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Methods to Obtain Approximate Responses of a Non-Linear Vibration Isolation System

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비선형 진동절연 시스템의 근사적 응답을 구하는 방법

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ABSTRACT

A non-linear vibration isolation system composed of a non-linear spring and a linear damper was presented in a previous study. The advantage of the proposed isolator is the simple structure of the system. When the base of the isolator is harmonically excited, the response component of the mass at the excitation frequency was approximated using three different methods: linear approximation, harmonic balance, and higher-order frequency response functions (FRFs). The method using higher-order FRFs produces significantly more accurate results compared with the other methods. The error between the exact and approximate responses does not increase monotonously with the excitation amplitude and is less than 2%.

Keywords : Frequency Component(주파수성분), Harmonic Balance(하모닉 밸런스), Higher-order Frequency Response Function(고차 주파수응답함수), Non-linear Vibration Isolation System(비선형 진동절연 시스템), Volterra Series(Volterra 급수)

1. Introduction

Vibration isolation is a procedure by which the undesirable effects of vibration are reduced. It involves the insertion of a resilient member (or isolator) between the vibrating mass and the source of vibration to achieve a reduction in the dynamic response of the system under specified vibration excitation[1]. The suspension system of a car seat is an example of a vibration isolator.

To resolve the conflict between small transmissibility and large static deflection, the use of non-linear springs is proposed. A vibration isolator with a non-linear spring was proposed and its characteristics investigated in previous research[2]. The proposed non-linear spring is composed of two symmetric linear springs depicted in Fig. 1. The structure of the spring is very simple compared to other non-linear springs(e.g., disc springs[3]), and the force-displacement relationship can be expressed easily, resulting in a straightforward dynamic analysis

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of systems equipped with the spring.

If the non-linear relationship between the spring force and the displacement can be represented by a polynomial, the equation of motion of the vibration isolator with this non-linear spring can be solved approximately.

In this study, the equation of motion of the non-linear isolator was solved approximately using three different methods: linear approximation, harmonic balance, and higher-order frequency response functions(FRFs), and the results were compared with numerical integration results which are assumed to be exact.

2. Description of Non-linear Equation-solving Methods

2.1 Harmonic balance method

Because the harmonic balance method[4] is based on an analysis primarily applicable to periodic functions, it is particularly useful for computing approximations to periodic solutions. In this method the steady-state response of a non-linear system subjected to a harmonic input is expressed by a sum of harmonics. In a simple case the response is expressed as

$$y(t) = A\cos\omega t \tag{1}$$

The expression for the response is inserted into a non-linear equation, and by equating coefficients of each harmonic on both sides the amplitude of each harmonic is obtained.

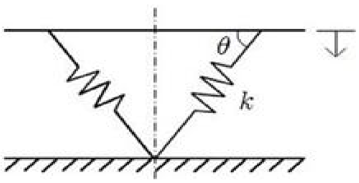

Fig. 1 Proposed non-linear spring

2.2 Higher-order FRFs

When a system with polynomial non-linearities is subjected to harmonic excitation

$$x(t) = X\cos\omega t = \frac{X}{2}e^{j\omega t} + \frac{X}{2}e^{-j\omega t}, \tag{2}$$

the response can be expressed using higher-order FRFs as follows. Detailed explanation on the Volterra series can be found in References [5,6].

$$\begin{aligned} y(t) = {} & H_1(\omega)\frac{X}{2}e^{j\omega t} + H_1(-\omega)\frac{X}{2}e^{-j\omega t} \\ & + H_2(\omega,\omega)\left(\frac{X}{2}\right)^2 e^{j2\omega t} + 2H_2(\omega,-\omega)\left(\frac{X}{2}\right)^2 \\ & + H_2(-\omega,-\omega)\left(\frac{X}{2}\right)^2 e^{-j2\omega t} + H_3(\omega,\omega,\omega)\left(\frac{X}{2}\right)^3 e^{j3\omega t} \\ & + 3H_3(\omega,\omega,-\omega)\left(\frac{X}{2}\right)^3 e^{j\omega t} + 3H_3(\omega,-\omega,-\omega)\left(\frac{X}{2}\right)^3 e^{-j\omega t} \\ & + H_3(-\omega,-\omega,-\omega)\left(\frac{X}{2}\right)^3 e^{-j3\omega t} + \cdots \end{aligned} \tag{3}$$

where $H_n(\omega,\cdots,\omega,-\omega,\cdots,-\omega)$ is a higher-order FRF. The arguments of the FRF are composed of ω and $-\omega$. Denoting the FRF by $H_{n,i}$ when ω is repeated $n-i$ times and $-\omega$ is repeated i times in the argument list, the above response can be expressed as:

