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Analytic Solution for an Eaton Lens for Rotating 90°

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The Eaton lens, with spherical symmetry to its refractive index, was described by Eaton in 1952 and was found recently in the design of an invisible sphere for cloaking. In this paper, an Eaton lens for rotating 90° was designed using Luneburg theory, by which we found it was a fourth-order equation in the refractive index n. Therefore, the refractive index n has four roots. The equation in n was solved and studied using mathematical technology. The unsuitable complex roots of the equation should be dropped; consequently, only one of the four roots remained. To verify the refractive-index profile, the only root was solved for, before a simulation using finite-element analysis (FEA) was performed. The simulation showed that all rays will bend 90° to the right. The result of the simulation is identical to our expectation. This treatment provides a possible method for rotating light at many other angles.

Keywords : Eaton lens, Finite element analysis, Refractive-index profile
OCIS codes : (080.3630) Lenses; (160.3918) Metamaterials; (160.4670) Optical materials

I. INTRODUCTION

In history, there have been several famous lenses with a spherically symmetric gradient-index profile, including Maxwell's fisheye lens [1], the Luneburg lens [2] and the Eaton lens [3]. Maxwell's fisheye lens possesses the refractive-index profile $n(r)=2/(1+r^2)$, where r is the radial spherical coordinate (the same below) that can perfectly image from one point at the radius distance to the opposite point at the radius. The Luneburg lens possesses the refractive-index profile $n(r)=\sqrt{2-r^2}$; light rays will be focused on the surface of the Luneburg lens when the rays coming from ∞ enter the Luneburg lens, for $r<1$. The Eaton lens possesses the refractive-index profile $n(r)=\sqrt{2/r-1}$; when light rays of parallel optical axis coming from ∞ propagate to the Eaton lens, for $r<1$ the rays leave in the same direction from which they came. That is, the rays are retroreflected and rotated by 180°. In 1964 Luneburg studied spherical gradient-index lenses in detail, and developed effective mathematical tools to design analogous lenses [2, 4].

In the literature [5], Miñano reported that the Eaton lens can be used to design an invisible sphere for rotating rays 360°. From the deflection angle -2π, he obtained a cubic equation in n, $(1+n)^2nr=4$ and consequently obtained the solution $n=(Q-1/3Q)^2$ (with $Q=Q(r)$ [5]). The Eaton lens rotates rays 180° and Miñano's lens rotates rays 360°. Has any other deflection angle been designed? In fact, a lens for rotating 90° can be used for beam splitting, optical manufacturing, optical communication, endoscopy, and image stabilization.

In a previous paper [6], Sang-Hoon Kim studied a 90°-rotating Eaton lens (right-bender) with refractive-index profile $n^2=1/(nr)+\sqrt{1/(n^2r^2)-1}$. Due to the complexity of finding the solution, Sang-Hoon Kim only deduced the approximate solution $n(r)\cong(2/r-1)^{\theta/(\pi+\theta)}$, but not the analytic solution. Fabrications of the 90° rotating Eaton lens were performed [7-9], all based on the equation in the literature [6]. However, it was very regrettable that the analytic solution still did not appear. Therefore, it is most important that the analytic solution should be solved first, and that is the mission of this paper.

In this paper, another equation for the Eaton lens with rotating 90° was deduced, using Lunebrug theory. We found that the equation for the refractive-index profile is a fourth-order equation in n. The analytic solution of the

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equation was then found and researched using mathematical software (such as Mathematica, MathCAD, or Maple). To verify our analytic solution, finite-element analysis (using FEA software, such as COMSOL or Ansys) was conducted. The simulation proved that our analytic solution of the equation was correct.

