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BARRIER OPTIONS UNDER THE MFBM WITH JUMPS : APPLICATION OF THE BDF2 METHOD

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Abstract. In this paper we consider a mixed fractional Brownian motion (mfBm) with jumps. The prices of European barrier options can be evaluated by solving a partial integro-differential equation (PIDE) with variable coefficients, which is derived from the mfBm with jumps. The 2-step backward differentiation formula (BDF2 method) proposed in [6] is applied with the second-order convergence rate in the time and spatial variables. Numerical simulations are carried out to observe the convergence behaviors of the BDF2 method under the mfBm with the Kou model.

1. Introduction

A barrier option as one of path-dependent options is traded widely in the financial market. There are a variety of stochastic processes to model an underlying asset in order for us to price the barrier option. For example, the price of a down-and-out call option under the geometric Brownian motion was evaluated by Merton [8]. Empirical studies, however, exhibit that the Black-Scholes model is inconsistent to design the underlying asset because it cannot explain volatility smiles, heavy tails, long-range dependencies, and so on. We concentrate on a fractional Brownian motion (fBm) to incorporate these market phenomena.

The fBm B_t^H was proposed by Mandelbrot and Van Ness [7] and thereafter was modified by the following form

$$B_t^H = c_H \int_{\mathbb{R}} \left((t-s)_+^{H-1/2} - (-s)_+^{H-1/2} \right) dW_s$$

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with

$$c_H = \sqrt{\frac{2H\Gamma(\frac{3}{2}-H)}{\Gamma(\frac{1}{2}+H)\Gamma(2-2H)}},$$

where $(W_t)_{t\in\mathbb{R}}$ is a two-sided Brownian motion, H is the Hurst index with $H \in (0,1)$, and $\Gamma(\cdot)$ is the gamma function. We denote by $(x)_+^y$ the function given by

$$(x)_+^y = \begin{cases} x^y & \text{for } x \geq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

The fBm B_t^H is a self-similar process and has stationary increments. The covariance between two non-overlapping increments of B_t^H is negative for $H \in (0, \frac{1}{2})$ and is positive for $H \in (\frac{1}{2}, 1)$. The process B_t^H with $H = \frac{1}{2}$ corresponds to a two-sided Brownian motion. Moreover, the fBm B_t^H with $H \in (\frac{1}{2}, 1)$ has long-range dependence, that is, if a function $\gamma_H(n)$ is defined by the expectation

$$\gamma_H(n) = \mathbb{E}[B_1^H(B_{n+1}^H - B_n^H)],$$

then $\sum_{n=1}^{\infty} \gamma_H(n) = \infty$ for $\frac{1}{2} < H < 1$.

When the underlying asset follows the geometric fBm which reflects the stylized facts observed in the financial market, Cheridito [4] showed that there exist arbitrage strategies and how arbitrage opportunities can be eliminated under the exponential fBm. It is incompatible with the economic explanation discussed by Björk and Hult [2].

In order to conquer the shortcomings of the fBm, a mixed fractional Brownian motion (mfBm) is considered with the log return of the underlying asset. The mfBm is the linear combination of the standard Brownian motion and the fBm independently and it is proved in [3] that there are no arbitrage opportunities for $H \in (\frac{3}{4}, 1)$. The mfBm is popularly employed to price financial derivatives in the literature [1, 10, 11].

In this paper we are interested in barrier options when the underlying asset follows the mfBm with jumps. Although there are a number of numerical methods to evaluate barrier options, they have not been considered under the mfBm with jumps to our knowledge. The 2-step backward differentiation formula (BDF2 method) proposed by Lee and Lee [6] is used to solve a partial integro-differential equation (PIDE) for pricing barrier options under the mfBm with the Kou model.

The remainder of this paper is organized as follows. In section 2 we briefly introduce the mfBm with jumps and the PIDE for an up-and-out European barrier call option. In section 3 the BDF2 method is employed to price the barrier option. A variety of numerical simulations are performed to show the convergence behaviors of the BDF2 method in section 4. This paper ends with conclusions in section 5.

2. Option pricing model under the mfBm with jumps

A fractional Brownian motion $B^H = (B^H_t)_{t\in\mathbb{R}}$ with Hurst parameter $H \in (0, 1)$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a continuous and centered Gaussian process with mean zero and covariance

$$E[B^H(t)B^H(s)] = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right) \quad \text{for } s, t \in \mathbb{R}.$$

A mfBm $M^H_t(a, b)$ for $t \in \mathbb{R}$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is defined by

$$M^H_t(a, b) = aB_t + bB^H_t,$$

where B_t is a Brownian motion, B_t and B^H_t are independent, and a and b are two nonzero real constants.

