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MODULE DERIVATIONS ON COMMUTATIVE BANACH MODULES

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Abstract. In this paper, the commutative module amenable Banach algebras are characterized. The hereditary and permanence properties of module amenability and the relations between module amenability of a Banach algebra and its ideals are explored. Analogous to the classical case of amenability, it is shown that the projective tensor product and direct sum of module amenable Banach algebras are again module amenable. By an application of Ryll-Nardzewski fixed point theorem, it is shown that for an inverse semigroup S, every module derivation of $l^1(S)$ into a reflexive module is inner.

1. Introduction

The concept of module amenability for a class of Banach algebras which is in fact a generalization of the classical amenability (Johnson's amenability) [17] has been developed by the first author in [1]. He showed that for every inverse semigroup S with a subsemigroup E of idempotents, the $l^1(E)$-module amenability of $l^1(S)$ is equivalent to the amenability of S. Recall that a discrete semigroup S is called an *inverse semigroup* if for each $s \in S$, there is a unique element $s^* \in S$ such that $ss^*s = s$ and $s^*ss^* = s^*$. An element $e \in S$ is called an *idempotent* if $e = e^* = e^2$. The set of idempotents of S is denoted by E. The mentioned notion was modified in [7] and [5], by using module homomorphisms between Banach algebras. In [22], Aghababa and the second author introduced the notions of module approximate amenability and contractibility of Banach algebras that are modules over another Banach algebra. They proved that $l^1(S)$ is $l^1(E)$-module approximately amenable (contractible) if and only if S is amenable; for the module character amenability, generalized notions of module character amenability, weak module amenability and the n-weak module amenability of inverse semigroup algebras, we refer to [8], [12], [2] and [10], respectively. Furthermore, permanent weak module amenability of the triangular Banach algebras (resp. the module projective tensor product of

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Banach algebras) and applications to inverse semigroup algebras are studied in [13] (resp. [11]). Recently, Asgari et al. in [4] studied the module amenability of the weighted semigroup algebras and showed that for an inverse semigroup S equipped with a weight ω and the set of idempotents E, when $l^1(E)$ acts on $l^1(S,\omega)$ trivially from left and by multiplication from right, the weighted semigroup algebra $l^1(S,\omega)$ is $l^1(E)$-module amenable if and only if S is amenable and sup $\{\Omega(s) : s \in S\} < \infty$ where $\Omega(s) = \omega(s)\omega(s^*)$ for all $s \in S$.

In this paper, we characterize the module amenability of a commutative Banach $\mathfrak{A}$-module $\mathcal{A}$, where $\mathfrak{A}$ and $\mathcal{A}$ are Banach algebras. We prove that for every ideal of a commutative module amenable Banach algebra, its module amenability is equivalent to having a bounded approximate identity. In addition, we investigate the module amenability of tensor product and direct sum of Banach algebras. As an application of Ryll-Nardzewski fixed point theorem, we prove that for an inverse semigroup S, every module derivation of $l^1(S)$ into a reflexive module is inner.

2. Preliminaries

Let us introduce some notations that will be used throughout this paper. Let $\mathcal{A}$ and $\mathfrak{A}$ be Banach algebras such that $\mathcal{A}$ is a Banach $\mathfrak{A}$-bimodule with compatible actions as follows:

$$\alpha \cdot (ab) = (\alpha \cdot a)b, \quad (ab) \cdot \alpha = a(b \cdot \alpha) \qquad (a, b \in \mathcal{A}, \alpha \in \mathfrak{A}).$$

Let X be a Banach $\mathcal{A}$-bimodule and a Banach $\mathfrak{A}$-bimodule with compatible actions, that is

$$\alpha \cdot (a \cdot x) = (\alpha \cdot a) \cdot x, \; a \cdot (\alpha \cdot x) = (a \cdot \alpha) \cdot x, \; (\alpha \cdot x) \cdot a = \alpha \cdot (x \cdot a)$$

for all $a \in \mathcal{A}, \alpha \in \mathfrak{A}, x \in X$ and similarly for the right or two-sided actions. Then, we say that X is a Banach $\mathcal{A}$-$\mathfrak{A}$-module. Moreover, if $\alpha \cdot x = x \cdot \alpha$ for all $\alpha \in \mathfrak{A}, x \in X$, then X is called a *commutative* $\mathcal{A}$-$\mathfrak{A}$-module. If X is a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module, then so is X^*, the first dual space of X, where the actions of $\mathcal{A}$ and $\mathfrak{A}$ on X^* are defined as usual:

$$\langle \alpha \cdot f, x \rangle = \langle f, x \cdot \alpha \rangle, \qquad \langle a \cdot f, x \rangle = \langle f, x \cdot a \rangle$$

for all $a \in \mathcal{A}, \; \alpha \in \mathfrak{A}, \; x \in X, \; f \in X^*$ and similarly for the right actions. Note that in general, $\mathcal{A}$ is not an $\mathcal{A}$-$\mathfrak{A}$-module because $\mathcal{A}$ does not satisfy the compatibility condition $a \cdot (\alpha \cdot b) = (a \cdot \alpha) \cdot b$ for $\alpha \in \mathfrak{A}, a, b \in \mathcal{A}$. But when $\mathcal{A}$ is a commutative $\mathfrak{A}$-module and acts on itself by multiplication from both sides, then it is also a Banach $\mathcal{A}$-$\mathfrak{A}$-module.

Let $\mathcal{A}$ and $\mathfrak{A}$ be as in the above and X be a Banach $\mathcal{A}$-$\mathfrak{A}$-module. A bounded map $D : \mathcal{A} \longrightarrow X$ is called a *module derivation* if

$$D(a \pm b) = D(a) \pm D(b), \;\; D(ab) = D(a) \cdot b + a \cdot D(b) \qquad (a, b \in \mathcal{A}),$$

and

$$D(\alpha \cdot a) = \alpha \cdot D(a), \quad D(a \cdot \alpha) = D(a) \cdot \alpha \qquad (a \in \mathcal{A}, \alpha \in \mathfrak{A}).$$

One should remember that D is not necessarily linear, but its boundedness (defined as the existence of $M > 0$ such that $\|D(a)\| \leq M\|a\|$ for all $a \in \mathcal{A}$) still implies its continuity, as it preserves subtraction. For a commutative $\mathcal{A}$-$\mathfrak{A}$-module X, each $x \in X$ defines a module derivation

$$D_x(a) = a \cdot x - x \cdot a \quad (a \in \mathcal{A}).$$

These are called *inner* module derivations. A Banach algebra $\mathcal{A}$ is called *module amenable* (as an $\mathfrak{A}$-module) if for any commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module X, each module derivation $D : \mathcal{A} \longrightarrow X^*$ is inner. We use the notations $Z_{\mathfrak{A}}(\mathcal{A}, X^*)$ and $B_{\mathfrak{A}}(\mathcal{A}, X^*)$ for the space of all module derivations and inner module derivations from $\mathcal{A}$ to X^*, respectively. We also use the notation $H_{\mathfrak{A}}(\mathcal{A}, X^*)$ for the quotient space $Z_{\mathfrak{A}}(\mathcal{A}, X^*)/B_{\mathfrak{A}}(\mathcal{A}, X^*)$ which we call the first relative (to $\mathfrak{A}$)-cohomology group of $\mathcal{A}$ with coefficients in X^*. Hence, $\mathcal{A}$ is module amenable if and only if $H_{\mathfrak{A}}(\mathcal{A}, X^*) = \{0\}$ for all commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module X [1].

