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THE CHAIN RECURRENT SET ON COMPACT TVS-CONE METRIC SPACES

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Abstract. Conley introduced attracting sets and repelling sets for a flow on a topological space and showed that if f is a flow on a compact metric space, then $\mathcal{R}(f) = \bigcap\{A_U \cup A_U^* | U$ is a trapping region for $f\}$. In this paper we introduce chain recurrent set, trapping region, attracting set and repelling set for a flow f on a TVS-cone metric space and prove that if f is a flow on a compact TVS-cone metric space, then $\mathcal{R}(f) = \bigcap\{A_U \cup A_U^* | U$ is a trapping region for $f\}$.

1. Introduction and Preliminaries

Recently Long-Guang and Xian [2] generalized the notion of metric space by replacing the set of real numbers by an ordered Banach space, defined a cone metric space. I. Beg, A. Abbas, and M. Arshad [4] introduced a topological vector space valued cone metric space(or shortly TVS-cone metric space). Conley [1] introduced attracting sets and repelling sets for a flow on a topological space and showed that if f is a flow on a compact metric space, then $\mathcal{R}(f) = \bigcap\{A_U \cup A_U^* | U$ is a trapping region for $f\}$. In this paper we introduce chain recurrent set, trapping region, attracting set and repelling set for a flow f on a TVS-cone metric space and prove that if f is a flow on a compact TVS-cone metric space, then $\mathcal{R}(f) = \bigcap\{A_U \cup A_U^* | U$ is a trapping region for $f\}$.

Definition 1.1. Let E be a topological vector space. A subset P of E is called a topological vector space cone(abbr. TVS-cone) if the following are satisfied.

(1) P is closed and $\mathrm{Int}(P) \neq \emptyset$.
(2) $\alpha, \beta \in P$ and $a, b \in \mathbb{R}^+ \Rightarrow a\alpha + b\beta \in P$.

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(3) $\alpha, -\alpha \in P \Rightarrow \alpha = 0$.

Definition 1.2. Let P be a TVS-cone of a topological vector space E. Some partial ordering $\preceq$, $\prec$ and $\ll$ on E with respect to P are defined as follows respectively.

(1) $\alpha \preceq \beta$ if $\beta - \alpha \in P$.

(2) $\alpha \prec \beta$ if $\alpha \preceq \beta$ and $\alpha \neq \beta$.

(3) $\alpha \ll \beta$ if $\beta - \alpha \in \mathrm{Int}P$.

Lemma 1.3. [3] *Let P be a TVS-cone of a topological vector space E. Then the following hold.*

(1) If $\alpha \gg 0$, then $r\alpha \gg 0$ for each $r \in \mathbb{R}^+$.

(2) If $\alpha_1 \gg \beta_1$ and $\alpha_2 \gg \beta_2$, then $\alpha_1 + \alpha_2 \gg \beta_1 + \beta_2$.

(3) If $\alpha \gg 0$ and $\beta \gg 0$, then there is $\gamma \gg 0$ such that $\gamma \ll \alpha$ and $\gamma \ll \beta$.

Definition 1.4. Let X be a set and let E be a topological vector space equipped with partial order $\preceq$ given by an order cone P. A vector-valued function $d : X \times X \to E$ is called a TVS-come metric on X, and (X, d) is called a TVS-cone metric space if the following are satisfied.

(1) $d(x, y) \succeq 0$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$.

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(3) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

For $x \in X$ and $\epsilon \gg 0$, let $B(x, \epsilon) = \{y \in X | d(x, y) \ll \epsilon\}$. Then $\mathfrak{B} = \{B(x, \epsilon) | x \in X \text{ and } \epsilon \gg 0\}$ is a basis for some topology $\Im$ on X.[3]

In this paper, we always suppose that a cone P is a TVS-cone of a topological vector space E and a TVS-cone metric space (X, d) is a topological space with the above topology $\Im$.

