Commun. Korean Math. Soc. **35** (2020), No. 3, pp. 855–866
https://doi.org/10.4134/CKMS.c190403
pISSN: 1225-1763 / eISSN: 2234-3024

PROPERTIES OF k^{th}-ORDER (SLANT TOEPLITZ + SLANT HANKEL) OPERATORS ON $H^2(\mathbb{T})$

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Abstract. For two essentially bounded Lebesgue measurable functions ϕ and ξ on unit circle $\mathbb{T}$, we attempt to study properties of operators $S^k_{\mathcal{M}(\phi,\xi)} = S^k_{T_\phi} + S^k_{H_\xi}$ on $H^2(\mathbb{T})$ $(k \geq 2)$, where $S^k_{T_\phi}$ is a k^{th}-order slant Toeplitz operator with symbol ϕ and $S^k_{H_\xi}$ is a k^{th}-order slant Hankel operator with symbol ξ. The spectral properties of operators $S^k_{\mathcal{M}(\phi,\phi)}$ (or simply $S^k_{\mathcal{M}(\phi)}$) are investigated on $H^2(\mathbb{T})$. More precisely, it is proved that for $k = 2$, the Coburn's type theorem holds for $S^k_{\mathcal{M}(\phi)}$. The conditions under which operators $S^k_{\mathcal{M}(\phi)}$ commute are also explored.

1. Introduction and preliminaries

Since the beginning of nineteenth century, the theory of Toeplitz and Hankel operators is being studied extensively on several spaces like Hardy space, Bergman space, Fock space, etc. These operators have lots of applications in mathematics and mathematical physics and therefore they got a prominent place in the study of operator theory. In 1999, Basor and Ehrhardt [4, 5] studied the sum of Toeplitz and Hankel operators on the Hardy space and defined it as $\mathcal{M}(\phi) = T_\phi + H_\phi$ for functions $\phi \in L^\infty(\mathbb{T})$, where T_ϕ and H_ϕ are Toeplitz and Hankel operators, respectively, with symbol ϕ and evaluated its several properties. Later on, they investigated the connections between Fredholmness and invertibility of $\mathcal{M}(\phi)$ (see [6]). These developments [6] were also extended to the study of operators $\mathcal{M}(\phi,\xi) = T_\phi + H_\xi$ for functions $\phi, \xi \in L^\infty(\mathbb{T})$ (denote $\mathcal{M}(\phi,\phi)$ by $\mathcal{M}(\phi)$). In 1996, the notion of slant Toeplitz operators on $L^2(\mathbb{T})$ and its compression to $H^2(\mathbb{T})$ were introduced by Ho [8]. Then, Arora, Batra and Singh [2] introduced the class of slant Hankel operators on $L^2(\mathbb{T})$ and extended his ideas to define the compression of slant Hankel operators to $H^2(\mathbb{T})$.

Received November 22, 2019; Accepted January 16, 2020.
2010 *Mathematics Subject Classification.* Primary 47B35; Secondary 47B30.
Key words and phrases. k^{th}-order slant Toeplitz operator, k^{th}-order slant Hankel operator, k^{th}-order (slant Toeplitz + slant Hankel) operator, Fredholm operator.
Support of CSIR-UGC Research Grant(UGC) [Ref. No. 21/12/2014(ii) EU-V] to second author for carrying out the research work is gratefully acknowledged.

After that k^{th}-order slant Toeplitz and k^{th}-order slant Hankel operators were defined and studied on $H^2(\mathbb{T})$ (see [1,3]). These developments motivated us to study k^{th}-order (slant Toeplitz + slant Hankel) operators on $H^2(\mathbb{T})$. Algebraic as well as spectral properties of these operators are investigated. Along with this, it is shown that if $k = 2$, then the Coburn type theorem holds for k^{th}-order (slant Toeplitz + slant Hankel) operators on $H^2(\mathbb{T})$. Meanwhile, it is shown that if $S^k_{M(\phi)}$ is Fredholm, then ϕ is invertible. Also, the conditions under which the operators $S^k_{M(\phi)}$ commute have been explored, where we denote the operator $S^k_{M(\phi,\phi)}$ by simply $S^k_{M(\phi)}$ for $\phi \in \mathscr{L}^\infty(\mathbb{T})$.

Let $L^2(\mathbb{T})$ denote the space of all complex valued square-integrable Lebesgue measurable functions f on the unit circle $\mathbb{T}$ with respect to normalized Lebesgue measure $d\theta$. It forms a Hilbert space with respect to the inner product

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}d\theta$$

for all $f, g \in L^2(\mathbb{T})$. In other words, the space $L^2(\mathbb{T})$ can also be expressed as

$$L^2(\mathbb{T}) = \{f : \mathbb{T} \to \mathbb{C} : f(e^{i\theta}) = \sum_{n\in\mathbb{Z}} f_n e^{in\theta} \text{ and } \sum_{n\in\mathbb{Z}} |f_n|^2 < \infty\}.$$

The Hardy space $H^2(\mathbb{T})$ is defined as

$$H^2(\mathbb{T}) = \{f \in L^2(\mathbb{T}) : f_n = 0 \text{ for all } n < 0\}$$

and it is a closed subspace of $L^2(\mathbb{T})$. The space $L^\infty(\mathbb{T})$ is the Banach space of all essentially bounded Lebesgue measurable functions on $\mathbb{T}$. Denote by P, the orthogonal projection of $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$. For $f \in L^2(\mathbb{T})$ and for $\phi \in L^\infty(\mathbb{T})$, the operator $M_\phi : L^2(\mathbb{T}) \longrightarrow L^2(\mathbb{T})$ is the multiplication operator induced by ϕ and is defined as $M_\phi(f) = \phi f$ and $J : L^2(\mathbb{T}) \longrightarrow L^2(\mathbb{T})$ defined as $J(f(z)) = \bar{z}f(\bar{z})$ is the unitary self-adjoint operator.