Let $X\widehat{\otimes}Y$ denote the projective tensor product of two Banach spaces X and Y and let $\mathfrak{A}$ be a Banach algebra. Let also X be a Banach right $\mathfrak{A}$-module and Y be a Banach left $\mathfrak{A}$-module. Put N as the closed linear span of

$$\{x \cdot \alpha \otimes y - x \otimes \alpha \cdot y : \alpha \in \mathfrak{A}, x \in X, y \in Y\}.$$

We also consider the Banach space $X\widehat{\otimes}_{\mathfrak{A}}Y = (X\widehat{\otimes}Y)/N$ (for more details refer to [24]). If now X and Y are Banach $\mathfrak{A}$-modules, then $X\widehat{\otimes}Y$ is a Banach $\mathfrak{A}$-module by the following usual actions:

$$\alpha \cdot (x \otimes y) = (\alpha \cdot x) \otimes y, \quad (x \otimes y) \cdot \alpha = x \otimes (y \cdot \alpha) \quad (\alpha \in \mathfrak{A}, x \in X, y \in Y).$$

Hence, N is clearly an $\mathfrak{A}$-submodule of $X\widehat{\otimes}Y$, and so $X\widehat{\otimes}_{\mathfrak{A}}Y$ is a Banach $\mathfrak{A}$-module.

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are Banach algebras and Banach $\mathfrak{A}$-bimodules. Then, $\mathcal{A}\widehat{\otimes}\mathcal{B}$ is a Banach algebra with respect to the canonical multiplication that satisfies that

$$(a \otimes b)(c \otimes d) = (ac \otimes bd) \quad (a, c \in \mathcal{A}, b, d \in \mathcal{B}).$$

Consider $I_{\mathcal{A}}$ as the closed ideal of the projective tensor product $\mathcal{A}\widehat{\otimes}\mathcal{A}$ generated by elements of the form

$$\{a \cdot \alpha \otimes b - a \otimes \alpha \cdot b \mid \alpha \in \mathfrak{A}, a, b \in \mathcal{A}\}.$$

Let $\omega_{\mathcal{A}} : \mathcal{A}\widehat{\otimes}\mathcal{A} \longrightarrow \mathcal{A}$ be the product map, i.e., $\omega_{\mathcal{A}}(a \otimes b) = ab$. Let $J_{\mathcal{A}}$ be the closed ideal of $\mathcal{A}$ generated by

$$\omega_{\mathcal{A}}(I) = \{(a \cdot \alpha)b - a(\alpha \cdot b) \mid \alpha \in \mathfrak{A}, a, b \in \mathcal{A}\}.$$

We note that $J_{\mathcal{A}}$ is indeed equal to the closed subspace generated by $\omega_{\mathcal{A}}(I)$ [10, Lemma 3.1]. Furthermore, the module projective tensor product $\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A} \cong (\mathcal{A}\widehat{\otimes}\mathcal{A})/I_{\mathcal{A}}$ and the quotient Banach algebra $\mathcal{A}/J_{\mathcal{A}}$ are both Banach $\mathcal{A}$-modules and Banach $\mathfrak{A}$-modules. Moreover, $\mathcal{A}/J_{\mathcal{A}}$ is always $\mathcal{A}$-$\mathfrak{A}$-module with compatible actions when $\mathcal{A}$ acts on $\mathcal{A}/J_{\mathcal{A}}$ canonically. We have $(\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A})^* = \mathcal{L}_{\mathfrak{A}}(\mathcal{A}, \mathcal{A}^*)$ where the right hand side is the space of all $\mathfrak{A}$-module morphism from A to

A^* [24]. Consider the map $\widetilde{\omega}_{\mathcal{A}} : \mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A} \cong (\mathcal{A}\widehat{\otimes}\mathcal{A})/I_{\mathcal{A}} \longrightarrow \mathcal{A}/J_{\mathcal{A}}$ defined via $\widetilde{\omega}_{\mathcal{A}}(a\otimes b + I_{\mathcal{A}}) = ab + J_{\mathcal{A}}$ and extended by linearity and continuity. It is clear that $\widetilde{\omega}_{\mathcal{A}}, \widetilde{\omega}_{\mathcal{A}}^*$ and second adjoint $\widetilde{\omega}_{\mathcal{A}}^{**} : (\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A})^{**} \longrightarrow \mathcal{A}^{**}/J_{\mathcal{A}}^{\perp\perp}$ are $\mathcal{A}$-module and $\mathfrak{A}$-module homomorphisms. We shall denote $I_{\mathcal{A}}, J_{\mathcal{A}}$ and $\widetilde{\omega}_{\mathcal{A}}$ by I, J and $\widetilde{\omega}$ respectively, if there is no risk of confusion. Obviously, I and J are $\mathcal{A}$-submodules and $\mathfrak{A}$-submodules of $\mathcal{A}\widehat{\otimes}\mathcal{A}$ and $\mathcal{A}$ respectively, and the quotients $\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A}$ and $\mathcal{A}/J$ are $\mathcal{A}$-modules and $\mathfrak{A}$-modules.

Let $\mathcal{A}$ be a Banach algebra and X be a Banach $\mathcal{A}$-bimodule. We say X is $\mathcal{A}$-pseudo-unital if $X = \mathcal{A}\cdot X\cdot\mathcal{A} = \{a\cdot x\cdot b : a, b\in\mathcal{A},\, x\in X\}$. Recall that $\mathfrak{A}$ has a bounded approximate identity for $\mathcal{A}$ if there is a bounded net $\{\gamma_i\}$ in $\mathfrak{A}$ such that for each $a\in\mathcal{A}$, $\|\gamma_i\cdot a - a\|\to 0$ and $\|a\cdot\gamma_i - a\|\to 0$ as $i\to\infty$.

It is proved in [1, Proposition 2.1] that if $\mathfrak{A}$ has a bounded approximate identity for $\mathcal{A}$, then amenability of $\mathcal{A}$ implies its module amenability. Indeed, the above result shows that every $\mathfrak{A}$-module derivation is also a linear derivation when $\mathfrak{A}$ has a bounded approximate identity for $\mathcal{A}$. This condition is strong and this is reduced to a weaker condition, as left or right essential $\mathfrak{A}$-module [22]. Recall that a left Banach $\mathcal{A}$-module X is called a *right essential* $\mathcal{A}$-module if the linear span of $X\cdot\mathcal{A} = \{x\cdot a : a\in\mathcal{A},\, x\in X\}$ is dense in X. Left essential $\mathcal{A}$-modules and (two-sided) essential $\mathcal{A}$-bimodules are defined similarly. We will use this fact several times in the current work.

Example 2.1. (i) Let G be a locally compact group. Suppose $1\le p\le\infty$ and $f\in L^1(G)$ and $g\in L^p(G)$. Then, $f*g\in L^p(G)$ and $\|f*g\|_p\le\|f\|_1\|g\|_p$. In the case that G is a compact group, $g*f\in L^p(G), L^1(G)$ and $\|g*f\|\le\|f\|_1\|g\|_p$. Now, if G is an abelian compact group, $\mathfrak{A} = l^1(G)$, the Banach algebra of discrete measures, $\mathcal{A} = L^p(G)$ and $X = L^1(G)$, then $L^1(G)$ is an $L^p(G)$-$l^1(G)$-module which is commutative as an $l^1(G)$-module. For a concrete example, we illustrate this for $G = \mathbb{T}$, the compact abelian group of unit complex numbers. Let $1 < p < \infty$ and $\frac{1}{p}+\frac{1}{q} = 1$. Consider $f = \Sigma_{n=-\infty}^{\infty}\frac{1}{n}e^{2\pi int}\in L^p(\mathbb{T})$ (which is basically the Fourier transform of the above function f used in the discrete case). Then, for each $g\in L^q(\mathbb{T})$, $g*f = \Sigma_{n=-\infty}^{\infty}\frac{1}{n}\hat{g}(n)e^{2\pi int}\in L^1(\mathbb{T})$ where $\hat{g}\in c_0(G)$ is the Fourier transform of g.

