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A NOTE ON COMPLETE MOMENT CONVERGENCE FOR m-PNQD RANDOM VARIABLES

Mi-Hwa Ko*

Abstract. In this paper, we established the complete moment convergence for sequences of m-pairwise negatively quadrant dependent random variables which are weakly upper bounded by a random variable X.

1. Introduction

Let (Ω, F, P) be a probability space and $\{X_n, n \geq 1\}$ be a sequence of random variables. A sequence $\{X_n, n \geq 1\}$ is said to converge completely to a constant C if

$$\sum_{n=1}^{\infty} P(|X_n - C| > \epsilon) < \infty \text{ for all } \epsilon > 0. \tag{1.1}$$

The notion of complete convergence was introduced by Hsu and Robbins(1947).

By the Borel-Cantelli lemma (1.1) implies that $X_n \to C$ almost surely. The converse is true if $\{X_n, n \geq 1\}$ are independent. Hsu and Robbins(1947) proved that the sequence of arithmetic means of *i.i.d.* random variables converges completely to the expected value, provided the variance is finite. The converse was proved by Erdös(1949). The result of Hsu-Robbins-Erdös has been generalized and extended in several directions by many authors.

One of the results has been generalized and extended by Baum and Katz (1965) as follows: If $\{X_n, n \geq 1\}$ is a sequence of *i.i.d.* random variables with $E|X_1| < \infty$, then $E|X_1|^{pr} < \infty$ for $1 \leq p < 2$ and $r \geq 1$

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is equivalent to

$$(1.2) \qquad \sum_{n=1}^{\infty} n^{r-2} P(|\sum_{i=1}^{n}(X_i - EX_i)| > \epsilon n^{\frac{1}{p}}) < \infty \text{ for all } \epsilon > 0.$$

Let $\{X_n, n \geq 1\}$ be a sequence of random variables and $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be a sequence of positive numbers. If $\sum_{n=1}^{\infty} a_n E\{b_n^{-1}|X_n| - \epsilon\}_+^q < \infty$ for $q > 0$ and all $\epsilon > 0$, then the sequence $\{X_n, n \geq 1\}$ is said to be complete moment convergent.

Chow(1988) showed the complete moment convergence for a sequence of *i.i.d.* random variables by generalizing the result of Baum and Katz(1965) as follows: If $\{X_n, n \geq 1\}$ is a sequence of *i.i.d.* random variables with $E|X_1|^{pr} < \infty$ for some $1 \leq p < 2$ and $r \geq 1$, then

$$(1.3) \qquad \sum_{n=1}^{\infty} n^{r-2-\frac{1}{p}} E(|\sum_{n=1}^{n}(X_i - EX_i)| - \epsilon n^{\frac{1}{p}})^+ < \infty \text{ for all } \epsilon > 0,$$

where $a^+ = \max(a, 0)$.

It is well known that complete moment convergence implies complete convergence. Thus, complete moment convergence is stronger than complete convergence. For more details about complete moment convergence, we refer to Guo et al.(2013), Wang and Hu(2014), Wu et al.(2014), Shen et al.(2016) and among others.

A sequence $\{X_n, n \geq 1\}$ of random variables is said to be pairwise negatively quadrant dependent(PNQD) if for every real numbers x_i, x_j and $i \neq j$

$$P(X_i < x_i, X_j < x_j) \leq P(X_i < x_i)P(X_j < x_j).$$

The concept of negative quadrant dependence was introduced by Lehman (1966).

Wu and Rosalsky(2015) defined a more general dependence structure which contains PNQD as a special case as follows:

Definition 1.1. Let $m \geq 1$ be a fixed integer. A sequence of random variables $\{X_n, n \geq 1\}$ is said to be m-PNQD if for all positive integers j and k with $|j - k| \geq m$, X_j and X_k are NQD.

Clearly, PNQD is the special case $m = 1$ of the concept of m-PNQD. Indeed if $\{X_n, n \geq 1\}$ is m-PNQD for some $m \geq 1$, then $\{X_n, n \geq 1\}$ is m'-PNQD for all $m' > m$.

We provide an example of sequence of m-PNQD random variables(see Wu and Rosalsky(2015)).

EXAMPLE 1.2. *Let* $\{Y_n, n \geq 1\}$ *be a sequence of PNQD random variables and let* $m \geq 2$. *For* $n \geq 1$, *let* $r \geq 1$ *be such that* $(r-1)m+1 \leq n \leq rm$ *and let* $X_n = Y_r$. *Then* $\{X_n, n \geq 1\}$ *is a sequence of* m*-PNQD random varibles.*

We introduce a sequence of random variables $\{X, X_n, n \geq 1\}$ satisfying the following inequalities

$$c_1 P(|X| > t) \leq \frac{1}{n}\sum_{k=1}^{n} P(|X_k| > t) \leq c_2 P(X > t), \tag{1.4}$$

where c_1 and c_2 are positive constants.