II. DESIGN OF AN EATON LENS WITH ROTATION ANGLE OF 90°

The definition of the rotation angle (or deflection angle) χ is shown in Fig. 1, which denotes the angle between the exit ray and the incident ray. The sign rule for the rotation angle is positive for clockwise rotation from the starting ray to the ending ray; conversely, negative is for anticlockwise direction. Because the incident ray is the ending ray and the refractive ray is the starting ray, the right-bender of the ray represents the minus sign of the anticlockwise direction from the refractive ray to the incident ray, as shown in Fig. 1. The refractive index at the center of the Eaton lens is greater than that at the surface, so the ray should bend rightward toward the center of the Eaton.

Now we design an Eaton lens that rotates light 90°. Using Luneburg theory, the rotation angle χ satisfies the following relations:

$$\chi = -2\pi\nu, \tag{1}$$

$$\mu = 1/(2\nu), \tag{2}$$

$$r = \frac{2n^{\mu-1}}{1+n^{2\mu}}, \tag{3}$$

where ν and μ are only intermediate variables. For the rotation angles, we have $\chi = -\pi/2$, $\nu = 1/4$, and $\mu = 2$; substituting into Eq. (3), we obtain

$$r = \frac{2n}{1+n^4} \Rightarrow n^4 - \frac{2}{r}n + 1 = 0. \tag{4}$$

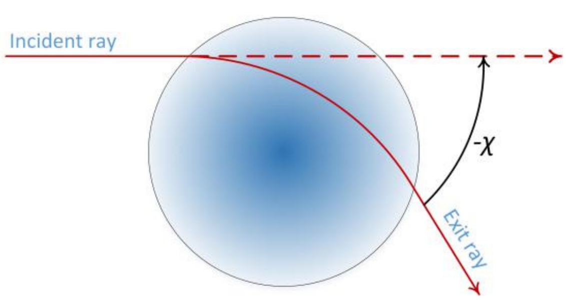

FIG. 1. Rotation angle χ, the angle between the exit ray and the incident ray. The ray will bend rightward toward the center of the Eaton lens, because the center possesses higher refractive index.

Eq. (4) is the distribution equation for the refractive-index profile needed for the Eaton lens to rotate 90°, where r is the radial spherical coordinate and n is the distribution of the refractive-index profile. n should be solved as a function of r, namely $n = n(r)$. When solving Eq. (4), r is treated as a constant. In fact, by squaring both sides of $n^2 - 1/(nr) = \sqrt{1/(n^2r^2) - 1}$ the same equation can be obtained.

It is somewhat complicated to solve Eq. (4) for $n(r)$, which is a fourth-order equation in n. One can solve it by referring to a mathematical dictionary in which the ready-made solutions of some special fourth-order equations are listed. Nevertheless, it is not an easy thing to do. In fact, we have a strong tool for doing so, which is mathematical software such as Mathematica, MathCAD, or Maple. For analytic solutions, we use mathematical software to research Eq. (4). For example, in Mathematica one can apply the command line "Solve $[n^4 - 2n/r + 1 = 0, n]$" to solve for the roots of Eq. (4) in n. The four roots of Eq. (4) are solved for, of which we found that three roots should be dropped, and only one root remains. There are two complex roots and two real roots. For the distribution of the refractive index, $n(r)$ should take only real-number values, so the two complex roots should be dropped first. One of the real roots varies from small to large when r varies from 0 to 1. This is not in accord with the properties of the Eaton lens, so that real root should be dropped. Now only one real root remains, as follows:

