lambda_1^3$. Now taking the values of different parameters as $\lambda_1 = 0.02$, $\lambda_A = 0.03$, $\lambda_B = 0.021$, $\lambda_{s_1} = 0.022$, and $\lambda_{s_2} = 0.025$ and varying $\lambda_1, \lambda_2, \lambda_{s_1}$ and λ_{s_1} one by one respectively as $0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09, 0.10$ in (5.3), the variation in MTTF, with respect to failure rates, can be obtained as per Table 4 and Figure 5.

Table 4. Computation of MTTF corresponding to the failure rates

Failure rates	λ_1	λ_A	λ_B	λ_{s_1}	λ_{s_2}
0.01	14.388	12.732	12.750	14.048	14.624
0.02	12.157	12.375	12.201	12.434	12.876
0.03	10.601	12.157	11.834	11.167	11.518
0.04	9.482	12.030	11.585	10.141	10.428
0.05	8.654	11.963	11.415	9.289	9.529
0.06	8.027	11.937	11.302	8.570	8.774
0.07	7.542	11.940	11.229	7.954	8.129
0.08	7.163	11.963	11.185	7.420	7.572
0.09	6.865	12.001	11.164	6.952	7.086
0.10	6.629	12.048	11.158	6.538	6.657

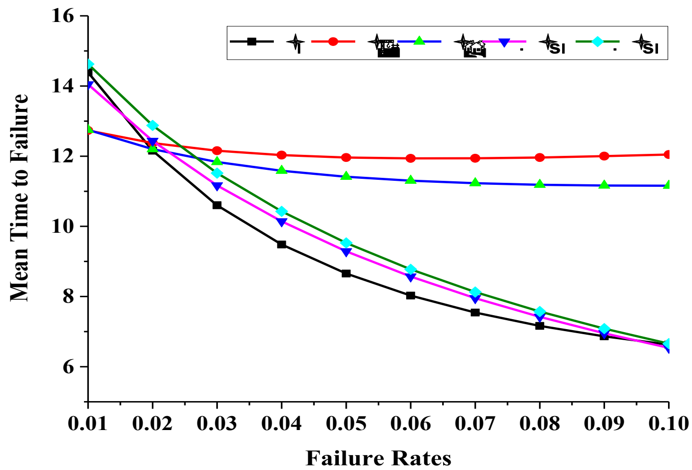

Figure 5. MTTF as a function of failure rates

5.4. Cost Analysis If the service facility is always available, then expected profit during the interval $[0, t)$

$$E_p(t) = K_1 \int_0^t P_{up}(t)dt - K_2 t \tag{5.4}$$

For the same set of parameters defined in (4.44), one can obtain (50). Therefore,

$$\begin{aligned} E_p(t) &= K_1\{0.149e^{-1.0000t} - 0.043e^{-3.480t} + 0.0420e^{-2.119t} - 0.037e^{-1.471t} \\ &\quad -13.394e^{-0.077t} + 13.283\} - K_2 t \end{aligned} \tag{5.5}$$

Setting $K_1 = 1$, and $K_2 = 0.6, 0.5, 0.4, 0.3$ and 0.2 respectively and varying $t = 0, 10, 20, 30, 40, 50, 60, 70, 80, 90$ and 100 units of time, the results for expected profit can be seen in Table 5 and Figure 6.

Table 5. Profit computation for different values of time

Time	K_2	K_2	K_2	K_2	K_2
0	0.000	0.000	0.000	0.000	0.000
10	5.124	6.124	7.124	8.124	9.124
20	12.522	14.522	16.522	18.522	20.522
30	22.621	25.621	28.621	31.621	34.621
40	35.966	39.966	43.966	47.966	51.966
60	75.135	81.135	86.135	92.135	98.135
70	102.695	109.695	116.695	123.695	130.695
80	137.061	145.061	153.185	161.061	169.061
90	179.465	188.465	197.465	206.465	215.465
100	231.679	241.679	251.679	261.679	271.679

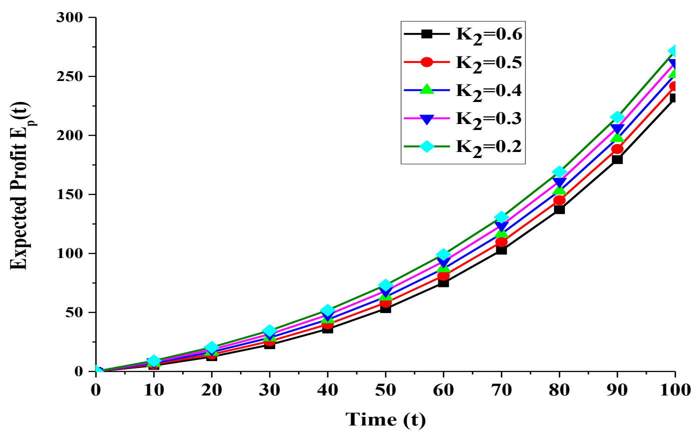

Figure 6. Expected profit as a function of time

6. Conclusion

Warm-standby redundancy has been used as an effective technique for improving the availability and reliability of the system while attaining the equilibrium between fast repair and low process cost. Depending on the level of operation promptness of standby units, there can exist many standby modes, each categorized by altered standby maintenance and startup costs. In this paper, we have considered two subsystems in a series configuration with a switching device. The subsystems-1 has five identical units, while subsystem-2 has two non-identical units, and both are connected via switching device.

The availability of the system can be seen from Table 2 and Figure 3 when failure rates are fixed at $\lambda_1 = 0.02, \lambda_A = 0.03, \lambda_B = 0.021$. It is moving down as the value of t increases in all the three cases considered based on switching device and eventually become stable after a sufficiently long interval of time. On the other hand, the reliability experiences a steep fall in all three cases for the same failure rates, as it is evident from Table 3 and Figure 4. Moreover, resultant values of availability are more significant than the values of reliability, which highlights the necessity of regular repair for repairable systems. Furthermore, the study for availability and reliability reveals that switching devices for both the subsystems have a significant effect on the output.

Table 4 and Figure 5 yield the MTTF of the system concerning variation in failure rates λ_1, λ_A, λ_B, λ_{s_1} and λ_{s_2} respectively when other parameters are fixed. We observe that on average basis MTTF for the failure rate λ_{s_1} and λ_{s_2} is maximum with little variation, while similar variation for subsystem-1 and switching device. Thus, the failure rates λ_1, λ_A and λ_B are more responsible for the effective operation of the system. The expected profit can be revealed from Table 5 and Figure 6, which is maximum for $K_1 = 0.2$ and minimum for $K_2 = 0.6$. Conclusively, we can observe that as service cost decreases, profit increases with a variation of time. The model given in this paper is suitable for several real systems such as power plant and transmission system, server design for the network, and so on.

Conflict of Interest

The work is original and has not been submitted anywhere for publication.

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