If there exists a positive constant $c_1(c_2)$ such that the left-hand side(right-hand side) of (1.4) is satisfied for all $k \geq 1$, $n \geq 1$ and $t \geq 0$, then the sequence $\{X_n, n \geq 1\}$ is said to be weakly lower(upper) bounded by X. Moreover, the sequence $\{X_n, n \geq 1\}$ is said to be weakly bounded by X if it is both lower and upper bounded by X.

In this paper we establish the complete moment convergence and the Mar cinkiewicz - Zygmund strong law of large numbers for a sequence of m-PNQD random variables which is weakly bounded by random variable X.

2. Some lemmas

Inspired by Lehmann(1966) we obtain the following lemma.

LEMMA 2.1. *Let* $\{X_n, n \geq 1\}$ *be a sequence of* m*-PNQD random variables. Then we have the following properties.*

(i) $Cov(X_i, X_j) \leq 0$ *for all* i, j *such that* $|i-j| \geq m$.

(ii) *If* $\{f_n, n \geq 1\}$ *are nondecreasing functions, then for all* j, k *such that* $|j-k| \geq m$, $f_j(X_j)$ *and* $f_k(X_k)$ *are PNQD.*

LEMMA 2.2 ([13]). *Let* $\{X_n, n \geq 1\}$ *be a sequence of* m*-PNQD random variables with* $EX_n = 0$ *and* $EX_n^2 < \infty$, $n \geq 1$. *Then there exists a positive constant* C *depending only on* m *such that*

$$E(\max_{1 \leq j \leq n}(\sum_{k=1}^{j} X_k)^2) \leq C(\log 2n)^2 \sum_{k=1}^{n} EX_k^2, n \geq 1, \tag{2.1}$$

where $\log n = \log_e(\max\{e, n\}), n \geq 1$.

LEMMA 2.3 ([6]). *Let $\{X_n, n \geq 1\}$ be a sequence of random variables which is weakly bounded by random variable X. Let $r > 0$ and for some $A > 0$ $X_i^{'} = X_i I(|X_i| \leq A), X_i^{''} = X_i I(|X_i| > A)$, and $X^{'} = XI(|X| \leq A), X^{''} = XI(|X| > A)$. Then*

(i) $E|X|^r < \infty$ *implies* $(n^{-1}) \sum_{i=1}^n E|X_i|^r \leq CE|X|^r$,
(ii) $n^{-1} \sum_{i=1}^n E|X_i^{'}|^r \leq C(E|X^{'}|^r + A^r P(|X| > A))$,
(iii) $n^{-1} \sum_{i=1}^n E|X_i^{''}|^r \leq CE|X^{''}|^r$.

LEMMA 2.4 ([10]). *Let $\{Y_n, n \geq 1\}$ and $\{Z_n, n \geq 1\}$ be sequences of random variables. Then, for any $q > 1, \epsilon > 0$ and $a > 0$,*

$$\begin{aligned} & E(\max_{1\leq k\leq n} |\sum_{i=1}^k (Y_i + Z_i)| - \epsilon a)^+ \\ \leq & (\frac{1}{\epsilon^q} + \frac{1}{q-1})\frac{1}{a^{q-1}} E(\max_{1\leq k\leq n} |\sum_{i=1}^k Y_i|^q) + E(\max_{1\leq k\leq n} |\sum_{i=1}^k Z_i|). \end{aligned}$$

3. Main results

THEOREM 3.1. *Let $\{X_n, n \geq 1\}$ be a mean zero sequence of m-PNQD random variables which is weakly upper bounded by a random variable X with $EX^2 < \infty$. If for $r > 1$ and $0 < p < 2$ such that $1 < pr < 2$*

$$E(|X|^{pr}(\log(1 + |X|))^2) < \infty, \tag{3.1}$$

then for every $\epsilon > 0$

$$\sum_{n=1}^{\infty} n^{r-2-\frac{1}{p}} E(\max_{1\leq k\leq n} |\sum_{i=1}^k X_i| - \epsilon n^{\frac{1}{p}})^+ < \infty. \tag{3.2}$$

Proof. For fixed $m \geq 1$, and $1 \leq i \leq n$, let

$$\begin{aligned} X_{ni} &= -n^{\frac{1}{p}} I(X_i < -n^{\frac{1}{p}}) + X_i I(|X_i| \leq n^{\frac{1}{p}}) + n^{\frac{1}{p}} I(X_i > n^{\frac{1}{p}}), \\ X_{ni}^{'} &= X_i - X_{ni} \text{ and } \tilde{X}_{ni} = X_{ni} - EX_{ni}. \end{aligned}$$