The Toeplitz operator on $H^2(\mathbb{T})$ with symbol ϕ is a bounded linear operator $T_\phi : H^2(\mathbb{T}) \longrightarrow H^2(\mathbb{T})$ defined by

$$T_\phi(f) = PM_\phi(f)$$

and the Hankel operator $H_\phi : H^2(\mathbb{T}) \longrightarrow H^2(\mathbb{T})$ is defined as

$$H_\phi(\phi) = PM_\phi J(f)$$

for all $f \in H^2(\mathbb{T})$. Also, $\|T_\phi\| \le \|\phi\|_\infty$ and $\|H_\phi\| \le \|\phi\|_\infty$.

Throughout the paper we assume that $k \ge 2$. Define $W_k : L^2(\mathbb{T}) \longrightarrow L^2(\mathbb{T})$ by

$$W_k(z^{kn+p}) = \begin{cases} z^n & \text{if } p = 0 \\ 0 & \text{if } p > 0 \end{cases}$$

for all integers n, p such that $0 \le p < k$. One can refer [1] to find that W_k is a bounded linear operator with $\|W_k\| = 1$ and the adjoint of W_k is given by $W_k^*(z^n) = z^{kn}$ for all integers n.

The k^{th}-order slant Toeplitz operator $S^k_{T_\phi}$ [1] and k^{th}-order slant Hankel operator $S^k_{H_\phi}$ [3] on $H^2(\mathbb{T})$ with symbol ϕ are defined as

$$S^k_{T_\phi} = W_k T_\phi \qquad \text{and} \qquad S^k_{H_\phi} = W_k H_\phi.$$

Let $B(X, Y)$ be the set of all bounded linear operators from a complex Banach space X to a complex Banach space Y (if $X = Y$, then denote it by $B(X)$). An operator $T \in B(X, Y)$ is Fredholm operator if Range T is closed, dimensions of Kernel T and cokernel T are finite. In this case, index of T is defined as

$$\text{ind } T = \dim(\text{Kernel } T) - \dim(\text{cokernel } T).$$

2. Generalized (slant Toeplitz + slant Hankel) operators on $H^2(\mathbb{T})$

In this section, the k^{th}-order (slant Toeplitz + slant Hankel) operators on $H^2(\mathbb{T})$ are defined and their basic properties are studied.

Definition. Let $\phi, \xi \in L^\infty(\mathbb{T})$. For all $f \in H^2(\mathbb{T})$, a linear operator $S^k_{\mathcal{M}(\phi,\xi)}$: $H^2(\mathbb{T}) \longrightarrow H^2(\mathbb{T})$ defined by

$$S^k_{\mathcal{M}(\phi,\xi)}(f) = S^k_{T_\phi}(f) + S^k_{H_\xi}(f) = W_k P(M_\phi + M_\xi J)(f)$$

is said to be k^{th}-order (slant Toeplitz + slant Hankel) on $H^2(\mathbb{T})$. Evidently the operator $S^k_{\mathcal{M}(\phi,\xi)}$ is bounded and

$$\|S^k_{\mathcal{M}(\phi,\xi)}\| = \|S^k_{T_\phi} + S^k_{H_\xi}\| \leq \|\phi\|_\infty + \|\xi\|_\infty.$$

Since T_ϕ and H_ϕ can be represented as $T_\phi = PM_\phi P$ and $H_\phi = PM_\phi JP$, therefore, we shall use the notation $S^k_{\mathcal{M}(\phi,\xi)}(f) = W_k P(M_\phi + M_\xi J)P$ throughout the paper. If $\phi = \xi$, then it is denoted by $S^k_{\mathcal{M}(\phi)}$ simply. Also, for $k = 2$ the operator $S^k_{\mathcal{M}(\phi,\xi)}$ is denoted by $S_{\mathcal{M}(\phi,\xi)}$.

Define $\widehat{g}(z) = g(\bar{z})$ and $\overline{g}(z) = \overline{g(z)}$ for all functions g. In [4], it is proved that

$$\mathcal{M}(\phi\xi) = \mathcal{M}(\phi)\mathcal{M}(\xi) + H_\phi \mathcal{M}(\widehat{\xi} - \xi). \tag{1}$$

Now applying W_k on both sides of the Eq. (1), we get

$$S^k_{\mathcal{M}(\phi\xi)} = S^k_{\mathcal{M}(\phi)}\mathcal{M}(\xi) + S^k_{H_\phi}\mathcal{M}(\widehat{\xi} - \xi). \tag{2}$$