$$r' = r/R,\ \ s = \sqrt[3]{2},\ \ t = \sqrt[3]{3}, \tag{5}$$

$$p = \sqrt[3]{9r' + \sqrt{3} \cdot \sqrt{27r'^2 - 16r'^6}}, \tag{6}$$

$$q = \sqrt{\frac{s^5 r'}{tp} + \frac{sp}{t^2 r'}}, \tag{7}$$

$$n(r) = \frac{1}{2}q + \frac{1}{2}\sqrt{-q^2 + \frac{4}{r'q}}, \tag{8}$$

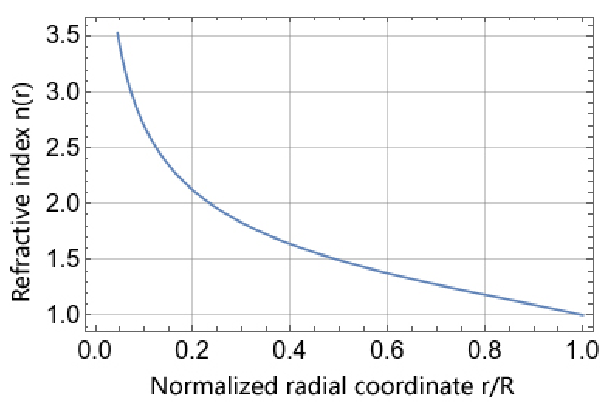

FIG. 2. The curve of the calculated distribution. The refractive-index curve $n(r)$ versus r is plotted from Eq. (8), which was solved using mathematical software.

where R is the radius of the Eaton lens for rotating 90°, and the intermediate variables r', s, t, p, q are used to help in abbreviating the refractive-index expression $n(r)$. The curve of $n(r)$ versus r was plotted, as shown in Fig. 2; we call this the *calculated distribution*. In Fig. 2 the x axis plots the normalized radial coordinate r/R, ranging from 0 to 1, and the y axis plots the refractive index $n(r)$, ranging from 1 to ∞. According to the properties of the Eaton lens, $n(r)$ goes to ∞ when $r=0$, and to 1 when $r=R$. When light rays encounter the surface of the Eaton lens, they are not refracted.

III. SIMULATION USING FINITE-ELEMENT ANALYSIS (FEA)

There are many important, popular numerical methods in computational electromagnetics, for which exclusive computer programs have been designed. Among them, the finite-difference time-domain (FDTD) method and the finite-element analysis (FEA) method are two of the most famous methods. In this paper, we choose the FEA method. Representative FEA software packages are EM Solution, COMSOL Multiphysics, and HFSS.

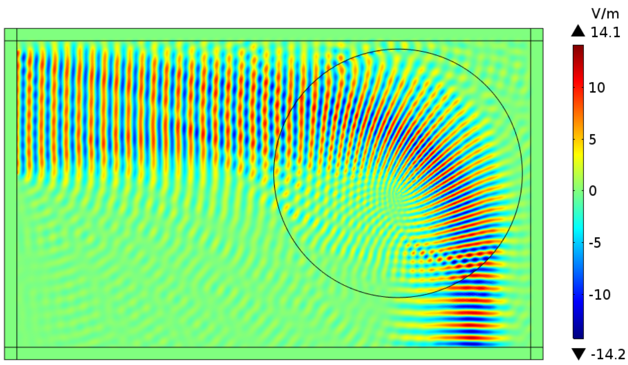

FIG. 3. The simulation of the Eaton lens with rotation angle of 90°. Parallel light rays are released from the left at ∞, enter the Eaton lens, bend 90° to the right, and exit from the bottom of the lens.

The simulated Eaton lens is simply a sphere containing a graded-index medium satisfying the distribution of Eq. (8). In the FEA software, the geometric picture of an Eaton lens with spherical symmetry should be modeled, and its domain should be appointed to the refractive-index $n(r)$, as shown in Fig. 3. After that, parallel light rays are released from the left at ∞, and then enter the Eaton lens with refractive index profile $n(r)$ (see Eq. (8)). Once the rays enter the Eaton lens with a gradient index, they bend toward the larger refractive index at the center. Therefore, all rays will bend 90° to the right and will exit from the bottom of the Eaton lens. The light trajectories of the simulation are identical to our expectations.