Then, we obtain

$$X_{ni}^{'} = n^{\frac{1}{p}} I(X_i < -n^{\frac{1}{p}}) - n^{\frac{1}{p}} I(X_i > n^{\frac{1}{p}}) + X_i I(|X_i| > n^{\frac{1}{p}})$$

and

$$X_i = \tilde{X_{ni}} + X_{ni}^{'} + EX_{ni}.$$

Letting $a = n^{\frac{1}{p}}$ and $q = 2$ in Lemma 2.4 we obtain

$$\begin{aligned}
& \sum_{n=1}^{\infty} n^{r-2-\frac{1}{p}} E(\max_{1\le k\le n} |\sum_{i=1}^{k} X_i| - \epsilon n^{\frac{1}{p}})^+ \\
\le\ & (\frac{1}{\epsilon^2}+1) \sum_{n=1}^{\infty}\sum_{n=1}^{\infty} n^{r-2-\frac{2}{p}} E(\max_{1\le k\le n} |\sum_{i=1}^{k} \tilde{X_{ni}}|^2) \\
& +\sum_{n=1}^{\infty} n^{r-2-\frac{1}{p}} E(\max_{1\le k\le n} |\sum_{i=1}^{k} X_{ni}^{'}|) + \sum_{n=1}^{\infty} n^{r-2-\frac{1}{p}} (\max_{1\le k\le n} |\sum_{i=1}^{k} EX_{ni}|) \qquad (3.3)\\
\le\ & I_1 + I_2 + I_3.
\end{aligned}$$

Note that $\{\tilde{X_{ni}}, 1 \le i \le n, n \ge 1\}$ is m-PNQD by Lemma 2.1 (ii). From (1.4) and (2.1) we have

$$\begin{aligned}
I_1 =\ & (\frac{1}{\epsilon^2}+1) \sum_{n=1}^{\infty} n^{r-2-\frac{2}{p}} E(\max_{1\le k\le n} |\sum_{i=1}^{k} \tilde{X_{ni}}|^2) \\
\le\ & C \sum_{n=1}^{\infty} n^{r-2-\frac{2}{p}} (\log n)^2 \sum_{i=1}^{n} (E\tilde{X_{ni}}^2) \\
\le\ & C \sum_{n=1}^{\infty} n^{r-2-\frac{2}{p}} (\log n)^2 \sum_{i=1}^{n} EX_{ni}^2 \\
\le\ & C \sum_{n=1}^{\infty} n^{r-2-\frac{2}{p}} (\log n)^2 \sum_{i=1}^{n} EX_i^2 I(|X_i| \le n^{\frac{1}{p}}) \\
& +C \sum_{n=1}^{\infty} n^{r-2-\frac{2}{p}} (\log n)^2 \sum_{i=1}^{n} n^{\frac{2}{p}} P(|X_i| > n^{\frac{1}{p}}) \qquad (3.4)\\
\le\ & C \sum_{n=1}^{\infty} n^{r-1-\frac{2}{p}} (\log n)^2 EX^2 I(|X| \le n^{\frac{1}{p}}) \\
& +C \sum_{n=1}^{\infty} n^{r-1} (\log n)^2 P(|X| > n^{\frac{1}{p}}) \\
=\ & I_{11} + I_{12}.
\end{aligned}$$

Since the function $(\log x)^2$ is slowly varying at x we have

$$
\begin{aligned}
I_{11} &= C\sum_{n=1}^{\infty} n^{r-1-\frac{2}{p}}(\log n)^2 \sum_{m=1}^{n} E(X^2 I(m-1 < |X|^p \le m)) \\
&= C\sum_{m=1}^{\infty} EX^2 I(m-1 < |X|^p \le m) \sum_{n=m}^{\infty} n^{r-1-\frac{2}{p}}(\log n)^2 \\
&\le C\sum_{n=1}^{\infty} n^{r-\frac{2}{p}}(\log n)^2 E(X^2 I((n-1) < |X|^p \le n)) \qquad (3.5) \\
&\le CE|X|^{pr}(\log(1+|X|))^2 < \infty.
\end{aligned}
$$

$$
\begin{aligned}
I_{12} &= C\sum_{n=1}^{\infty} n^{r-1}(\log n)^2 P(|X| > n^{\frac{1}{p}}) \\
&\le C\sum_{n=1}^{\infty} n^{r-1-\frac{1}{p}}(\log n)^2 E|X|I(|X| > n^{\frac{1}{p}}) \\
&= C\sum_{n=1}^{\infty} n^{r-1-\frac{1}{p}}(\log n)^2 \sum_{m=n}^{\infty} E(|X|I(m < |X|^p \le m+1)) \\
&= C\sum_{m=1}^{\infty} E(|X|I(m < |X|^p \le m+1)) \sum_{n=1}^{m} n^{r-1-\frac{1}{p}}(\log n)^2 \qquad (3.6) \\
&= C\sum_{n=1}^{\infty} n^{r-\frac{1}{p}}(\log n)^2 E(|X|I(n < |X|^p \le n+1)) \\
&\le CE(|X|^{pr}(\log(1+|X|))^2) < \infty.
\end{aligned}
$$