(ii) For a locally compact group G, $L^1(G)$ is a closed two sided ideal in $M(G)$, the measure algebra of G and hence, we can consider it as a Banach $M(G)$-module with convolution. Therefore, by the convolution action, $M(G)$ is an $L^1(G)$-$l^1(G)$-module which is commutative as an $l^1(G)$-module if and only if G is abelian [20].

Definition 2.2. A bounded net $\{\widetilde{u}_j\}$ in $\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A}$ is called a *module approximate diagonal* if $\widetilde{\omega}_{\mathcal{A}}(\widetilde{u}_j)$ is a bounded approximate identity of $\mathcal{A}/J$ and $\lim_j\|\widetilde{u}_j\cdot a - a\cdot\widetilde{u}_j\| = 0$ for each $a\in\mathcal{A}$. An element $\widetilde{M}\in(\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A})^{**}$ is called a *module virtual diagonal* if

$$\widetilde{\omega}_{\mathcal{A}}^{**}(\widetilde{M})\cdot a = \widetilde{a},\quad \widetilde{M}\cdot a = a\cdot\widetilde{M}\quad (a\in\mathcal{A}),$$

where $\widetilde{a} = a + J^{\perp\perp}$.

3. Main results

In this section, we characterize the module amenable commutative Banach modules. We also study hereditary and permanence properties of module amenability for these modules.

Recall that a Banach algebra $\mathcal{A}$ is called *module biflat* (as an $\mathfrak{A}$-module) if $\widetilde{\omega}_{\mathcal{A}}^*$ has a bounded left inverse which is an $\mathcal{A}/J$-$\mathfrak{A}$-module homomorphism [9].

Theorem 3.1. *Let $\mathcal{A}$ be a commutative Banach $\mathfrak{A}$-bimodule. Then, the following statements are equivalent*:

(i) *$\mathcal{A}$ is module amenable*;

(ii) *$\mathcal{A}$ has a module virtual diagonal*;

(iii) *$\mathcal{A}$ has a module approximate diagonal*;

(iv) *$\mathcal{A}$ is module biflat and has a bounded approximate identity*;

(v) *$\mathcal{A}$ has a bounded approximate identity and $\ker \widetilde{\omega}_{\mathcal{A}}$ has a bounded right approximate identity*;

(vi) *$\mathcal{A}$ has a bounded approximate identity and $H_{\mathfrak{A}}(\mathcal{A}, X^{**}) = \{0\}$ for each commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module X*;

(vii) *$\mathcal{A}$ has a bounded approximate identity and every bounded module derivation $D : \mathcal{A} \longrightarrow (\ker \widetilde{\omega}_{\mathcal{A}})^{**}$ is inner.*

Proof. Since $\mathcal{A}$ is a commutative Banach $\mathfrak{A}$-bimodule, so is $\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A}$ and $J = \{0\}$. If $\mathcal{A}$ is module amenable, then $\mathcal{A}$ has a bounded approximate identity by [1, Proposition 2.2]. Now, the equivalence of (i) and (ii) follows from [20, Theorem 2.1].

(ii)$\Leftrightarrow$(iii) Let $(\widetilde{u}_j)_j$ be a module approximate diagonal for $\mathcal{A}$, and regard $(\widetilde{u}_j)_j$ as a bounded net in $(\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A})^{**}$. Then, $(\widetilde{u}_j)_j$ has a weak*-accumulation point, and each such point is a module virtual diagonal for $\mathcal{A}$. Let now $\widetilde{M}$ be a module virtual diagonal for $\mathcal{A}$, and $(\widetilde{u}_j)_j$ be a bounded net in $\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A}$ with $M = w^*\text{-}\lim_j \widetilde{u}_j$. Thus

$$w - \lim_j (a \cdot \widetilde{u}_j - \widetilde{u}_j \cdot a) = 0 \quad \text{and} \quad w - \lim_j a\omega_{\mathcal{A}}(\widetilde{u}_j) = a \quad (a \in \mathcal{A}).$$

By passing to convex combinations, we obtain an approximate diagonal. Equivalence of (i) and (iv) is the consequence of [9, Theorem 2.1]. Besides, [22, Theorem 4.4] shows that (i) and (v) are equivalent.

(i)$\Rightarrow$(vi) The first part follows from [1, Proposition 2.2] and the vanishing of the module cohomology group is clear.

(vi)$\Rightarrow$(vii) It is trivial.

(vii)$\Rightarrow$(iii) If $\{e_j\}$ is a bounded approximate identity for $\mathcal{A}$, then passing to a subnet we may assume that $(e_j \otimes e_j + I)$ is w^*-convergent to F in $(\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A})^{**} = (\mathcal{A}\widehat{\otimes}\mathcal{A})^{**}/I^{\perp\perp}$. For each $a \in \mathcal{A}$ and $f \in \mathcal{A}^*$, we have

$$\begin{aligned} \langle \widetilde{\omega}_{\mathcal{A}}^{**}(a \cdot F - F \cdot a), f\rangle &= \langle a \cdot F - F \cdot a, \widetilde{\omega}_{\mathcal{A}}^*(f)\rangle \\ &= \lim_j \langle \widetilde{\omega}_{\mathcal{A}}^*(f), ae_j \otimes e_j - e_j \otimes e_j a + I\rangle \end{aligned}$$

$$= \lim_j \langle f, ae_j^2 - e_j^2 a\rangle = 0.$$

Thus, $a \cdot F - F \cdot a \in \ker \widetilde{\omega}_{\mathcal{A}}^{**}$. Consider the map $T : \mathcal{A} \longrightarrow \ker \widetilde{\omega}_{\mathcal{A}}^{**}; a \mapsto a \cdot F - F \cdot a$. Then T is a bounded module derivation. By assumption, there exists $E \in \ker \widetilde{\omega}_{\mathcal{A}}^{**}$ such that

$$a \cdot F - F \cdot a = a \cdot E - E \cdot a \tag{3.1}$$

for all $a \in \mathcal{A}$. We show that $\widetilde{M} = F - E \in (\mathcal{A}\widehat{\otimes}\mathcal{A})^{**}/I^{\perp\perp}$ is a module virtual diagonal. We have $\widetilde{\omega}_{\mathcal{A}}^{**}(\widetilde{M}) = \widetilde{\omega}_{\mathcal{A}}^{**}(F) - \widetilde{\omega}_{\mathcal{A}}^{**}(E) = \widetilde{\omega}_{\mathcal{A}}^{**}(F)$. Take $a \in \mathcal{A}$ and $f \in \mathcal{A}^*$. Then

$$\langle \widetilde{\omega}_{\mathcal{A}}^{**}(F)\cdot a, f\rangle = \langle F, \widetilde{\omega}_{\mathcal{A}}^{*}(a\cdot f)\rangle = \lim_j \langle \widetilde{\omega}_{\mathcal{A}}^{*}(a\cdot f), e_j \otimes e_j + I\rangle = \lim_j \langle f, e_j^2 a\rangle = \langle f, a\rangle.$$

Therefore, $\widetilde{\omega}_{\mathcal{A}}^{**}(\widetilde{M}) \cdot a = a$. Once more, it follows from (3.1) that $\widetilde{M} \cdot a = a \cdot \widetilde{M}$. This completes the proof. □

The following result is proved in [1, Proposition 2.5] by using the definition of module actions. Here, we prove it using module approximate diagonal. The proof is similar to the proof of [16, Theorem 2.5] but we include it for the sake of completeness.