Eq. (8) should been appointed to the refractive index $n(r)$ of the Eaton lens. Is the refractive-index profile just the Eq. (8) for the perfect simulation? We can analyze the results obtained through FEA software. Figure 4 is the simulated distribution of the refractive-index profile. The left panel shows the pseudocolor distribution of the refractive-index profile with spherical symmetry. The colors from white to black represent refractive-index values from 1 to ∞. The variations are smoother in the outer lane and sharper in the central zone. The right panel shows the simulated distribution curve of the refractive index taken along an arbitrary radius. The curve matches the calculated distribution curve in Fig. 2 very well.

One application of the Eaton lens with 90° rotation is image stabilization. The pentagonal prism has ever been used to maintain image stabilization in camera systems [10]. The antishake principle of the pentagonal prism relies on the action of a double-sided mirror. The incident ray will rotate 90° to the exit ray, which just equals double the included angle of the double-sided mirror. No matter the direction in which the incident ray enters the pentagonal prism, the exit ray will always rotate 90° as shown in

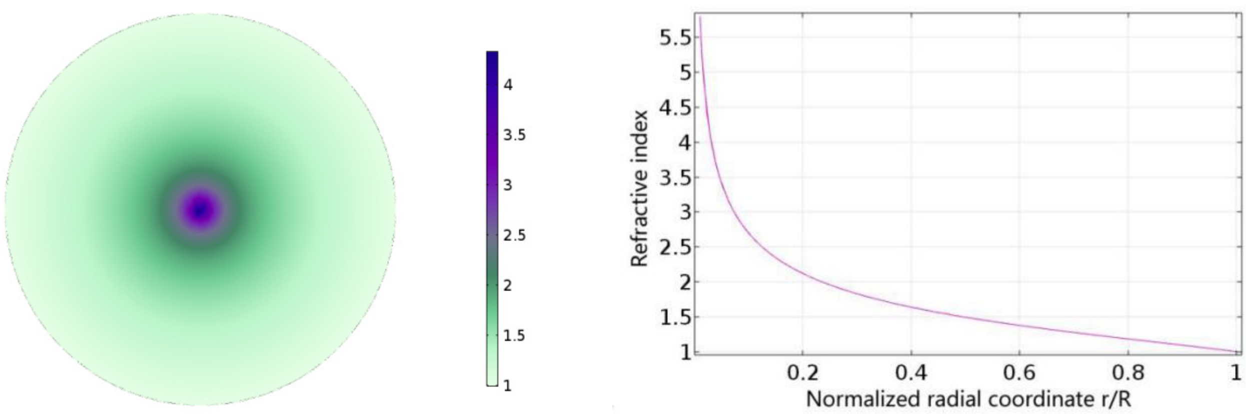

FIG. 4. The simulated distribution of the refractive-index profile. The left panel shows a pseudocolor graph of the refractive-index profile. The colors from white to black represent refractive-index values from 1 to ∞. The right panel shows the simulated distribution curve of the refractive index, which matches very well the calculated distribution curve in Fig. 2.

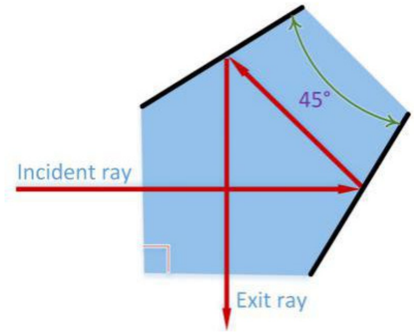

FIG. 5. A pentagonal prism can also rotate ray 90°.

Fig. 5. The Eaton lens with 90° rotation can accomplish the same thing, so it can be used to stabilize an image. Therefore, a device including an Eaton lens with 90° rotation is potently applied in an image-stabilization system.

IV. CONCLUSION

The Eaton lens for rotating 90° was designed using Luneburg theory. A fourth-order equation in n was deduced. The equation was solved and studied by means of mathematical software. Only one of the four roots of the equation was kept. To verify the refractive-index profile, simulation using FEA software was performed. The results of the simulation proved to be identical to our expectations. The treatment provides a possible method for rotation by many other angles.

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