Let $a_{m,n}$ denote $(m, n)^{th}_{m,n\geq 0}$ entry of the matrix representation of $S^k_{\mathcal{M}(\phi,\xi)}$ with respect to the standard orthonormal basis $\{e_n = z^n\}^\infty_{n=0}$ of $H^2(\mathbb{T})$. Let $\phi(z) = \sum_{n\in\mathbb{Z}} \phi_n z^n$ and $\xi(z) = \sum_{n\in\mathbb{Z}} \xi_n z^n$ be functions in $L^\infty(\mathbb{T})$, then for all non-negative integers m, n, we have $\left\langle S^k_{T_\phi} z^n, z^m \right\rangle = \phi_{km-n}$ and $\left\langle S^k_{H_\xi} z^n, z^m \right\rangle = \xi_{km+n+1}$. Therefore,

$$\begin{aligned} a_{m,n} = \left\langle S^k_{\mathcal{M}(\phi,\xi)} z^n, z^m \right\rangle &= \left\langle S^k_{T_\phi} z^n, z^m \right\rangle + \left\langle S^k_{H_\xi} z^n, z^m \right\rangle \\ &= \phi_{km-n} + \xi_{km+n+1} \end{aligned}$$

and hence the matrix of k^{th}-order (slant Toeplitz + slant Hankel) operator on $H^2(\mathbb{T})$ is given by

$$(3)\quad [S^k_{\mathcal{M}(\phi,\xi)}] = \begin{bmatrix} \phi_0+\xi_1 & \phi_{-1}+\xi_2 & \phi_{-2}+\xi_3 & \cdots \\ \phi_k+\xi_{k+1} & \phi_{k-1}+\xi_{k+2} & \phi_{k-2}+\xi_{k+3} & \cdots \\ \phi_{2k}+\xi_{2k+1} & \phi_{2k-1}+\xi_{2k+2} & \phi_{2k-2}+\xi_{2k+3} & \cdots \\ \phi_{3k}+\xi_{3k+1} & \phi_{3k-1}+\xi_{3k+2} & \phi_{3k-2}+\xi_{3k+3} & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{bmatrix}.$$

Proposition 2.1. *Let $\phi, \xi \in L^\infty(\mathbb{T})$. Then the following hold*:

(1) $\|\phi\|_\infty \leq \|S^k_{\mathcal{M}(\phi)}\| \leq \sqrt{2}\|\phi\|_\infty$.

(2) *The mapping $\Gamma : L^\infty(\mathbb{T}) \to B(H^2(\mathbb{T}))$ defined by $\Gamma(\phi) = S^k_{\mathcal{M}(\phi)}$ is linear and one-one.*

(3) *$a_{m,n} + a_{m+2,n} = a_{m+1,n-k} + a_{m+1,n+k}$ for all integers $m \geq 0, n \geq k$, where $a_{m,n}$ is the $(m,n)^{th}_{m,n\geq 0}$ entry of the matrix representation of $S^k_{\mathcal{M}(\phi,\xi)}$ with respect to basis $\{e_n\}_{n=0}^\infty$ of $H^2(\mathbb{T})$.*

Proof. (1) By ([4], Proposition 2.1), we have $\|\phi\|_\infty \leq \|\mathcal{M}(\phi)\| \leq \sqrt{2}\|\phi\|_\infty$. Since $\|W_k\| = 1$, therefore, $\|S^k_{\mathcal{M}(\phi)}\| \leq \sqrt{2}\|\phi\|_\infty$. Let $U_n = M_{z^n}$ on $L^2(\mathbb{T})$. Then

$$\begin{aligned} U_{-n}S_{\mathcal{M}^k(\phi)}U_n &= U_{-n}W_kPM_\phi PU_n + U_{-n}W_kPM_\phi JPU_n \\ &= (U_{-n}W_kU_n)(U_{-n}PU_n)(U_{-n}M_\phi U_n)(U_{-n}PU_n) \\ &\quad + (U_{-n}W_kU_n)(U_{-n}PU_{-n})(U_nM_\phi JU_n)(U_{-n}PU_n). \end{aligned}$$

Since $U_{-n}M_\phi U_n = M_\phi$, $U_nM_\phi JU_n = M_\phi J$, $U_{-n}PU_n \to I$ and $U_{-n}PU_{-n} \to 0$ strongly on $L^2(\mathbb{T})$ as $n \to \infty$, therefore we have

$$U_{-n}S^k_{\mathcal{M}(\phi)}U_n \to U_{-n}W_kU_nM_\phi = U_{-n}W_kM_\phi U_n$$

strongly as $n \to \infty$. Since $U_{\pm n}$ are isometries on $L^2(\mathbb{T})$, therefore, $\|S^k_{\mathcal{M}(\phi)}\| \geq \|\phi\|_\infty$.

(2) From the linearity of Γ and by part (1), the result follows.

(3) From matrix 3, it follows that $a_{m,n} + a_{m+2,n} = a_{m+1,n-k} + a_{m+1,n+k}$ for all integers $m \geq 0, n \geq k$. $\square$

3. Fredholm properties of k^{th}-order (slant Toeplitz + slant Hankel) operators

This section is devoted to the study of spectral properties of k^{th}-order (slant Toeplitz + slant Hankel) operators on $H^2(\mathbb{T})$. In the following theorem, we show that a non-invertible function $\phi \in \mathscr{L}^\infty(\mathbb{T})$ cannot induce a Fredhom k^{th}-order (slant Toeplitz + slant Hankel) operator. As a consequence of this, it follows that if $S^k_{\mathcal{M}(\phi)}$ is an invertible operator, then ϕ is invertible.