Definition 1.5. Let (X, d) be a TVS-cone metric space. A flow on X is the triplet $(X, \mathbb{R}, f)$, where f is a continuous function from the product space $X \times \mathbb{R}$ into the space X satisfying the following axioms;

(1) (Identity axiom) $f(x, 0) = x$ for every $x \in X$;

(2) (Group axiom) $f(f(x, t_1), t_2) = f(x, t_1 + t_2)$ for every $x \in X$ and $t_1, t_2 \in \mathbb{R}$;

For each $t \in \mathbb{R}$, the map $f^t : X \to X$, $x \mapsto f(x, t)$ is a homeomorphism.

For a flow f on X, $\overline{f}$ is the flow on X defined by $\overline{f}(x, t) = f(x, -t)$ for all $x \in X$ and $t \in \mathbb{R}$.

Definition 1.6. Let f be a flow on a TVS-cone metric space (X, d). Given $\epsilon \gg 0$, $T > 0$, and $x, y \in X$, an (ϵ, T)-chain from x to y with respect to f^t and d is a pair of finite sequences $x = x_0, x_1, \cdots, x_n = y$ in X

and $t_0, \cdots, t_{n-1}$ in $[T, \infty)$, denoted together by $(x_0, \cdots, x_n; t_0, \cdots, t_{n-1})$, such that $d(f^{t_i}(x_i), x_{i+1}) \ll \epsilon$ for $i = 0, 1, \cdots, n-1$.

Definition 1.7. Let f be a flow on a TVS-cone metric space (X, d). A point $x \in X$ is called chain recurrent point with respect to f and d if there is an (ϵ, T)-chain from x to x for each $\epsilon \gg 0$ and $T > 0$. The set of all chain recurrent points of f, denoted by $\mathcal{R}(f)$, is called the chain recurrent set of f.

It is clear that $\mathcal{R}(\overline{f}) = \mathcal{R}(f)$.

Definition 1.8. Let f be a flow on a TVS-cone metric space (X, d). A set $U \subset X$ is called a trapping region for f if U is positively invariant with respect to f and there exists a positive real number T such that $f^T(\overline{U}) \subset \mathrm{Int}(U)$.

A trapping region determines two sets called the attracting set and the repelling set, respectively, associated to the trapping region.

Definition 1.9. Let f be a flow on a TVS-cone metric space (X, d). A set $A \subset X$ is called an attracting set for f if there exists a trapping region U for f such that $A = \bigcap_{t \geq 0} f^t(U)$. A set $A^* \subset X$ is called a repelling set for f if there exists a trapping region V for f such that $A^* = \bigcap_{t \leq 0} f^t(X - V)$. When we wish to emphasize the dependence of an attracting set A or a repelling set A^* on the trapping region U from which it arises, we denote it by A_U or A^*_U, respectively.

Conley [1] introduced attracting sets and repelling sets, referring to attracting sets as attractors and to repelling sets as repellers. We reserve the terms attractor and repeller for specific types of attracting sets and repelling sets.

Definition 1.10. Let f be a flow on a TVS-cone metric space (X, d). A pair (A, A^*) of subsets of X is an attracting-repelling pair for f if there exists a trapping region U for f such that $A = \bigcap_{t \geq 0} f^t(U)$, and $A^* = \bigcap_{t \leq 0} f^t(X - U)$

2. Main theorem

Now, we characterize the chain recurrent set of a flow on a compact TVS-cone metric space via the attracting-repelling pairs of the flow.

Theorem 2.1. *Let f be a flow on a TVS-cone metric space (X, d). If U is a trapping region for f associated to the attracting-repelling pair*

(A, A^*) for f, then $X - U$ is a trapping region for $\overline{f}$ and (A^*, A) is the attracting-repelling pair for $\overline{f}$ associated to $X - U$.

Proof. Let $t \geq 0$. Since U is positively invariant, $f^t(U) \subset U$. So $X - U \subset X - f^t(U) = f^t(X - U)$ and $f^{-t}(X - U) \subset X - U$. Thus $X - U$ is negatively invariant. This means that $X - U$ is positively invariant with respect to $\overline{f}$. Since U is a trapping region for f, there exists a $T > 0$ such that $f^T(\overline{U}) \subset \mathrm{Int}(U)$. Thus
$X - \mathrm{Int}(U) \subset X - f^T(\overline{U}) = f^T(X - \overline{U})$ so $f^{-T}(X - \mathrm{Int}(U)) \subset X - \overline{U}$.