We are interested in the Hurst parameter $H \in (\frac{3}{4}, 1)$ in which the mfBm is equivalent to a Brownian motion and it allows us to exclude arbitrage strategies. For more details, see [3]. In order to take into account the long-range dependence observed in the financial market, an underlying asset in a risk-neutral measure $\mathbb{Q}$ is considered to satisfy

$$dS_t/S_{t-} = (r - d - \lambda\zeta)dt + M^H_t(\sigma_1, \sigma_2) + \eta dN_t, \tag{2.1}$$

where r is a risk-free interest rate, d is a continuous dividend yield, N_t is a Poisson process with intensity λ, $\ln(\eta + 1)$ is a random variable to determine jump sizes of the log return, and ζ is the expectation of η. all random processes B_t, B^H_t, η, and N_t are assumed to be independent.

Let us consider the price of an up-and-out European barrier call option $u(\tau, x)$ that satisfies the following PIDE, which is described in [9], by using the Wick-Itô integral

$$\begin{aligned} \frac{\partial u}{\partial \tau}(\tau, x) &= \alpha(\tau)\frac{\partial^2 u}{\partial x^2}(\tau, x) + \beta(\tau)\frac{\partial u}{\partial x}(\tau, x) \\ &\quad -(r+\lambda)u(\tau, x) + \lambda\int_{\mathbb{R}} u(\tau, z)f(z - x)dz, \end{aligned} \tag{2.2}$$

$$h(x) = \max(0, S_0e^x - K) \cdot 1_{\{S_0e^x < \bar{B}\}}, \tag{2.3}$$

where $\tau = T - t$ is the time to maturity, $x = \ln(S_t/S_0)$ is the log price, K is a strike price, $\bar{B}$ is a barrier level, 1_Ω is the indicator function of

Ω, and $\alpha(\tau)$ and $\beta(\tau)$ are given by

$$\begin{aligned}\alpha(\tau) &= \left(\frac{\sigma_1^2}{2} + H\sigma_2^2(T-\tau)^{2H-1}\right),\\ \beta(\tau) &= \left(r - d - \frac{\sigma_1^2}{2} - H\sigma_2^2(T-\tau)^{2H-1} - \lambda\zeta\right).\end{aligned}$$

3. The BDF2 method for barrier option pricing

The three implicit methods in [6] are proposed for pricing options under a regime-switching jump-diffusion model where coefficients are considered as functions in the time and spatial variables. It is shown in [6] that these implicit methods have the second-order accuracy in the time and spatial variables. We note that these implicit methods can be also employed to solve the PIDE (2.2)–(2.3) because $\alpha(\tau)$ and $\beta(\tau)$ are the functions in the time variable for $H \in (\frac{3}{4}, 1)$. One of the three implicit methods called the BDF2 method is based on the 2-step backward differentiation formula. In this paper, we introduce the BDF2 method when the number of states of economy is 1. The discrete equation of the BDF2 method with $U_m^n = U(\tau_n, x_m)$ is given by

$$\frac{1}{\Delta\tau}\left(\frac{3}{2}U_m^{n+1} - 2U_m^n + \frac{1}{2}U_m^{n-1}\right) = \mathcal{L}_\Delta U_m^{n+1}, \tag{3.1}$$

where the discrete integro-differential operator $\mathcal{L}_\Delta U_m^{n+1}$ is

$$\mathcal{L}_\Delta U_m^{n+1} = \mathcal{D}_\Delta U_m^{n+1} + \mathcal{I}_\Delta(2U_m^n - U_m^{n-1}) - (r+\lambda)U_m^{n+1}, \tag{3.2}$$

the discrete differential operator $\mathcal{D}_\Delta U_m^{n+1}$ is

$$\mathcal{D}_\Delta U_m^{n+1} = \alpha^{n+1}\frac{U_{m+1}^{n+1} - 2U_m^{n+1} + U_{m-1}^{n+1}}{\Delta x^2} + \beta^{n+1}\frac{U_{m+1}^{n+1} - U_{m-1}^{n+1}}{2\Delta x}, \tag{3.3}$$

and the discrete integral operator $\mathcal{I}_\Delta U_m^n$ is

$$\begin{aligned}\mathcal{I}_\Delta U_m^n = \frac{\lambda\Delta x}{2}&\left(U_0^n f_{m,0} + 2\sum_{i=1}^{M-1} U_i^n f_{m,i} + U_M^n f_{m,M}\right)\\ &+\lambda R(\tau_n, x_m, X, Y)\end{aligned} \tag{3.4}$$

with $f_{m,i} = f(x_i - x_m)$ and $R(\tau_n, x_m, X, Y)$ which is given by

$$R(\tau_n, x_m, X, Y) = \int_{\mathbb{R}\backslash(X,Y)} h(z)f(z - x_m)dz.$$

TABLE 1. The prices of the up-and-out European barrier call option obtained by the BDF2 method under the mfBm with the Kou model. N is the number of time steps and M is the number of spatial steps.