Proposition 3.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and Banach $\mathfrak{A}$-modules. If $\mathcal{A}$ is module amenable and $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ is a continuous module homomorphism and algebra homomorphism with dense range, then $\mathcal{B}$ is module amenable.*

Proof. By [20, Theorem 2.1], $\mathcal{A}$ has a module approximate diagonal, say $\widetilde{u}_j = \sum_k a_k^j \otimes b_k^j + I_{\mathcal{A}}$, in $\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A}$. Define the map $\widetilde{\phi} : \mathcal{A}/J_{\mathcal{A}} \longrightarrow \mathcal{B}/J_{\mathcal{B}}$ via $\widetilde{\phi}(a + J_{\mathcal{A}}) = \phi(a) + J_{\mathcal{B}}$. The map $\widetilde{\phi}$ is well-defined because for each $a, b \in \mathcal{A}$ and $\alpha \in \mathfrak{A}$ we have

$$\phi((a \cdot \alpha)b - a(\alpha \cdot b)) = (\phi(a) \cdot \alpha)\phi(b) - \phi(a)(\alpha \cdot \phi(b)) \in J_{\mathcal{B}}.$$

Set $\widetilde{v}_j = \sum_k \phi\left(a_k^j\right) \otimes \phi\left(b_k^j\right) + I_{\mathcal{B}}$. For each $a \in \mathcal{A}$, we get

$$\begin{aligned}
\lim_j (\phi(a) + J_{\mathcal{B}}) \cdot \widetilde{\omega}_{\mathcal{B}}(\widetilde{v}_j) &= \lim_j (\phi(a) + J_{\mathcal{B}}) \cdot \left(\sum_k \phi\left(a_k^j\right)\phi\left(b_k^j\right) + J_{\mathcal{B}}\right)\\
&= \lim_j \left(\sum_k \phi\left(a a_k^j b_k^j\right) + J_{\mathcal{B}}\right)\\
&= \lim_j \widetilde{\phi}\left((a + J_{\mathcal{A}}) \cdot \left(\sum_k a_k^j b_k^j + J_{\mathcal{A}}\right)\right)\\
&= \lim_j \widetilde{\phi}((a + J_{\mathcal{A}}) \cdot \widetilde{\omega}_{\mathcal{B}}(\widetilde{u}_j)) = \widetilde{\phi}(a + J_{\mathcal{A}}) = \phi(a) + J_{\mathcal{B}}.
\end{aligned}$$

The density of the range of ϕ and its continuity show that for any $b \in \mathcal{B}$, $\lim_j (b + J_{\mathcal{B}}) \cdot \widetilde{\omega}_{\mathcal{B}}(\widetilde{v}_j) = b + J_{\mathcal{B}}$. Define the map $\overline{\phi} : \mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A} \cong (\mathcal{A}\widehat{\otimes}\mathcal{A})/I_{\mathcal{A}} \longrightarrow$

$\mathcal{B}\widehat{\otimes}_{\mathfrak{A}}\mathcal{B} \cong (\mathcal{B}\widehat{\otimes}\mathcal{B})/I_{\mathcal{B}}$ through

$$\overline{\phi}(a \otimes b + I_{\mathcal{A}}) = \phi(a) \otimes \phi(b) + I_{\mathcal{B}}, \quad (a, b \in \mathcal{A}).$$

Similar to $\widetilde{\phi}$, the map $\overline{\phi}$ is well-defined and it is also a module homomorphism. For each $a \in \mathcal{A}$ we obtain

$$\begin{aligned}
&\lim_j (\widetilde{v}_j \cdot \phi(a) - \phi(a) \cdot \widetilde{v}_j) \\
&= \lim_j \left(\sum_k \left(\phi\left(a_k^j\right) \otimes \phi\left(b_k^j a\right) - \phi\left(a a_k^j\right) \otimes \phi\left(b_k^j\right)\right) + I_{\mathcal{B}} \right) \\
&= \overline{\phi}\left(\lim_j \left(\sum_k (a_k^j \otimes b_k^j a - a a_k^j \otimes b_k^j) + I_{\mathcal{A}} \right)\right) \\
&= \overline{\phi}\left(\lim_j (\widetilde{u}_j \cdot a - a \cdot \widetilde{u}_j)\right) = 0.
\end{aligned}$$

Thus, for $b \in \mathcal{A}$, $\lim_j (\widetilde{v}_j \cdot b - b \cdot \widetilde{v}_j) = 0$. Therefore, $\widetilde{v}_j = \sum_{k=1}^{p} \phi\left(a_k^j\right) \otimes \phi\left(b_k^j\right) + I_{\mathcal{B}}$ is a module approximate diagonal for $\mathcal{B}$, and so $\mathcal{B}$ is module amenable. □

Corollary 3.3. *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module and $\mathcal{I}$ be a closed ideal in $\mathcal{A}$ which is an $\mathfrak{A}$-submodule of $\mathcal{A}$. Then, the module amenability of $\mathcal{A}$ implies module amenability of $\mathcal{A}/\mathcal{I}$.*

Proposition 3.4. *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module and $\mathcal{I}$ be a closed ideal and an $\mathfrak{A}$-submodule of $\mathcal{A}$. If $\mathcal{I}$ and $\mathcal{A}/\mathcal{I}$ are module amenable, then so is $\mathcal{A}$.*

Proof. Suppose that X is a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module with compatible actions and $D : \mathcal{A} \longrightarrow X^*$ is a bounded module derivation. Since $\mathcal{I}$ is module amenable, there exists $f_1 \in X^*$ such that $D\mid_{\mathcal{I}} = D_{f_1}$. Thus, the map $\widetilde{D} = D - D_{f_1}$ vanishes on $\mathcal{I}$. This map induces a module derivation into X^*, which we again denote by $\widetilde{D}$. Let Y be the closed linear span of

$$\{a \cdot x - y \cdot b \mid a, b \in \mathcal{I}, x, y \in X\},$$

in X. It follows immediately that Y is a closed $\mathcal{A}$-submodule and $\mathfrak{A}$-submodule of X, and so X/Y is a Banach $\mathcal{A}/\mathcal{I}$-$\mathfrak{A}$-module with compatible actions. Since $D\mid_{\mathcal{I}} = \{0\}$, we have $a \cdot \widetilde{D}(b) = \widetilde{D}(ab) - \widetilde{D}(a) \cdot b = 0$ for all $a \in \mathcal{I}$ and $b \in \mathcal{A}$. Similarly, $\widetilde{D}(b) \cdot a = 0$. This implies $\widetilde{D}(\mathcal{A}/\mathcal{I}) \subset Y^{\perp} = (X/Y)^*$. Due to module amenability of $\mathcal{A}/\mathcal{I}$, there is $f_2 \in Y^{\perp} \subset X^*$ such that $\widetilde{D} = D_{f_2}$. Consequently, $D = D_{f_1+f_2}$. □

The upcoming result is a direct consequence of Corollary 3.3 and Proposition 3.4.

Corollary 3.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach $\mathfrak{A}$-modules. Then, $\mathcal{A}$ and $\mathcal{B}$ are module amenable if and only if $\mathcal{A} \oplus \mathcal{B}$ is module amenable.*

We denote the character space of a Banach algebra $\mathfrak{A}$ by $\Phi_{\mathfrak{A}}$. We say a Banach algebra $\mathfrak{A}$ acts trivially from left on a Banach algebra $\mathcal{A}$ if there is $f \in \Phi_{\mathfrak{A}}$ such that

$$\alpha \cdot a = f(\alpha)a$$

for each $\alpha \in \mathfrak{A}$ and $a \in \mathcal{A}$. Right trivial action is defined similarly.