Since $X - \overline{U} \subset X - U$ and $X - \overline{U}$ is open, $X - \overline{U} \subset \mathrm{Int}(X - U)$. Since $X - U \subset X - \mathrm{Int}(U)$ and $X - \mathrm{Int}(U)$ is closed, $\overline{X - U} \subset X - \mathrm{Int}(U)$. Thus $\overline{f}^T(\overline{X - U}) = f^{-T}(\overline{X - U}) \subset f^{-T}(X - \mathrm{Int}(U)) \subset X - \overline{U} \subset \mathrm{Int}(X - U)$.

Therefore $X - U$ is a trapping region for $\overline{f}$.

Since (A, A^*) is the attracting-repelling pair for f associated to U,
$A_{X-U} = \bigcap_{t \geq 0} \overline{f}^t(X - U) = \bigcap_{t \geq 0} f^{-t}(X - U) = \bigcap_{t \leq 0} f^t(X - U) = A^*$
and
$A^*_{X-U} = \bigcap_{t \leq 0} \overline{f}^t(X - (X - U)) = \bigcap_{t \leq 0} f^{-t}(U) = \bigcap_{t \geq 0} f^t(U) = A.$

Therefore (A^*, A) is the attracting-repelling pair for $\overline{f}$ associated to $X - U$. □

LEMMA 2.2. *Let f be a flow on a compact TVS-cone metric space (X, d). Let $y \in X$, $\epsilon \gg 0$ and $T > 0$. Define*
$U = \{x \in X | \forall s \in [0, T]$ there is an (ϵ, T)-chain from y to $f^{-s}(x)\}$.
Then U is a trapping region for f.

Proof. To show that U is positively invariant with respected to f, let $\tau > 0$ and let $u \in U$. It suffices to show that for every $s \in [0, T]$ there exists an (ϵ, T)-chain from y to $f^{-s}(f^\tau(u)) = f^{\tau - s}(u)$. If $s - \tau \geq 0$, then $s - \tau \in [0, T]$. By the definition of U there exists an (ϵ, T)-chain from y to $f^{-(s-\tau)}(u) = f^{\tau - s}(u)$. Let $\tau - s > 0$. Since $u \in U$, there is an (ϵ, T)-chain

$$(y = x_0, x_1, \cdots, x_n = f^{-T}(u); t_0, t_1, \cdots, t_{n-1})$$

from y to $f^{-T}(u)$, in which case

$$(y = x_0, x_1, \cdots, x_n = f^{-T}(u), f^{\tau - s}(u); t_0, t_1, \cdots, t_{n-1}, T + \tau - s)$$

is an (ϵ, T)-chain from y to $f^{\tau - s}(u)$. Thus U is positively invariant.

Next we shall show that $f^{2T}(\overline{U}) \subset \mathrm{Int}(U)$. Let $u \in \overline{U}$ and let $z = f^{2T}(u)$. Since $[-T, 0]$ and $[T, 2T]$ are compact subsets of $\mathbb{R}$, by the integral continuity theorem, there exists $\delta \gg 0$ such that if $d(z, w) \ll \delta$, then $d(f^\tau(z), f^\tau(w)) \ll \frac{1}{2}\epsilon$ for all $\tau \in [-T, 0]$, and that if $d(u, v) \ll \delta$, then $d(f^\tau(u), f^\tau(v)) \ll \frac{1}{2}\epsilon$ for all $\tau \in [T, 2T]$. Let $w \in B(z, \delta)$. We will show that $w \in U$ so that $z \in \mathrm{Int}(U)$. Let $s \in [0, T]$. We shall produce an (ϵ, T)-chain from y to $f^{-s}(w)$. Since $u \in \overline{U}$ there exists $v \in U$ such that $d(u, v) \ll \delta$. Since $v \in U$, there is an (ϵ, T)-chain

$$(y = x_0, x_1, \cdots, x_n = v; t_0, t_1, \cdots, t_{n-1})$$

from y to $f^{-0}(v) = v$. Notice that $2T - s \in [T, 2T]$. According to the choice of δ,