$$\begin{aligned} y(t) &= \sum_{n=1}^{\infty} y_n(t) \\ &= \sum_{n=1}^{\infty}\sum_{i=0}^{n}\binom{n}{i}H_{n,i}\left(\frac{X}{2}\right)^n e^{j(n-2i)\omega t} \end{aligned} \tag{4}$$

where $\binom{n}{i}$ is a binomial coefficient equal to $n!/i!(n-i)!$. Denote the complex ratio of the response component at the excitation frequency ω to the excitation component at the same frequency as $\hat{H}_1(\omega)$. By considering only the terms associated with the input $\frac{X}{2}e^{j\omega t}$ in Eq. (3), an estimate of $\hat{H}_1(\omega)$

for a non-linear system can be obtained:

$$\begin{aligned} \hat{H}_1(\omega) &= \frac{Y(\omega)}{X(\omega)} \qquad (5)\\ &= H_1(\omega) + \binom{3}{1} H_3(\omega,\omega,-\omega)\left(\frac{X}{2}\right)^2 \\ &+ \binom{5}{2} H_5(\omega,\omega,\omega,-\omega,-\omega)\left(\frac{X}{2}\right)^4 + \cdots \\ &= \sum_{m=0}^{\infty} \binom{2m+1}{m} H_{2m+1,m}\left(\frac{X}{2}\right)^{2m} \end{aligned}$$

The equation of motion of a single degree-of-freedom system with cubic stiffness subjected to harmonic excitation can be written

$$\begin{aligned} &m\ddot{y}(t) + c\dot{y}(t) + k_1 y(t) + k_2 y(t)^2 + k_3 y(t)^3 \qquad (6)\\ &\quad = \frac{X}{2}\left(e^{j\omega t} + e^{-j\omega t}\right) \end{aligned}$$

Substituting Eq. (3) into the above equation and equating the coefficients of the $\left(\frac{X}{2}\right)e^{j\omega t}$ terms on both sides, one can obtain

$$H_1(\omega) = \frac{1}{k_1 - m\omega^2 + jc\omega} \qquad (7)$$

Similarly, equating the coefficients of the $\left(\frac{X}{2}\right)^2 e^{j2\omega t}$ terms, one can obtain

$$(-4m\omega^2 + jc2\omega + k_1)H_2(\omega,\omega) + k_2 H_1(\omega)^2 = 0 \qquad (8)$$

$$H_2(\omega,\omega) = \frac{-k_2 H_1(\omega)^2}{k_1 - m(2\omega)^2 + jc2\omega} = -k_2 H_1(\omega)^2 H_1(2\omega) \qquad (9)$$

Since a higher-order FRF is a complex valued function, it can be represented by magnitude and phase. Fig. 2 shows $H_2(\omega,\omega)$ for a non-linear system which will be considered later.

Extending this method of harmonic probing[7], all

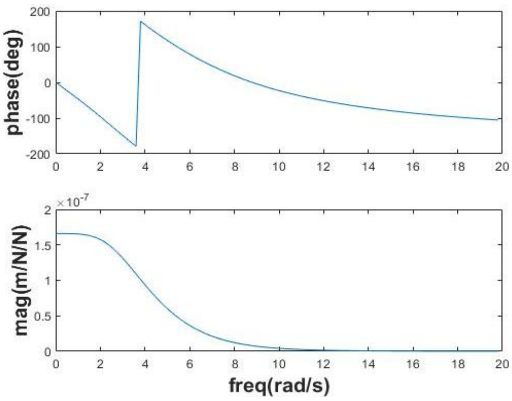

Fig. 2 Magnitude and phase of $H_2(\omega,\omega)$ of a non-linear system

higher-order FRFs in Eq. (5) can be obtained. Based on this result a higher-order FRF can be expressed in terms of lower-order FRFs. After all the FRFs in Eq. (5) are calculated, the response component at the excitation frequency, $Y(\omega)$ can be obtained from Eq. (5).