Let $\mathfrak{A}$ be a Banach algebra and let $f \in \Phi_{\mathfrak{A}}$ be fixed. Suppose that $\mathfrak{A}$ act on $\mathbb{C}$ trivially from both sides. That is

$$\alpha \cdot \lambda = \lambda \cdot \alpha = f(\alpha)\lambda, \quad (\alpha \in \mathfrak{A}, \lambda \in \mathbb{C}). \tag{3.2}$$

With the above actions, $\mathbb{C}$ is a two-sided essential commutative $\mathfrak{A}$-bimodule. Here, we mean that $\mathbb{C}$ is a Banach $\mathfrak{A}$-bimodule for which the module actions are given by (3.2). It is well known that $\mathbb{C}$ is amenable. It is shown in [22] that if $\mathcal{A}$ is a left (right) essential $\mathfrak{A}$-module, then every $\mathfrak{A}$-module derivation is also a derivation. Since $\mathbb{C}$ is a two-sided essential $\mathfrak{A}$-bimodule, it is module amenable and so we have the following result by Corollary 3.5.

Proposition 3.6. *Let $n \in \mathbb{N}$. Then, $\mathbb{C}^n$ is module amenable* (*as an $\mathfrak{A}$-module*).

Assume that $\mathcal{A}$ is a unital and module amenable Banach algebra. We wish to show that $\mathcal{A}$ has a module approximate diagonal $(\widetilde{u}_j)$ such that $\widetilde{\omega}(\widetilde{u}_j) = e + J$ for all $\widetilde{u}_j$, where e is an identity for $\mathcal{A}$. Put $T = e \otimes e + I$. Define the map $D : \mathcal{A} \longrightarrow (\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A})^{**}$ by $D(a) := a \cdot T - T \cdot a \quad (a \in \mathcal{A})$. Clearly, D is a module derivation and $D(a) \subseteq \ker\, \widetilde{\omega}^{**} \subseteq (\ker\, \widetilde{\omega})^{**}$. By assumption there exists a $N \in (\ker\, \widetilde{\omega})^{**}$ so that $D(a) = a \cdot N - N \cdot a$. Letting $M = T - N$, we obtain $a \cdot \widetilde{\omega}^{**}(M) = a + J^{\perp\perp}$ and

$$a \cdot M - M \cdot a = D_M(a) = D_{T-N}(a) = D_T(a) - D_N(a) = 0$$

for all $a \in \mathcal{A}$, and so M is a module virtual diagonal for $\mathcal{A}$. On the other hand, there exists a bounded net $(u_j) \subseteq \ker\, \widetilde{\omega}$ such that $N = w^*\text{-}\lim_j(u_j)$. Putting $\widetilde{u}_j = T - u_j$, we have $a \cdot \widetilde{u}_j - \widetilde{u}_j \cdot a \to 0$ and $(a + J) \cdot \widetilde{\omega}(\widetilde{u}_j) \to (a + J)$. In particular, $\widetilde{\omega}(\widetilde{u}_j) = e + J$.

Let $\mathcal{A}$ be a Banach algebra which is Banach $\mathfrak{A}$-module. A Banach $\mathcal{A}$-$\mathfrak{A}$-module X is called *module pseudo-unital* if

$$X = \{a \cdot x \cdot b : a, b \in \mathcal{A}, x \in X\}.$$

Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras such that $\mathcal{A}$ is contained as a closed ideal in $\mathcal{B}$. Then, the strict topology on $\mathcal{B}$ with respect to $\mathcal{A}$ is defined by the family of seminorms $(\mathcal{T}_a)_{a\in\mathcal{A}}$, where

$$\mathcal{T}_a(b) := \|ba\| + \|ab\| \quad (b \in \mathcal{B}).$$

To prove the next theorem, we need the following lemma which is analogous to [25, Proposition 2.1.6]. Since the proof is similar, we only include some parts of the proof related to the module actions.

Lemma 3.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be commutative Banach algebras which are $\mathfrak{A}$-modules such that $\mathcal{A}$ has a bounded approximate identity which is contained as closed ideal in $\mathcal{B}$. If X is a pseudo-unital Banach $\mathcal{A}$-$\mathfrak{A}$-module and $D \in Z_{\mathfrak{A}}(\mathcal{A}, X^*)$, then X is a Banach $\mathcal{B}$-$\mathfrak{A}$-module in a canonical way, and there is a unique $\widetilde{D} \in Z_{\mathfrak{A}}(\mathcal{B}, X^*)$ such that $\widetilde{D}|_{\mathcal{A}} = D$ and $\widetilde{D}$ is continuous with respect to the strict topology on $\mathcal{B}$ and the w^*-topology on X^*.*

Proof. For $x \in X$, let $a \in \mathcal{A}$ and $y \in X$ be such that $x = a \cdot y$. For $b \in \mathcal{B}$, define $b \cdot x := ba \cdot y$. It is shown in [25, Proposition 2.1.6] that $b \cdot x$ is well defined, i.e., independent of the choices for a and y, and so X becomes a Banach $\mathcal{B}$-bimodule. Once more, let $x \in X$, $a \in \mathcal{A}$ and $y \in X$ be such that $x = y \cdot a$. For each $b \in \mathcal{B}$, we have

$$\alpha \cdot (b \cdot x) = \alpha \cdot (b \cdot (a \cdot y)) = \alpha \cdot (ba \cdot y) = (\alpha \cdot ba) \cdot y = (\alpha \cdot b)a \cdot y = (\alpha \cdot b) \cdot x.$$

Similarly, we can show that $b \cdot (\alpha \cdot x) = (b \cdot \alpha) \cdot x$ and $(\alpha \cdot x) \cdot b = \alpha \cdot (x \cdot b)$ for all $b \in \mathcal{A}, \alpha \in \mathfrak{A}, x \in X$. Thus, X is $\mathcal{B}$-$\mathfrak{A}$-module satisfying the compatible actions. Define the map $\widetilde{D} : \mathcal{B} \longrightarrow X^*$ by $\widetilde{D}(b) = w^* - \lim_j (D(be_j) - b \cdot D(e_j))$, where (e_j) is a bounded approximate identity for $\mathcal{A}$. It is proved in [25, Proposition 2.1.6] that $\widetilde{D}$ is well-defined which is also continuous with respect to the strict topology on $\mathcal{B}$ and the w^*-topology on X. In addition, $\widetilde{D}(ab) = a \cdot \widetilde{D}(b) + \widetilde{D}(a) \cdot b$. It remains to be shown that $\widetilde{D}$ is an $\mathfrak{A}$-module homomorphism. For each $b \in \mathcal{B}$ and $\alpha \in \mathfrak{A}$, we find

$$\begin{aligned}\langle \widetilde{D}(\alpha \cdot b), x\rangle &= \lim_j \langle D(\alpha \cdot be_j) - \alpha \cdot b \cdot D(e_j), x\rangle \\ &= \lim_j \langle D(be_j) - b \cdot D(e_j), x \cdot \alpha\rangle \\ &= \lim_j \langle \alpha \cdot (D(be_j) - b \cdot D(e_j)), x\rangle = \langle \alpha \cdot \widetilde{D}(b), x\rangle.\end{aligned}$$

The above relations show that $\widetilde{D}$ is a left $\mathfrak{A}$-module homomorphism. Similarly, $\widetilde{D}$ is a right $\mathfrak{A}$-module homomorphism. □

Theorem 3.8. *Let $\mathcal{A}$ be a commutative module amenable Banach algebra (as an $\mathfrak{A}$-bimodule), $\mathcal{I}$ be a closed ideal of $\mathcal{A}$ and an $\mathfrak{A}$-submodule of $\mathcal{A}$. Then, $\mathcal{I}$ is module amenable if and only if it has a bounded approximate identity.*

Proof. If $\mathcal{I}$ is module amenable, then it has a bounded approximate identity by [1, Proposition 2.2].