$d(f^{2T-s}(v), f^{-s}(w)) \preceq d(f^{2T-s}(v), f^{2T-s}(u)) + d(f^{2T-s}(u), f^{-s}(w)) = d(f^{2T-s}(v), f^{2T-s}(u) + d(f^{-s}(z), f^{-s}(w)) \ll \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon.$

Thus $(y = x_0, x_1, \cdots, x_n = v, f^{-s}(w); t_0, t_1, \cdots, t_{n-1}, 2T - s)$ is an (ϵ, T)-chain from y to $f^{-s}(w)$. So, $w \in U$ and $z \in \mathrm{Int}(U)$. Consequently $f^{2T}(\overline{U}) \subset \mathrm{Int}(U)$. Therefore U is a trapping region for f. □

Proposition 2.3. *Let f be a flow on a compact TVS-cone metric space (X, d). If A is the attracting set for f associated to the trapping region U, then $A = \bigcap_{t \geq 0} f^t(\overline{U}) = \bigcap_{t \geq 0} f^t(\mathrm{Int}(U))$.*

Proof. Let A be the attracting set for f associated to a trapping region U. Since U is a trapping region, there exists $T > 0$ such that $f^T(\overline{U}) \subset \mathrm{Int}(U)$. Then

$A = \bigcap_{t \geq 0} f^t(U) \subset \bigcap_{t \geq 0} f^t(\overline{U}) \subset \bigcap_{t \geq T} f^t(\overline{U}) = \bigcap_{t \geq 0} f^{t+T}(\overline{U}) = \bigcap_{t \geq 0} f^t(f^T(\overline{U})) \subset \bigcap_{t \geq 0} f^t(\mathrm{Int}(U)) \subset \bigcap_{t \geq 0} f^t(U) = A.$

Therefore $A = \bigcap_{t \geq 0} f^t(\overline{U}) = \bigcap_{t \geq 0} f^t(\mathrm{Int}(U))$. □

Lemma 2.4. *Let f be a flow on a compact TVS-cone metric space (X, d). If U is a trapping region for f, then $\mathcal{R}(f) \cap U \subset A_U$.*

Proof. Let $x \in \mathcal{R}(f) \cap U$. Since U is a trapping region for f, there exists $T > 0$ such that $f^T(\overline{U}) \subset \mathrm{Int}(U)$. As we can see from the proof of the Proposition 2.3, since $\bigcap_{t \geq T} f^t(\overline{U}) \subset A_U$, it suffices to prove that $x \in f^\tau(\overline{U})$ for every $\tau \geq T$. Let $\tau \geq T$. Since $f^\tau(\overline{U})$ is closed and X is compact, $f^\tau(\overline{U})$ is compact. Since

$$f^\tau(\overline{U}) = f^T(f^{\tau-T}(\overline{U})) = f^T(\overline{f^{\tau-T}(U)}) \subset f^T(\overline{U}) \subset \mathrm{Int}(U),$$

there exists $\eta \gg 0$ such that $B(f^\tau(\overline{U}), \eta) \subset \mathrm{Int}(U)$. Suppose that $x \notin f^\tau(\overline{U})$. Then there exists $\epsilon \gg 0$ such that $\epsilon \ll \eta$ and $x \notin B(f^\tau(\overline{U}), \epsilon)$. Since $x \in \mathcal{R}(f)$, there is an (ϵ, τ)-chain

$$(x = x_0, x_1, \cdots, x_n = x; t_0, t_1, \cdots, t_{n-1})$$

from x to x. Since U is positively invariant, $x_0 = x \in U$ and $t_0 \geq \tau$, we have

$$f^{t_0}(x_0) = f^\tau(f^{t_0-\tau}(x_0)) \in f^\tau(f^{t_0-\tau}(U)) \subset f^\tau(U) \subset f^\tau(\overline{U}).$$