3. Application of the Solving Methods

3.1 Proposed non-linear isolator

The proposed non-linear isolator is composed of two symmetric linear springs, a linear damper, and a base as depicted in Fig. 3. The linear springs have spring constant k, free length l_0, and are inclined by θ from the horizontal base. When the upper support of the springs moves downward by y, the vertical force applied on the support from the springs can be calculated easily. Fig. 4 illustrates the variation of the

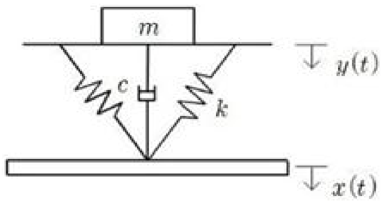

Fig. 3 Considered vibration isolation system

spring force with y for $k = 10{,}000$ N/m, $l_0 = 0.7$ m, and $\theta = 0.7752$ rad(44.42°). The spring force reaches a maximum value of 1261 N at y = 0.239 m. The figure shows that the stiffness of the spring decreases with the displacement of the support and becomes negative for large displacements.

When the base of the vibration isolator is excited as depicted in Fig. 3, the motion of the mass is described by the following equation.

$$m\ddot{y} = -c(\dot{y} - \dot{x}) - f(y - x) \tag{10}$$

In Eq. (10) a dot on a letter represents differentiation with respect to time. c and $f(x)$ represent the damping constant and the spring force, respectively. The excitation of the base $x(t)$ is given by $X\sin\omega t$. The displacement of the mass, $y(t)$ is measured from the equilibrium position due to gravity. Letting $z = y - x$, Eq. (10) becomes

$$m\ddot{z} + c\dot{z} + f(z) = m\omega^2 X \sin\omega t \tag{11}$$

Solving Eq. (11) for z numerically and adding x, the response of the mass is obtained. Computing a Fourier Transform of $y(t)$, its frequency component at the excitation frequency ω is obtained. The considered system is equipped with the above non-linear spring and has the parameters : $m = 100$ kg, $c = 1000$ Ns/m, $\omega = 10$ rad/s.

3.2 Approximate solutions

The response of the mass for the harmonic excitation of the base was calculated by solving the equation of motion numerically. By computing a Fourier transform of the response, the response component at the excitation frequency ω is obtained, which is assumed to be exact. The amplitude of the response component at the excitation frequency was also calculated approximately and the results compared based on three methods : linear approximation, harmonic balance, and higher-order FRFs.

3.2.1 Linear approximation

At the static equilibrium of the system, the deflection of the mass and slope of the tangent to the curve in Fig. 4 are found to be 0.13 m and 5028 N/m, respectively. If the non-linear spring is approximated by a linear spring with this stiffness, the ratio of the amplitude of the mass to that of the base becomes 1.0022 based on linear theory. Thus the amplitude of the mass is $Y = 0.0501$ m for $X = 0.05$

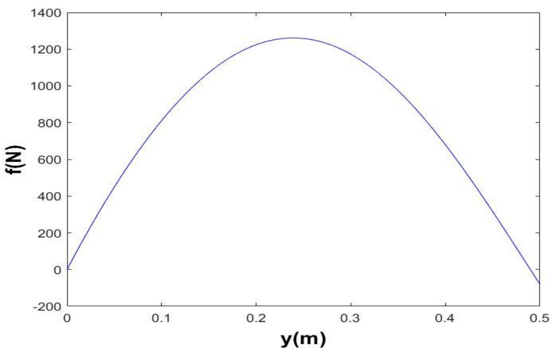

Fig. 4 Variation of the spring force with the deflection

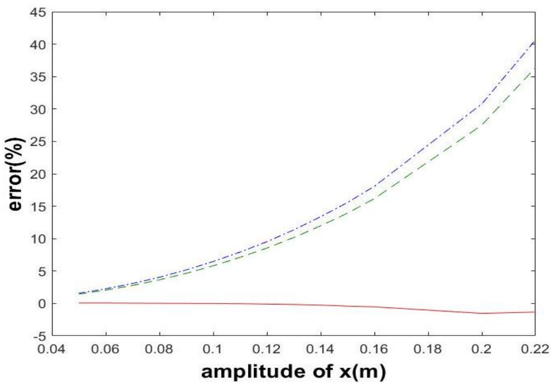

Fig. 5 Variation of the error with the excitation amplitude for linear approximation(dash-dot line), harmonic balance(dashed line), and higher-order FRFs(solid line)

m. Comparing this amplitude with the exact response component 0.049328 m, the error is 1.57%. Because the system is approximated as a linear system, the ratio does not change with the excitation amplitude. The response component of the mass at the excitation frequency and the error were calculated repeatedly for various excitation amplitudes. As the excitation amplitude increases, the error increases monotonously and reaches 40.65% for $X=0.22$ m as depicted in Fig. 5.