Conversely, assume that X is a Banach $\mathcal{I}$-$\mathfrak{A}$-module with compatible actions. Since $\mathcal{I}$ has a bounded approximate identity, by [1, Lemma 2.1] we may suppose that X is an $\mathcal{I}$-pseudo-unital Banach $\mathcal{I}$-$\mathfrak{A}$-module. If $D \in Z_{\mathfrak{A}}(\mathcal{I}, X^*)$, the proof of Lemma 3.7 shows that the module actions of $\mathcal{I}$ on X extend to $\mathcal{A}$ and D has an extension $\widetilde{D} \in Z_{\mathfrak{A}}(\mathcal{A}, X^*)$. Due to module amenability of $\mathcal{A}$, $\widetilde{D}$ is an inner derivation, and thus $D = \widetilde{D}|_{\mathcal{I}}$ is an inner derivation. □

Lemma 3.9. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach $\mathfrak{A}$-modules. If $\mathcal{B}$ is a right essential $\mathfrak{A}$-module, then so is $\mathcal{A}\widehat{\otimes}\mathcal{B}$.*

Proof. Suppose that $f = \Sigma_i a_i \otimes b_i \in \mathcal{A}\widehat{\otimes}\mathcal{B}$, where $a_i \in \mathcal{A}$ and $b_i \in \mathcal{B}$ for all i. Since $\mathcal{B}$ is an essential right $\mathfrak{A}$-module, we may assume that $b_i = \lim_j \left(\Sigma_j b_i^j \cdot \alpha^j\right)$ in which $b_i^j \in \mathcal{B}$ and $\alpha^j \in \mathfrak{A}$. We have

$$f = \Sigma_i a_i \otimes \left(\lim_j \left(\Sigma_j b_i^j \cdot \alpha^j\right)\right) = \lim_j \left(\Sigma_i a_i \otimes \left(\Sigma_j b_i^j \cdot \alpha^j\right)\right)$$
$$= \lim_j \left(\Sigma_j \Sigma_i a_i \otimes b_i^j \cdot \alpha^j\right).$$

The above equalities show that f belongs to the closed linear span $(\mathcal{A}\widehat{\otimes}\mathcal{B}) \cdot \mathfrak{A}$, i.e., $\mathcal{A}\widehat{\otimes}\mathcal{B}$ is a right essential $\mathfrak{A}$-module. □

Remark 3.10. If $\mathcal{A}$ and $\mathcal{B}$ are amenable such that $\mathcal{B}$ is an essential right $\mathfrak{A}$-module, then by Lemma 3.9, $\mathcal{A}\widehat{\otimes}\mathcal{B}$ is an essential right $\mathfrak{A}$-module. Hence, every $\mathfrak{A}$-module derivation on $\mathcal{A}\widehat{\otimes}\mathcal{B}$ is also a derivation. Therefore, $\mathcal{A}\widehat{\otimes}\mathcal{B}$ is module amenable.

Let $\mathfrak{A}$ be a non-unital Banach algebra. Then, the unitization of $\mathfrak{A}$ which is $\mathfrak{A}^{\#} = \mathfrak{A} \oplus \mathbb{C}$ is a unital Banach algebra which contains $\mathfrak{A}$ as a closed ideal. Suppose that $\mathcal{A}$ is a Banach algebra and a Banach $\mathfrak{A}$-bimodule with compatible actions. Thus, $\mathcal{A}$ is a Banach algebra and a Banach $\mathfrak{A}^{\#}$-bimodule with compatible actions in the obvious way, i.e., the action of $\mathfrak{A}^{\#}$ on $\mathcal{A}$ is as follows:

$$(\alpha, \lambda) \cdot a = \alpha \cdot a + \lambda a, \quad a \cdot (\alpha, \lambda) = a \cdot \alpha + \lambda a \qquad (\lambda \in \mathbb{C},\ \alpha \in \mathfrak{A},\ a \in \mathcal{A}).$$

Let $\mathcal{A}$ be a Banach algebra and a Banach $\mathfrak{A}$-bimodule with compatible actions and let $\mathcal{A}^{\sharp} = (\mathcal{A} \oplus \mathfrak{A}^{\#}, \bullet)$, where the multiplication $\bullet$ is defined through

$$(a, u) \bullet (b, v) = (ab + av + ub, uv) \qquad (a, b \in \mathcal{A},\ u, v \in \mathfrak{A}^{\#}).$$

Then, with the actions defined by

$$u \cdot (a, v) = (u \cdot a, uv), \quad (a, v) \cdot u = (a \cdot u, vu) \qquad (a \in \mathcal{A},\ u, v \in \mathfrak{A}^{\#}),$$

we see that $\mathcal{A}^{\sharp}$ is a unital Banach algebra with identity $1_{\mathcal{A}}$ and a Banach $\mathfrak{A}^{\#}$-bimodule with compatible actions.

The next theorem is proved in [22, Theorem 3.1] and will be used in the proof of Theorem 3.12.

Theorem 3.11. *Let $\mathcal{A}$ be a Banach algebra and a Banach $\mathfrak{A}$-bimodule with compatible actions. Then, the following are equivalent*:

(i) *$\mathcal{A}$ is $\mathfrak{A}^{\#}$-module amenable*;
(ii) *$\mathcal{A}^{\sharp}$ is $\mathfrak{A}^{\#}$-module amenable.*

If, in addition, $\mathcal{A}$ is a left or right essential $\mathfrak{A}$-module, then (i) *and* (ii) *are equivalent to*

(iii) *$\mathcal{A}$ is $\mathfrak{A}$-module amenable.*

Theorem 3.12. *Let $\mathcal{A}$ and $\mathcal{B}$ be commutative Banach $\mathfrak{A}$-modules and $\mathcal{B}$ be a right essential $\mathfrak{A}$-module. If $\mathcal{A}$ and $\mathcal{B}$ are module amenable, then $\mathcal{A}\widehat{\otimes}\mathcal{B}$ is module amenable.*

Proof. Since $\mathcal{A}$ and $\mathcal{B}$ are commutative $\mathfrak{A}$-modules module amenable, they have bounded approximate identities by [1, Proposition 2.2], and so by [14, Proposition 2.9.21], $\mathcal{A}\widehat{\otimes}\mathcal{B}$ has a bounded approximate identity. Theorem 3.11 implies that $\mathcal{A}^\sharp$ and $\mathcal{B}^\sharp$ are amenable commutative $\mathfrak{A}^\#$-modules. Since $\mathcal{A}\widehat{\otimes}\mathcal{B}$ is a closed ideal in $\mathcal{A}^\sharp\widehat{\otimes}\mathcal{B}^\sharp$, by Theorem 3.8, we may suppose that the Banach algebras $\mathcal{A}$ and $\mathcal{B}$ are unital and Banach $\mathfrak{A}^\#$-modules. Assume that X is a commutative $\mathcal{A}\widehat{\otimes}\mathcal{B}$-$\mathfrak{A}$-module with compatible actions and $D : \mathcal{A}\widehat{\otimes}\mathcal{B} \longrightarrow X^*$ is module derivation. Hence, the restriction of D to $1_{\mathcal{A}}\widehat{\otimes}\mathcal{B}$ is also a module derivation. Thus, there exists $x^* \in X^*$ such that $D|_{(1_{\mathcal{A}}\widehat{\otimes}\mathcal{B})} = D_{x^*}$. Replacing D by $D - D_{x^*}$, we may suppose that $D|_{(1_{\mathcal{A}}\widehat{\otimes}\mathcal{B})} = 0$. Let now Y be the closed linear span