Theorem 3.1. *Let $\phi \in L^\infty(\mathbb{T})$. If the operator $S^k_{\mathcal{M}(\phi)}$ is a Fredholm operator, then ϕ is invertible.*

Proof. Let $S^k_{\mathcal{M}(\phi)}$ be a Fredholm operator, therefore, there exist $\epsilon > 0$ and a finite rank operator R on $KerS^k_{\mathcal{M}(\phi)}$ such that

$$\|S^k_{\mathcal{M}(\phi)}f\| + \|Rf\| \geq \epsilon\|f\|$$

for all $f \in H^2(\mathbb{T})$. For all $f \in L^2(\mathbb{T})$, we obtain that

$$\begin{aligned}\epsilon\|f\| &= \epsilon\|(I-P)f\| + \epsilon\|Pf\| \\ &\leq \epsilon\|(I-P)f\| + \|S^k_{\mathcal{M}(\phi)}Pf\| + \|PRPf\| \\ &= \|S^k_{\mathcal{M}(\phi)}f\| + \|PRPf\| + \epsilon\|(I-P)f\|.\end{aligned}$$

Now replacing f by U_nf and using the fact that $\|U_{\pm n}\| = 1$ for all integers n, we get that

$$\begin{aligned}\epsilon\|f\| &= \epsilon\|U_nf\| \\ &\leq \|S^k_{\mathcal{M}(\phi)}U_nf\| + \|PRPU_nf\| + \epsilon\|(I-P)U_nf\| \\ &= \|U_{-n}S^k_{\mathcal{M}(\phi)}U_nf\| + \|PRPU_nf\| + \epsilon\|U_{-n}(I-P)U_nf\|.\end{aligned}$$

Since $U_n \to 0$ weakly as $n \to 0$ and PRP is a compact operator, therefore, $PRPU_n \to 0$ strongly as $n \to 0$. Also, $U_{-n}PU_n \to I$ strongly as $n \to 0$, $\|W_k\| = 1$ and $\|U_{\pm n}\| = 1$. Hence, $\epsilon\|f\| \leq \|U_{-n}S^k_{\mathcal{M}(\phi)}U_nf\|$ but proceeding in the similar manner as in the proof of Proposition 2.1(1), we obtain that $U_{-n}S^k_{\mathcal{M}(\phi)}U_n \to U_{-n}W_kM_\phi U_n$ strongly on $L^2(\mathbb{T})$ as $n \to \infty$. Therefore, it follows that $\epsilon\|f\| \leq \|M_\phi f\|$. Hence, ϕ is invertible in $L^\infty(\mathbb{T})$. □

Let

$$H^2{}_c(\mathbb{T}) = \left\{\sum_{n\in\mathbb{Z}} a_ne^{in\theta} \in L^2(\mathbb{T}) \,|\, a_n = 0 \text{ for all integers } n \geq 0\right\}$$

and

$$\widehat{H^2}(\mathbb{T}) = \left\{\sum_{n\in\mathbb{Z}} a_ne^{in\theta} \in L^2(\mathbb{T}) \,|\, a_n = 0 \text{ for all integers } n > 0\right\}.$$

For $\phi \in L^\infty(\mathbb{T})$, define the set $A_\phi = \{z \in \mathbb{T} \,|\, \phi(z) = 0 = \widehat{\phi}(z)\}$ then the characteristic function χ_{A_ϕ} is real and $\chi_{A_\phi} = \widehat{\chi_{A_\phi}}$ on $\mathbb{T}$. Therefore, from Eq. (1), it follows that $0 = \mathcal{M}(\phi\chi_{A_\phi}) = \mathcal{M}(\phi)\mathcal{M}(\chi_{A_\phi})$ and hence, $\mathrm{Im}\mathcal{M}(\chi_{A_\phi}) \subseteq \mathrm{Ker}\mathcal{M}(\phi)$.

Theorem 3.2. *Let $\phi \in L^\infty(\mathbb{T})$. Then either $\mathrm{Ker}\mathcal{M}(\phi) = \mathrm{Im}\mathcal{M}(\chi_{A_\phi})$ or ${S^k_{\mathcal{M}(\phi)}}^*$ is one-one.*

Proof. Suppose $\mathrm{Ker}{S^k_{\mathcal{M}(\phi)}}^* \neq \{0\}$ then there exists a non-zero $h \in H^2(\mathbb{T})$ such that ${S^k_{\mathcal{M}(\phi)}}^*h = 0$. Let $g \in H^2(\mathbb{T})$ such that $g \in \mathrm{Ker}\mathcal{M}(\phi)$. Define the functions

$$g_1(z) = (I+J)g(z), \quad g_2(z) = M_\phi g_1(z),$$

$$h_1(z) = M_{\bar{\phi}} W_k^* h(z), \quad h_2(z) = (I+J)h_1(z).$$

Clearly $Pg_2 = 0$ and $Ph_2 = 0$ giving $g_2, h_2 \in H_c^2(\mathbb{T})$. Using $J^2 = I$, we have $Jg_1 = g_1$ and $Jh_2 = h_2$ but $h_2 \in H_c^2(\mathbb{T})$, therefore, $h_2 = 0$. This gives $Jh_1 = -h_1$ which means $g_1(z) = \bar{z}g_1(\bar{z})$ and $-h_1(z) = \bar{z}h_1(\bar{z})$. From this it follows that

$$g_1\overline{h_1}(\bar{z}) = -g_1\overline{h_1}(z) \tag{4}$$

but $g_1\overline{h_1}(z) = g_1\phi\overline{W_k^* h(z)} = g_2 W_k^* \overline{h}(z)$ where $h \in H^2(\mathbb{T})$ implies $W_k^*\overline{h} \in \widehat{H^2}(\mathbb{T})$. Since $g_2 \in H_c^2(\mathbb{T})$, therefore, $g_2 W_k^* \overline{h} \in H_c^1(\mathbb{T})$ and hence by Eq. (4) it follows that $g_2 W_k^* \overline{h} = 0$, but $h \in H^2(\mathbb{T})$ and $h \neq 0$ implies $W_k^* h \neq 0$, therefore, by F. and M. Riesz Theorem [9], we have $W_k^* h \neq 0$ a.e on $\mathbb{T}$. This gives $g_2 = 0$ implies $\phi g_1 = 0$. Now $\widehat{\phi}(z)g_1(z) = \bar{z}\phi(\bar{z})g_1(\bar{z}) = 0$. Therefore, $(1-\chi_{A_\phi})g_1(z) = 0$ implies $M\chi_{A_\phi} g = g$ and hence, $g \in \mathrm{Im}\mathcal{M}(\chi_{A_\phi})$. □