		$S = 90$		$S = 100$		$S = 110$	
N	M	Value	Error	Value	Error	Value	Error
50	64	0.556282	-	3.612688	-	9.054693	-
100	128	0.567844	0.011562	3.688933	0.076245	9.158832	0.104139
200	256	0.570841	0.002997	3.707951	0.019017	9.185099	0.026267
400	512	0.571598	0.000757	3.712709	0.004758	9.191666	0.006568
800	1024	0.571788	0.000190	3.713899	0.001190	9.193306	0.001640
1600	2048	0.571835	0.000048	3.714197	0.000298	9.193715	0.000409
3200	4096	0.571847	0.000012	3.714272	0.000075	9.193817	0.000102

In the up-and-out European barrier call option, $R(\tau_n, x_m, X, Y)$ can be regarded as 0 because $h(z) = 0$ on the domain $\mathbb{R} \setminus (X, Y)$.

4. Numerical simulations

In this section a number of numerical simulations with MATLAB on a computer are carried out to evaluate the prices of an up-and-out European barrier call option under the mfBm with jumps. The Kou model is considered with the probability density function of $\ln(\eta + 1)$ given by

$$f(x) = p\lambda_+ e^{-\lambda_+ x} 1_{x \geq 0} + (1 - p)\lambda_- e^{\lambda_- x} 1_{x<0},$$

where $\lambda_+ > 1, \lambda_- > 0$, $0 \leq p \leq 1$. The corresponding parameters in the simulation are

$$\sigma_1 = \sigma_2 = 0.15, \quad H = 0.85, \quad r = 0.05, \quad d = 0.02,$$

$$\lambda = 0.10, \quad p = 0.3445, \quad \lambda_+ = 3.0465, \quad \lambda_- = 3.0775,$$

$$T = 0.25, \quad K = 100, \quad \bar{B} = 130,$$

in which the parameters concerned with jumps are also used by d'Halluin, Forsyth, and Vetzal [5]. The truncated domain is taken with

$$(X, Y) = (-7 \ln(\bar{B}/S_0), \ln(\bar{B}/S_0)),$$

where $S_0 = K$.

In Table 1, the errors are computed by the consecutive changes of the prices of the up-and-out European barrier call option at each stock price. We can observe that the errors with the BDF2 method are reduced

TABLE 2. The rates of ℓ^2-errors obtained by the BDF2 method under the mfBm with the Kou model. N is the number of time steps and M is the number of spatial steps. ϵ is the rate of convergence calculated by (4.1).

N	M	$\lVert U(\Delta\tau,\Delta x) - U(\Delta\tau/2,\Delta x/2)\rVert_{\ell^2}$	ϵ
50	64		
		0.039889865794569	
100	128		
		0.010001800909309	1.996
200	256		
		0.002501155699599	2.000
400	512		
		0.000625051383142	2.001
800	1024		
		0.000156155707964	2.001
1600	2048		
		0.000039001906173	2.001
3200	4096		

to about a quarter as the numbers of time and spatial grid points are doubled. The ratio ϵ in Table 2 is calculated by

$$\epsilon = \log_2 \frac{\lVert U(\Delta\tau,\Delta x) - U(\Delta\tau/2,\Delta x/2)\rVert_{\ell^2}}{\lVert U(\Delta\tau/2,\Delta x/2) - U(\Delta\tau/4,\Delta x/4)\rVert_{\ell^2}}, \tag{4.1}$$

where $U(\Delta\tau,\Delta x)$ is the option price on $\tau = T$ and the discrete ℓ^2-norm of the numerical solution U^n is given by

$$\lVert U^n\rVert_{\ell^2} = \sqrt{\Delta x \sum_{m=1}^{M-1} (U_m^n)^2}.$$

The results in Table 2 show that the BDF2 method has the second-order convergence rate in the discrete ℓ^2-norm as the rates ϵ are almost quadratic.

5. Conclusion

In this paper we deal with the mfBm with jumps to evaluate the up-and-out European barrier call option. These prices under the mfBm with the Kou model can be obtained by solving the PIDE with variable coefficients. The BDF2 method proposed in [6] is applied to solve the

PIDE (2.2)–(2.3) numerically. A variety of simulations are performed to show the second-order convergence rate in the time and spatial variables.

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