$$\{x \cdot (1_{\mathcal{A}} \otimes b) - (1_{\mathcal{A}} \otimes b) \cdot x : b \in \mathcal{B}, x \in X\}$$

in X. Then, Y is a $\mathfrak{A}$-submodule of X, since for each $b \in \mathcal{B}$ and $\alpha \in \mathfrak{A}$,

$$\alpha \cdot (x \cdot (1_{\mathcal{A}} \otimes b) - (1_{\mathcal{A}} \otimes b) \cdot x) = (\alpha \cdot x) \cdot (1_{\mathcal{A}} \otimes b) - (1_{\mathcal{A}} \otimes b) \cdot (\alpha \cdot x) \in Y.$$

Besides, for each $a \in \mathcal{A}, b \in \mathcal{B}$, and $x \in X$,

$$\begin{aligned}(a \otimes 1_{\mathcal{B}}) \cdot (x \cdot (1_{\mathcal{A}} \otimes b) - (1_{\mathcal{A}} \otimes b) \cdot x) &= ((a \otimes 1_{\mathcal{B}}) \cdot x) \cdot (1_{\mathcal{A}} \otimes b) - (a \otimes b) \cdot x \\ &= ((a \otimes 1_{\mathcal{B}}) \cdot x) \cdot (1_{\mathcal{A}} \otimes b) \\ &\quad - (1_{\mathcal{A}} \otimes b) \cdot ((a \otimes 1_{\mathcal{B}}) \cdot x) \in Y.\end{aligned}$$

The above equalities show that Y is a left $(\mathcal{A}\widehat{\otimes}1_{\mathcal{B}})$-module. Similarly it is also a right $(\mathcal{A}\widehat{\otimes}1_{\mathcal{B}})$-module. Since $a \otimes b = (a \otimes 1_{\mathcal{B}})(1_{\mathcal{A}} \otimes b)$, we have

$$D(a \otimes b) = D(a \otimes 1_{\mathcal{B}}) \cdot (1_{\mathcal{A}} \otimes b) = (1_{\mathcal{A}} \otimes b) \cdot D(a \otimes b).$$

Hence, $\langle D(a \otimes 1_{\mathcal{B}}), x \cdot (1_{\mathcal{A}} \otimes b) - (1_{\mathcal{A}} \otimes b) \cdot x\rangle = 0$, and thus X/Y is a commutative Banach $(\mathcal{A}\widehat{\otimes}1_{\mathcal{B}})$-$\mathfrak{A}$-module and $D|_{(\mathcal{A}\widehat{\otimes}1_{\mathcal{B}})} : \mathcal{A}\widehat{\otimes}1_{\mathcal{B}} \longrightarrow (X/Y)^* = Y^\perp$ is a module derivation. Due to the module amenability of $\mathcal{A}$, there exists $y^* \in Y^\perp$ such that $D|_{(\mathcal{A}\widehat{\otimes}1_{\mathcal{B}})} = D_{y^*}$. Replacing D by $D - D_{y^*}$, we may assume that restriction of D to $\mathcal{A}\widehat{\otimes}1_{\mathcal{B}}$ is zero. Now, the original derivation is inner. Since $\mathcal{B}$ is a right essential $\mathfrak{A}$-module, the result follows from Theorem 3.11. □

Next, we have the next corollary which is a direct consequence of Corollary 3.3 and Theorem 3.12.

Corollary 3.13. *Let $\mathcal{A}$ be a commutative amenable Banach right essential $\mathfrak{A}$-module. Then, the projective tensor product $\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A}$ is module amenable.*

Let S be a (discrete) inverse semigroup with the set of idempotents E. We consider the natural partial order on E as $e \leq d \Leftrightarrow ed = e$ for all $e, d \in E$. The subsemigroup E of S is a semilattice [15, Theorem V.1.2], and so $l^1(E)$ could be regarded as a commutative subalgebra of $l^1(S)$. Thus, $l^1(S)$ is a Banach algebra and a Banach $l^1(E)$-module with compatible actions [1]. In [6], the second author showed that if S is an amenable inverse semigroup with the set of idempotents E, then the projective module tensor product $l^1(S)\widehat{\otimes}_{l^1(E)}l^1(S)$

is module amenable and so the commutativity condition for $l^1(S)$ in Corollary 3.13 is redundant.

Let X be a normed space and $A \subseteq X$. The *convex hull* of A denoted by $co(A)$, is the intersection of all convex sets that contain A. If X is a topological vector space, then the closed convex hull of A is the intersection of all closed convex subsets of X that contain A and is denoted by $\overline{co}(A)$. Now, assume that X is a locally convex space and K is a compact convex subset of X. A continuous map $T : K \longrightarrow K$ is called *affine* if $T(\lambda k_1 + (1-\lambda)k_2) = \lambda T(k_1) + (1-\lambda)T(k_2)$ for all $k_1, k_2 \in K$ and $\lambda \in [0, 1]$.

Theorem 3.14 (Ryll-Nardzewski Theorem)**.** *Let X be a locally convex space and K be a convex, weakly compact subset of X. Let Σ be a semigroup of weak continuous, affine maps from K to K such that Σ is distal in the sense that whenever $\xi, \eta \in K$ with $\xi \neq \eta$, then $0 \notin \{(T\xi - T\eta) : T \in \Sigma\}^-$. Then, there exists $\xi_0 \in K$ such that $T\xi_0 = \xi_0$ for all $T \in \Sigma$.*

Remark 3.15 ([19]). The last condition in Ryll-Nardzewski fixed point theorem requires that the semigroup Σ is distal (or so called, contracting). This is known to be equivalent to the condition that for each $\xi, \eta \in K$ with $\xi \neq \eta$, there is a seminorm ρ on X (depending on ξ and η with

$$\inf\{\rho(T\xi - T\eta) : T \in \Sigma\} > 0.$$

We consider the following actions of $l^1(E)$ on $l^1(S)$ as

$$\delta_e \cdot \delta_s = \delta_s, \ \delta_s \cdot \delta_e = \delta_{se} = \delta_s * \delta_e \qquad (s \in S, e \in E).$$

Let ϕ be the augmentation character on $l^1(E)$, that is, $\phi(\delta_e) = 1$ for each $e \in E$. It is shown in [22] that the action $l^1(E)$ over $l^1(S)$ from left is trivial. In the upcoming theorem, we assume that $l^1(E)$ acts trivially from the left on $l^1(S)$ (the above action) and act on the Banach space X trivially from both sides, that is $\delta_e \cdot x = x \cdot \delta_e = x$ and extended by continuity and linearity for all $e \in E$ and $x \in X$. Moreover, we assume that $l^1(S)$ is acting on X with arbitrary actions.