As a consequence of Theorem 3.2 and using the fact that if $\mathcal{M}(\phi)$ is Fredholm, then ϕ is invertible in $L^\infty(\mathbb{T})$ for $\phi \in L^\infty(\mathbb{T})$ [4], we obtain the following:

Corollary 3.3. *Let $\phi \in L^\infty(\mathbb{T})$. If $\mathcal{M}(\phi)$ is Fredholm, then either $\mathcal{M}(\phi)$ is one-one or ${S^k_{\mathcal{M}(\phi)}}^*$ is one-one.*

The well known Coburn's theorem states that for a non-zero Toeplitz operator T_ϕ on $H^2(\mathbb{T})$, either kernel of T_ϕ is $\{0\}$ or kernel of T_ϕ^* is $\{0\}$. One can refer to Chapter 7 of [7] for Coburn' theorem for Toeplitz operators and related results. The following result shows that under some conditions on function ϕ, the Coburn type theorem also holds for $S_{\mathcal{M}(\phi)}$.

Theorem 3.4. (Coburn Type Theorem for $S_{\mathcal{M}(\phi)}$) *If $\phi \in L^\infty(\mathbb{T})$ is such that $\phi(z) = \xi(z^2)$ for some $\xi \in L^\infty(\mathbb{T})$ and ϕ is invertible, then either $S_{\mathcal{M}(\phi)}$ is one-one or ${S_{\mathcal{M}(\phi)}}^*$ is one-one.*

Proof. If $\mathrm{Ker}{S_{\mathcal{M}(\phi)}}^* \neq \{0\}$, then there exists a non-zero $h \in H^2(\mathbb{T})$ such that ${S_{\mathcal{M}(\phi)}}^* h = 0$. Using the same argument as in Theorem 3.2 for $k = 2$, we get

$$g_1\overline{h_1}(\bar{z}) = -g_1\overline{h_1}(z) \tag{5}$$

but $g_1\overline{h_1}(z) = g_1\phi\overline{W_2^* h(z)} = g_1\phi W_2^*\overline{h}(z)$. Now applying W_2 on both sides of Eq. (5), we obtain that

$$W_2 g_1\overline{h_1}(\bar{z}) = -W_2 g_1\overline{h_1}(z). \tag{6}$$

Let P_2 denote the projection of $L^2(\mathbb{T})$ onto the closed linear span of $\{z^{2n} \mid n \in \mathbb{Z}\}$. Since $\phi \in L^\infty(\mathbb{T})$ and by reframing a property of W_2 [8] which states that if either f or g is in $L^\infty(\mathbb{T})$, then $W_2(fg) = W_2(f)W_2(g) + zW_2(\bar{z}f)W_2(\bar{z}g) = W_2(f)W_2(g) + W_2((I-P_2)f.(I-P_2)g)$ [8], we get

$$W_2 g_1\overline{h_1} = W_2(g_1\phi W_2^*\overline{h}) = W_2(g_1\phi)W_2(\overline{W_2^* h}) + W_2[(I-P_k)g_1\phi.(I-P_2)W_2^*\overline{h}].$$

By using the definition of W_2^*, we have $(I-P_2)W_2^*\overline{h} = 0$, therefore we obtain that $W_2 g_1\bar{h_1}(z) = W_2(g_1\phi)W_2(W_2^*\overline{h})(z) = W_2(g_1\phi)\overline{h})(z) = g_2(z)\overline{h}(z)$ and hence from Eq. (6) it follows that $g_2\overline{h}(\bar{z}) = -g_2\overline{h}(z)$ but $\overline{h} \in \widehat{H^2}(\mathbb{T})$ and $g_2 \in H^2_c(\mathbb{T})$, therefore, $g_2\overline{h} \in H^1_c(\mathbb{T})$ and hence, $g_2\overline{h} = 0$ but $h \in H^2(\mathbb{T})$ and $h \neq 0$, therefore, by F. and M. Riesz Theorem [9], we have $h \neq 0$ a.e on $\mathbb{T}$. This gives $g_2 = 0$ which implies $W_2(\phi g_1) = 0$. Since $W_2(\phi g_1) = W_2(\phi)W_2(g_1) + W_2[(I-P_2)\phi.(I-P_2)g_1]$, therefore,

$$W_2(\phi)W_2(g_1) = 0 \tag{7}$$

as $\phi_{2n+1} = 0$ for all integers n, where ϕ_{2n+1} is the Fourier coefficient of ϕ with respect to z^{2n+1} and this implies $(I-P_2)\phi = 0$. On applying W_2^* on both sides of Eq. (7), we get $0 = W_2^*(W_2(\phi)W_2(g_1)) = W_2^*W_2(\phi)W_2^*W_2(g_1) = \phi W_2^*W_2(g_1)$ but it is assumed that ϕ is invertible in $L^\infty(\mathbb{T})$. Hence, $W_2^*W_2(g_1) = 0$ gives $g = 0$. □

4. Commutativity of k^{th}-order (slant Toeplitz + slant Hankel) operators

In this section, we are dealing with the commutative property of k^{th}-order (slant Toeplitz + slant Hankel) operators and we show that these types of operators commute if and only if their symbol functions are scalar multiple of each other.