Since $d(f^{t_0}(x_0), x_1) \ll \epsilon \ll \eta$, we have $x_1 \in B(f^\tau(\overline{U}), \eta) \subset \mathrm{Int}(U) \subset U$. Similarly,

$$f^{t_1}(x_1) = f^\tau(f^{t_1-\tau}(x_1)) \in f^\tau(f^{t_1-\tau}(U)) \subset f^\tau(U) \subset f^\tau(\overline{U})$$

and $d(f^{t_1}(x_1), x_2) \ll \epsilon \ll \eta$. So $x_2 \in B(f^\tau(\overline{U}), \eta) \subset \mathrm{Int}(U) \subset U$. Continuing this fashion, we obtain $x_i \in U$ for $i = 0, 1, \cdots, n-1$. Consequently

$$f^{t_{n-1}}(x_{n-1}) = f^\tau(f^{t_{n-1}-\tau}(x_{n-1})) \in f^\tau(f^{t_{n-1}-\tau}(U)) \subset f^\tau(U) \subset f^\tau(\overline{U})$$

and $d(f^{t_{n-1}}(x_{n-1}), x) = d(f^{t_{n-1}}(x_{n-1}), x_n) \ll \epsilon$. Hence $x \in B(f^\tau(\overline{U}), \epsilon)$. This is a contradiction for $x \notin B(f^\tau(\overline{U}), \epsilon)$. Thus $x \in f^\tau(\overline{U})$ so $x \in A_U$. Therefore $\mathcal{R}(f) \cap U \subset A_U$. □

THEOREM 2.5. *If f be a flow on a compact TVS-cone metric space (X, d), then*

$$\mathcal{R}(f) = \bigcap\{A_U \cup A_U^* | U \text{ is a trapping region for } f\}.$$

Proof. Let $x \in \mathcal{R}(f)$, and let U be a trapping region for f. Either $x \in X$ or $x \in X - U$. If $x \in U$, then $x \in A_U$ by Lemma 2.4. Let $x \in X - U$. Since $\mathcal{R}(f) = \mathcal{R}(\overline{f})$,

$$x \in \mathcal{R}(\overline{f}) \cap (X - U) \subset A_{X-U} = A_U^*$$

by Theorem 2.1. So

$$\mathcal{R}(f) \subset \bigcap\{A_U \cup A_U^* | U \text{ is a trapping region for } f\}.$$

To prove the reverse inclusion, let $y \notin \mathcal{R}(f^t)$, so that there exist $\epsilon \gg 0$ and $T > 0$ such that there is no (ϵ, T)-chain from y to y. Define

$$U = \{x \in X | \forall s \in [0, T] \text{ there is an } (\epsilon, T)\text{-chain from } y \text{ to } f^{-s}(x)\}.$$

By Lemma 2.2 the set U is a trapping region for f. If $y \in U$, then there is an (ϵ, T)-chain from y to $f^{-0}(y) = y$. This is a contradiction. Thus $y \notin U$. Since $A_U \subset U$, we have $y \notin A_U$. For every $s \in [0, T]$, $(y, f^{2T-s}(y); 2T-s)$ is an (ϵ, T)-chain from y to $f^{2T-s}(y) = f^{-s}(f^{2T}(y))$. Thus $f^{2T}(y) \in U$ so $y \in f^{-2T}(U)$. Hence $y \notin X - f^{-2T}(U) = f^{-2T}(X - U)$. Since

$$A_U^* = \bigcap_{t \le 0} f^t(X - U) \subset f^{-2T}(X - U),$$

we have $y \notin A_U^*$ and so $y \notin A_U \cup A_U^*$. Hence

$$y \notin \bigcap\{A_U \cup A_U^* | U \text{ is a trapping region for } f\}$$

Therefore $\bigcap\{A_U \cup A_U^* | U$ is a trapping region for $f\} \subset \mathcal{R}(f)$. Consequently,

$$\mathcal{R}(f) = \bigcap\{A_U \cup A_U^* | U \text{ is a trapping region for } f\}.$$

□

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