Theorem 3.16. *Let S be an inverse semigroup with the set of idempotents E. Then every $l^1(E)$-module derivation from $l^1(S)$ into an $l^1(S)$-$l^1(E)$-module X whose underling Banach space is reflexive, is inner.*

Proof. Suppose that $D : l^1(S) \longrightarrow X$ is a $l^1(E)$-module derivation. Set $K = \overline{co}\{\delta_{s^*} \cdot D(\delta_s) : s \in S\}^{w^*} = \overline{co}\{\delta_{s^*} \cdot D(\delta_s) : s \in S\}^{w}$. It is easy to check that K is a weak-compact convex set. For each $s \in S$, define the map $T_s : K \longrightarrow K$ via

$$T_s(\phi) = \delta_{s^*} \cdot D(\delta_s) + \delta_{s^*} \cdot \phi \cdot \delta_s, \quad (\phi \in K).$$

Consider the set $\Sigma = \{T_s : s \in S\}$ having as binary operation the composition of functions. For each $s, t \in S$, and $\phi \in K$, we have

$$T_{st}(\phi) = \delta_{t^*s^*} \cdot D(\delta_{st}) + \delta_{t^*s^*} \cdot \phi \cdot \delta_{st}$$

$$
\begin{aligned}
&= \delta_{t^*s^*} \cdot D(\delta_s * \delta_t) + \delta_{t^*s^*} \cdot \phi \cdot \delta_{st} \\
&= (\delta_{t^*} * \delta_{s^*}) \cdot [\delta_s \cdot D(\delta_t) + D(\delta_s) \cdot \delta_t] + \delta_{t^*s^*} \cdot \phi \cdot \delta_{st} \\
&= \delta_{t^*} \cdot (\delta_{s^*s} \cdot D(\delta_t)) + \delta_{t^*} \cdot [\delta_{s^*} \cdot D(\delta_s)) + \delta_{s^*} \cdot \phi \cdot \delta_s] \cdot \delta_t. \\
&= \delta_{t^*} \cdot D(\delta_t) + \delta_{t^*} \cdot (T_s(\phi)) \cdot \delta_t = T_t(T_s(\phi)) = (T_t \circ T_s)(\phi).
\end{aligned}
$$

The above equalities show that Σ is a semigroup and indeed $T_s(K) \subseteq K$ for any $s \in S$. Now, suppose that $\lambda \in [0,1]$. Then, for each $s \in S$ and $\phi_1, \phi_2 \in \Sigma$, we get

$$
\begin{aligned}
&T_s(\lambda\phi_1 + (1-\lambda)\phi_2) \\
&= \delta_{s^*} \cdot D(\delta_s) + \delta_{s^*} \cdot (\lambda\phi_1 + (1-\lambda)\phi_2) \cdot \delta_s \\
&= \lambda(\delta_{s^*} \cdot D(\delta_s) + \delta_{s^*} \cdot \phi \cdot \delta_s) + (1-\lambda)(\delta_{s^*} \cdot D(\delta_s) + \delta_{s^*} \cdot \phi \cdot \delta_s) \\
&= \lambda T_s(\phi_1) + (1-\lambda) T_s(\phi_2).
\end{aligned}
$$

Thus, all of elements Σ are affine maps. Take a net $(\phi_j) \subseteq K$ such that $\phi_j \to \phi$, in the weak topology. For $x^* \in X^*$ and $s \in S$,

$$\langle s \cdot x^*, \phi \rangle = \langle x^*, \phi \cdot \delta_s \rangle \quad (\phi \in X)$$

defines a continuous linear functional $s \cdot x^* \in X^*$. Similarly one could define $x^* \cdot s \in X^*$. We have

$$\langle x^*, \delta_{s^*} \cdot \phi_j \cdot \delta_s - \delta_{s^*} \cdot \phi \cdot \delta_s \rangle = \langle x^*, \delta_{s^*} \cdot (\phi_j - \phi) \cdot \delta_s \rangle = \langle s \cdot x^* \cdot s^*, \phi_j - \phi \rangle \to 0$$

for each $x^* \in X^*$, and thus $T_s(\phi_j) \longrightarrow T_s(\phi)$, in the weak topology, for all $s \in S$. Hence, each element of Σ is weakly-continuous. Note that in any normed space, the norm closure and the weak closure of a convex set are the same. Furthermore, if $\phi_1, \phi_2 \in K$ such that $\phi_1 \neq \phi_2$, we find

$$
\begin{aligned}
\|T_s(\phi_1) - T_s(\phi_2)\| &= \|\delta_{s^*} \cdot (\phi_1 - \phi_2) \cdot \delta_s\| \\
&\geq \|\delta_s \cdot \big(\delta_{s^*} \cdot (\phi_1 - \phi_2) \cdot \delta_s\big) \cdot \delta_{s^*}\| \\
&= \|\delta_{ss^*} \cdot (\phi_1 - \phi_2) \cdot \delta_{ss^*}\| \\
&= \|\phi_1 - \phi_2\|
\end{aligned}
$$

for all $s \in S$. Hence, the semigroup Σ is distal by Remark 3.15. By Theorem 3.14, there exists $\phi \in K$ such that $\delta_{s^*} \cdot D(\delta_s) + \delta_{s^*} \cdot \phi \cdot \delta_s = \phi$ for all $s \in S$. Thus, $\delta_{ss^*} \cdot D(\delta_s) + \delta_{ss^*} \cdot \phi \cdot \delta_s = \delta_s \cdot \phi$. Therefore, $D(\delta_s) = \phi \cdot \delta_s - \delta_s \cdot \phi$. $\square$

For an inverse semigroup S, the ideal $J_{l^1(S)}$ (or simply J) is the closed linear span of $\{\delta_{set} - \delta_{st} : s, t \in S, e \in E\}$. We consider an equivalence relation on S as follows:

$$s \approx t \Longleftrightarrow \delta_s - \delta_t \in J \qquad (s, t \in S).$$

In this case the quotient $S/\approx$ is a discrete group (see [3] and [20]). In fact, $S/\approx$ is homomorphic to the maximal group homomorphic image G_S [18] of S [21]. In particular, S is amenable if and only if $S/\approx \, = G_S$ is amenable [18]. As in [23, Theorem 3.3], we may observe that $l^1(S)/J \cong l^1(G_S)$. With the notations of the previous section, $l^1(G_S)$ is a commutative $l^1(E)$-bimodule with the following actions

$$\delta_e \cdot \delta_{[s]} = \delta_{[s]},\ \delta_{[s]} \cdot \delta_e = \delta_{[se]} \quad (s \in S, e \in E),$$

where $[s]$ denotes the equivalence class of s in G_S. It is shown in [1, Lemma 3.3] that the above both actions of $l^1(E)$ on $l^1(G_S)$ are trivial. We recall that for a locally compact group G, the group algebra $L^1(G)$ is reflexive if and only if G is finite. As an application of Theorem 3.16, we bring the next example to show that trivial actions on X can be induced by non-trivial actions from $l^1(E)$ on $l^1(G_S)$.

Example 3.17. Let G be a group with identity e, and let Γ be a non-empty set. Then the Brandt inverse semigroup corresponding to G and Γ, denoted by $S = \mathcal{M}(G, \Gamma)$, is the collection of all $\Gamma \times \Gamma$ matrices $(g)_{ij}$ with $g \in G$ in the $(i, j)^{\text{th}}$ place and 0 (zero) elsewhere and the $\Gamma \times \Gamma$ zero matrix 0. Multiplication in S is given by the formula

$$(g)_{ij}(h)_{kl} = \begin{cases} (gh)_{il} & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases} \qquad (g, h \in G,\ i, j, k, l \in \Gamma),$$

and $(g)^*_{ij} = (g^{-1})_{ji}$ and $0^* = 0$. The set of all idempotents is $E_S = \{(e)_{ii} : i \in \Gamma\} \bigcup \{0\}$. It is shown in [20] that G_S is the trivial group. By Theorem 3.16, every $l^1(E)$-module derivation from $l^1(S)$ into $l^1(G_S)$ is inner.

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