Theorem 4.1. *Let $\phi(z) = \sum_{j=-m}^{n} a_j z^j$ and $\xi(z) = \sum_{j=-m}^{n} b_j z^j$ be such that $\phi, \xi \in L^\infty(\mathbb{T})$, where n and m are non-negative integers and $a_n, b_n, a_{-m}, b_{-m} \neq 0$. Then, $S^k_{\mathcal{M}(\phi)}$ and $S^k_{\mathcal{M}(\xi)}$ commute if and only if ϕ and ξ are scalar multiple of each other.*

Proof. If $\xi = \lambda\phi$ for some scalar λ, then it is obvious that $S^k_{\mathcal{M}(\phi)}$ and $S^k_{\mathcal{M}(\xi)}$ commute. Conversely, suppose that $S^k_{\mathcal{M}(\phi)}$ and $S^k_{\mathcal{M}(\xi)}$ commute. Therefore,

$${S^k_{\mathcal{M}(\phi)}}^* {S^k_{\mathcal{M}(\xi)}}^* (z^{n+m}) = {S^k_{\mathcal{M}(\xi)}}^* {S^k_{\mathcal{M}(\phi)}}^* (z^{n+m}). \tag{8}$$

Now

$$\begin{aligned}
&{S^k_{\mathcal{M}(\phi)}}^* {S^k_{\mathcal{M}(\xi)}}^* (z^{n+m}) \\
&= P(I+J)L_{\overline{\phi}}PW_k^*P(I+J)L_{\overline{\xi}}PW_k^*(z^{n+m}) \\
&= P(I+J)L_{\overline{\phi}}PW_k^*P(I+J)\left(\sum_{j=-m}^{n} \overline{b}_j \bar{z}^j z^{kn+km}\right) \\
&= P(I+J)L_{\overline{\phi}}PW_k^*P(I+J)\left(\sum_{j=(k-1)n+km}^{kn+(k+1)m} \overline{b}_{kn+km-j} z^j\right) \\
&= P(I+J)L_{\overline{\phi}}PW_k^*P\left(\sum_{j=(k-1)n+km}^{kn+(k+1)m} \overline{b}_{kn+km-j} z^j + \sum_{j=(k-1)n+km}^{kn+(k+1)m} \overline{b}_{kn+km-j} z^{-j-1}\right)
\end{aligned}$$

$$
\begin{aligned}
&= P(I+J)L_{\overline{\phi}}PW_k^*\left(\sum_{j=(k-1)n+km}^{kn+(k+1)m}\overline{b}_{kn+km-j}z^j\right)\\
&= P(I+J)L_{\overline{\phi}}\left(\sum_{j=(k-1)n+km}^{kn+(k+1)m}\overline{b}_{kn+km-j}z^{kj}\right)\\
&= P(I+J)\left(\sum_{i=-m}^{n}\overline{a}_i\overline{z}^i\sum_{j=(k-1)n+km}^{kn+(k+1)m}\overline{b}_{kn+km-j}z^{kj}\right)\\
&= \sum_{j=(k-1)n+km}^{kn+(k+1)m}\sum_{i=kj-n}^{kj+m}\overline{a}_{kj-i}\overline{b}_{kn+km-j}z^i\\
&= \sum_{i=(k^2-k-1)n+k^2m}^{k(k-1)n+(k^2+1)m}\overline{a}_{k(k-1)n+km-i}\overline{b}_n z^i\\
&\quad+\sum_{i=(k^2-k-1)n+k^2m+k}^{k(k-1)n+(k^2+1)m+k}\overline{a}_{k(k-1)n+km+k-i}\overline{b}_{n-1}z^i+\cdots\\
&\quad+\sum_{i=(k^2-1)n+k(k+1)m}^{k^2n+(k^2+k+1)m}\overline{a}_{k^2n+k(k+1)m-i}\overline{b}_{-m}z^i. \qquad (9)
\end{aligned}
$$

Similar calculations give

$$
\begin{aligned}
{S^k_{\mathcal{M}(\xi)}}^*{S^k_{\mathcal{M}(\phi)}}^*(z^{n+m}) &= \sum_{j=(k-1)n+km}^{kn+(k+1)m}\sum_{i=kj-n}^{kj+m}\overline{b}_{kj-i}\overline{a}_{kn+km-j}z^i\\
&= \sum_{i=(k^2-k-1)n+k^2m}^{k(k-1)n+(k^2+1)m}\overline{b}_{k(k-1)n+km-i}\overline{a}_n z^i\\
&\quad+\sum_{i=(k^2-k-1)n+k^2m+k}^{k(k-1)n+(k^2+1)m+k}\overline{b}_{k(k-1)n+km+k-i}\overline{a}_{n-1}z^i+\cdots\\
&\quad+\sum_{i=(k^2-1)n+k(k+1)m}^{k^2n+(k^2+k+1)m}\overline{b}_{k^2n+k(k+1)m-i}\overline{a}_{-m}z^i. \qquad (10)
\end{aligned}
$$