3.2.2 Harmonic balance method

In Eq. (11), $f(z)$ represents the non-linear spring force. If the origin of the coordinate axes is set to the static equilibrium position and the spring force around the origin is represented by a cubic polynomial $f(y)=k_1y+k_2y^2+k_3y^3$, the coefficients are $k_1=5031.6$ N/m, $k_2=-21126$ N/m^2, and $k_3=-13715$ N/m^3 based on a regression analysis in the range $0\leq y\leq 0.239$ m as depicted in Fig. 4. Then Eq. (11) can be written as

$$m\ddot{z}+c\dot{z}+k_1z+k_2z^2+k_3z^3=m\omega^2X\sin(\omega t+\phi) \tag{12}$$

In Eq. (12) the unknown phase ϕ is added to the excitation to obtain a single-term harmonic response $z(t)=Z\sin\omega t$[8].

Inserting $z(t)$ into Eq. (12), the following equation is obtained.

$$\begin{aligned}&-m\omega^2Z\sin\omega t+c\omega Z\cos\omega t+k_1Z\sin\omega t\\&+k_2Z^2\sin^2\omega t+k_3Z^3\sin^3\omega t=m\omega^2X\sin(\omega t+\phi)\end{aligned} \tag{13}$$

Using the following trigonometric relations

$$\sin^2\omega t=\frac{1}{2}-\frac{1}{2}\cos 2\omega t \tag{14}$$

$$\sin^3\omega t=\frac{3}{4}\sin\omega t-\frac{1}{4}\sin 3\omega t \tag{15}$$

$$\sin(\omega t+\phi)=\sin\omega t\cos\phi+\cos\omega t\sin\phi \tag{16}$$

and equating the coefficients of $\sin\omega t$ and $\cos\omega t$ terms on both sides, the following equations are obtained.

$$(k_1-m\omega^2)Z+\frac{3}{4}k_3Z^3=m\omega^2X\cos\phi \tag{17}$$

$$c\omega Z=m\omega^2X\sin\phi \tag{18}$$

Solving these equations simultaneously produces $Z=0.04474$ m and $\phi=2.03356$ rad. Because $z(t)=Z\sin\omega t$ is the response for the excitation $m\omega^2X\sin(\omega t+\phi)$, the response for the excitation $m\omega^2X\sin\omega t$ would be $z(t)=Z\sin(\omega t-\phi)$. Adding $x(t)$ to $z(t)$, $y(t)$ is obtained with an amplitude of the response component at the excitation frequency of 0.050044 m, resulting in an error of 1.45% which is slightly less than that for linear approximation. As the excitation amplitude increases, the error increases monotonously and reaches 36.41% for $X=0.22$ m as depicted in Fig. 4. The error varies with the excitation amplitude similarly to, but slightly less than the case of linear approximation. If more harmonics are included for the solution of Eq. (12), more accurate responses could be obtained. However, the procedure is very complicated as depicted in Reference [8]. Therefore, an approximate solution with more harmonics has not been attempted in this research.

3.2.3 Higher-order FRFs

If the spring force in Eq. (11) is approximated by a cubic polynomial, the equation of motion is expressed as

$$m\ddot{z}+c\dot{z}+k_1z+k_2z^2+k_3z^3=m\omega^2X\sin\omega t \tag{19}$$

The response component at the excitation frequency was obtained using higher-order FRFs, as explained in Section 2.2, and $x(t)$ was added to obtain the frequency component of $y(t)$. Increasing the maximum order of higher-order FRFs in Eq. (5), the above calculation was repeated. For the maximum order $n \geq 7$, the obtained results converged. Consequently the maximum order $n=9$ was used for future calculations. The response component at the excitation frequency was 0.049367 m for $X=0.05$ m, with an error of 0.079%. Comparing the result with the previous two cases, this method produces more accurate results. As the excitation amplitude increases, the error does not increase rapidly as in the previous cases. The error becomes a maximum of 1.53% when $X=0.2$ m as depicted in Fig. 5. This error is significantly smaller than those obtained by the two previous methods. When the excitation amplitude is increased beyond $X=0.22$ m, the response diverges.

4. Conclusion

A non-linear vibration isolator composed of a non-linear spring and a linear damper was proposed in previous research.

When the base of the isolator is excited harmonically, the response component of the mass at the excitation frequency was calculated approximately using three different methods: linear approximation, harmonic balance, and higher-order FRFs. For the linear approximation method the error between the exact and approximate responses increases monotonously with the excitation amplitude. For the harmonic balance method the error varies with the excitation amplitude similarly to, but slightly less than the case of linear approximation. The method using higher-order FRFs produces significantly more accurate results compared to the other methods. The error does not increase monotonously with the excitation amplitude, and is less than 2%.

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