Since $\{z^n\}_{n=0}^{\infty}$ forms an orthonormal basis, therefore, on using Eqs. (8), (9) and (10) and on comparing the coefficients of $z^{n+4m+1}, z^{n+4m+2}, z^{n+4m+3}, \ldots, z^{4n+7m-2}, z^{4n+7m-1}$ on both sides, it follows that

$$
\begin{aligned}
a_{n-1}b_n &= b_{n-1}a_n,\\
a_{n-2}b_n + a_nb_{n-1} &= b_{n-2}a_n + b_na_{n-1},
\end{aligned}
$$

$$a_{n-3}b_n + a_{n-1}b_{n-1} = b_{n-3}a_n + b_{n-1}a_{n-1},$$
$$\vdots$$
$$a_{-m}b_{-m+1} + a_{-m+2}b_{-m} = b_{-m}a_{-m+1} + b_{-m+2}a_{-m},$$
$$a_{-m+1}b_{-m} = b_{-m+1}a_{-m}.$$

This yields $\frac{b_p}{a_p} = \lambda$ for all $-m \le p \le n-1$, where $\lambda = \frac{b_n}{a_n}$. Hence, $\xi(z) = \lambda\phi(z)$. □

Now, we present a lemma which is instrumental in proving our main result (Theorem 4.4). The following lemma is true for $k = 2$ but we obtain the same result for $k \ge 2$.

Lemma 4.2. *Let $\phi(z) = \sum_{j=-m}^{n} a_j z^j$ and $\xi(z) = \sum_{j=-s}^{r} b_j z^j$ be such that $\phi, \xi \in L^\infty(\mathbb{T})$ where n, m, r, s are non-negative integers. If $S_{\mathcal{M}(\phi)}$ and $S_{\mathcal{M}(\xi)}$ commute, then $a_j = 0$ and $b_j = 0$ for all integers $j < -\min\{m, s\}$ or $\min\{n, r\} < j$ (if any such a_j or b_j exists).*

Proof. If any of ϕ and ξ is equal to zero, then the result follows trivially. Suppose none of ϕ and ξ is zero. Since $S_{\mathcal{M}(\phi)}$ and $S_{\mathcal{M}(\xi)}$ commute, therefore,

$$\begin{aligned} &S_{\mathcal{M}(\phi)}{}^* S_{\mathcal{M}(\xi)}{}^* (z^{\max\{n,r\}+\max\{m,s\}}) \\ &= S_{\mathcal{M}(\xi)}{}^* S_{\mathcal{M}(\phi)}{}^* (z^{\max\{n,r\}+\max\{m,s\}}). \end{aligned} \tag{11}$$

Then the following four cases arise:

(1) $n \ge r$ and $m \ge s$;
(2) $n \ge r$ and $m < s$;
(3) $n < r$ and $m \ge s$;
(4) $n < r$ and $m < s$.

Case I: If $n \ge r$ and $m \ge s$, then let $n - r = x \ge 0$. Now

$$\begin{aligned} &S_{\mathcal{M}(\phi)}{}^* S_{\mathcal{M}(\xi)}{}^* (z^{\max\{n,r\}+\max\{m,s\}}) \\ &= S_{\mathcal{M}(\phi)}{}^* S_{\mathcal{M}(\xi)}{}^* (z^{n+m}) \\ &= P(I+J)L_{\overline{\phi}}PW_2^*P(I+J)L_{\overline{\xi}}PW_2^*(z^{n+m}) \\ &= P(I+J)L_{\overline{\phi}}PW_2^*P(I+J)\left(\sum_{j=-s}^{r} \overline{b}_j \overline{z}^j z^{2n+2m}\right) \\ &= P(I+J)L_{\overline{\phi}}PW_2^*P(I+J)\left(\sum_{j=2n+2m-r}^{2n+2m+s} \overline{b}_{2n+2m-j} z^j\right) \\ &= P(I+J)L_{\overline{\phi}}PW_2^*P(I+J)\left(\sum_{j=n+2m+x}^{2n+2m+s} \overline{b}_{2n+2m-j} z^j\right) \end{aligned}$$

$$
\begin{aligned}
&= P(I+J)L_{\overline{\phi}}\left(\sum_{j=n+2m+x}^{2n+2m+s}\overline{b}_{2n+2m-j}z^{2j}\right)\\
&= P(I+J)\left(\sum_{i=-m}^{n}\overline{a}_i\overline{z}^i\sum_{j=n+2m+x}^{2n+2m+s}\overline{b}_{2n+2m-j}z^{2j}\right)\\
&= \sum_{j=n+2m+x}^{2n+2m+s}\sum_{i=2j-n}^{2j+m}\overline{a}_{2j-i}\overline{b}_{2n+2m-j}z^i\\
&= \sum_{i=n+4m+2x}^{2n+5m+2x}\overline{a}_{2n+4m+2x-i}\overline{b}_r z^i\\
&\quad+\sum_{i=n+4m+2x+2}^{2n+5m+2x+2}\overline{a}_{2n+4m+2x-i+2}\overline{b}_{r-1}z^i+\cdots\\
&\quad+\sum_{i=3n+4m+2s}^{4n+5m+2s}\overline{a}_{4n+4m+2s-i}\overline{b}_{-s}z^i. \qquad (12)
\end{aligned}
$$

Consider

$$
\begin{aligned}
&S_{\mathcal{M}(\xi)}{}^*S_{\mathcal{M}(\phi)}{}^*(z^{\max\{n,r\}+\max\{m,s\}})\\
&= S_{\mathcal{M}(\xi)}{}^*S_{\mathcal{M}(\phi)}{}^*(z^{n+m})\\
&= P(I+J)L_{\overline{\xi}}PW_2^*P(I+J)L_{\overline{\phi}}PW_2^*(z^{n+m})\\
&= P(I+J)L_{\overline{\xi}}PW_2^*P(I+J)\left(\sum_{j=-m}^{n}\overline{a}_j\overline{z}^j z^{2n+2m}\right)\\
&= P(I+J)L_{\overline{\xi}}PW_2^*P(I+J)\left(\sum_{j=n+2m}^{2n+3m}\overline{a}_{2n+2m-j}z^j\right)\\
&= P(I+J)L_{\overline{\xi}}\left(\sum_{j=n+2m}^{2n+3m}\overline{a}_{2n+2m-j}z^{2j}\right)\\
&= P(I+J)\left(\sum_{i=-s}^{r}\overline{b}_i\overline{z}^i\sum_{j=n+2m}^{2n+3m}\overline{a}_{2n+2m-j}z^{2j}\right)\\
&= \sum_{j=n+2m}^{2n+3m}\sum_{i=2j-r}^{2j+s}\overline{b}_{2j-i}\overline{a}_{2n+2m-j}z^i\\
&= \sum_{i=n+4m+x}^{2n+4m+s}\overline{b}_{2n+4m-i}\overline{a}_n z^i+\sum_{i=n+4m+x+2}^{2n+4m+s+2}\overline{b}_{2n+4m-i+2}\overline{a}_{n-1}z^i+\cdots
\end{aligned}
$$

$$+\sum_{i=3n+6m+x}^{4n+6m+s} \overline{b}_{4n+6m-i}\overline{a}_{-m}z^i. \tag{13}$$

Since $\{z^n\}_{n=0}^{\infty}$ forms an orthonormal basis, therefore, by using Eqs. (11), (12) and (13), it follows that the coefficients of $z^{n+4m+x+p}$ are zeroes for all integers $0 \le p < x$ for $x > 0$. This gives

$$\begin{aligned} b_{n-x}a_n &= 0, \\ b_{n-x-1}a_n &= 0, \\ b_{n-x-2}a_n + b_{n-x}a_{n-1} &= 0, \\ b_{n-x-3}a_n + b_{n-x-1}a_{n-1} &= 0, \\ b_{n-x-4}a_n + b_{n-x-2}a_{n-1} + b_{n-x}a_{n-2} &= 0, \\ &\vdots \quad . \end{aligned} \tag{14}$$

If $a_n \neq 0$, we have $b_{n-x} = b_r = 0$ and subsequently $b_r = b_{r-1} = \cdots = b_{r-x+1} = 0$. On comparing the coefficients of $z^{n+4m+2x}, z^{n+4m+2x+1}, \ldots$ etc., it is concluded that $b_p = 0$ for all $r \le p \le -s$. It means $\xi = 0$ which is a contradiction. Hence, $a_n = 0$. Observe that $(4n + 6m + s) - (4n + 5m + 2s) = m - s \ge 0$, so from Eqs. (12) and (13), it follows that the coefficients of $z^{4n+5m+2s+1}, z^{4n+5m+2s+2}, \ldots, z^{4n+6m+s}$ are zeroes. Hence by using Eq. (14), we obtain that $a_p = 0$ for all $p > r$ or $p < -s$.

The other 3 cases follow on similar lines. □

The above lemma also holds for $k \ge 2$ and it is stated as follows:

Lemma 4.3. *Let $\phi(z) = \sum_{j=-m}^{n} a_j z^j$ and $\xi(z) = \sum_{j=-s}^{r} b_j z^j$ be such that $\phi, \xi \in L^\infty(\mathbb{T})$ where n, m, r, s are non-negative integers. If $S^k_{\mathcal{M}(\phi)}$ and $S^k_{\mathcal{M}(\xi)}$ commute, then $a_j = 0$ and $b_j = 0$ for all integers $j < -\min\{m, s\}$ or $\min\{n, r\} < j$ (if any such a_j or b_j exists).*

Theorem 4.4. *Let $\phi(z) = \sum_{j=-m}^{n} a_j z^j$ and $\xi(z) = \sum_{j=-s}^{r} b_j z^j$ be such that $\phi, \xi \in L^\infty(\mathbb{T})$ where n, m, r, s are non-negative integers. Then $S^k_{\mathcal{M}(\phi)}$ and $S^k_{\mathcal{M}(\xi)}$ commute if and only if ξ is a scalar multiple of ϕ.*

Proof. Let $\phi = \lambda\xi$ for some scalar λ then it is obvious that $S^k_{\mathcal{M}(\phi)}$ and $S^k_{\mathcal{M}(\xi)}$ commute. Conversely, suppose that $S^k_{\mathcal{M}(\phi)}$ and $S^k_{\mathcal{M}(\xi)}$ commute. Then by Lemma 4.3, it follows that $a_j = 0$ and $b_j = 0$ for all integers $j < -\min\{m, s\}$ or $\min\{n, r\} < j$ (if any such a_j or b_j exists), that is, $\phi(z) = \sum_{j=-\min\{m,s\}}^{\min\{n,r\}} a_j z^j$ and $\xi(z) = \sum_{j=-\min\{m,s\}}^{\min\{n,r\}} b_j z^j$. Hence by Theorem 4.1, we conclude the result. □

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