$$
\begin{aligned}
I_2 &\le C\sum_{n=1}^{\infty} n^{r-2-\frac{1}{p}}\sum_{i=1}^{n} E|X'_{ni}| \\
&\le C\sum_{n=1}^{\infty} n^{r-2-\frac{1}{p}}\sum_{i=1}^{n} E(|X_i|I(|X_i|>n^{\frac{1}{p}})) \\
&\quad +C\sum_{n=1}^{\infty} n^{r-2-\frac{1}{p}}n^{\frac{1}{p}}\sum_{i=1}^{n} P(|X_i|>n^{\frac{1}{p}}) \\
&\le C\sum_{n=1}^{\infty} n^{r-1-\frac{1}{p}} E(|X|I(|X|>n^{\frac{1}{p}})) \qquad (3.7)\\
&= C\sum_{n=1}^{\infty} n^{r-1-\frac{1}{p}}\sum_{m=n}^{\infty} E(|X|I(m^{\frac{1}{p}}\le |X|\le (m+1)^{\frac{1}{p}})) \\
&= C\sum_{m=1}^{\infty} E(|X|I(m<|X|^p\le (m+1)))\sum_{n=1}^{m} n^{r-1-\frac{1}{p}} \\
&\le C\sum_{n=1}^{\infty} n^{r-\frac{1}{p}} E(|X|I(n<|X|^p\le n+1)) \\
&\le CE|X|^{pr}<\infty.
\end{aligned}
$$

$$
\begin{aligned}
I_3 &\le C\sum_{n=1}^{\infty} n^{r-2-\frac{1}{p}}\sum_{i=1}^{n} E|X_{ni}| \\
&\le C\sum_{n=1}^{\infty} n^{r-2-\frac{1}{p}}\sum_{i=1}^{n} E(|X_i|I(|X_i|\le n^{\frac{1}{p}})) + C\sum_{n=1}^{\infty} n^{r-2-\frac{1}{p}}\sum_{i=1}^{n} n^{\frac{1}{p}}P(|X_i|>n^{\frac{1}{p}}) \\
&\le C\sum_{n=1}^{\infty} n^{r-1-\frac{1}{p}} E(|X|I(|X|\le n^{\frac{1}{p}})) + C\sum_{n=1}^{\infty} n^{r-1-\frac{1}{p}} E(|X|I(|X|>n^{\frac{1}{p}})) \\
&\le I_{31}+I_{32}.
\end{aligned}
$$

$$
\begin{aligned}
I_{31} &= C\sum_{n=1}^{\infty} n^{r-1-\frac{1}{p}} E(|X|I(|X| \le n^{\frac{1}{p}})) \\
&= C\sum_{n=1}^{\infty} n^{r-1-\frac{1}{p}} \sum_{m=1}^{n} E|X|I((m-1)^{\frac{1}{p}} \le |X| \le m^{\frac{1}{p}}) \\
&= C\sum_{m=1}^{\infty} E(|X|I((m-1)^{\frac{1}{p}} \le |X| \le m^{\frac{1}{p}})) \sum_{n=1}^{m} n^{r-1-\frac{1}{p}} \\
&\le C\sum_{n=1}^{\infty} n^{r-\frac{1}{p}} E(|X|I((m-1) \le |X|^p \le m)) \qquad (3.8) \\
&\le CE(|X|^{pr}).
\end{aligned}
$$

By the similar method as in (3.7) we have

$$
\begin{aligned}
I_{32} &= C\sum_{n=1}^{\infty} n^{r-1-\frac{1}{p}} E(|X|I(|X| > n^{\frac{1}{p}})) \\
&\le CE(|X|^{pr}) < \infty. \qquad (3.9)
\end{aligned}
$$

Hence by combining (3.3)-(3.9), the proof is completed. □

COROLLARY 3.2. *Let $1 < r < 2$ and $0 < p < 2$ such that $1 < pr < 2$. Let $\{X_n, n \ge 1\}$ be a mean zero sequence of m-PNQD random variables which is weakly bounded by a random variable X such that $EX^2 < \infty$. If (3.1) holds, then, for every $\epsilon > 0$,*

$$
\sum_{n=1}^{\infty} n^{r-2} E(\sup_{k \ge n} |\frac{X_k}{k^{\frac{1}{p}}}| - \epsilon)^+ < \infty.
$$

Proof. Using the similar method in the proof of Theorem 3.1 we obtain that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-2} E(\sup_{k\geq n} |\frac{\sum_{i=1}^{k} X_i}{k^{\frac{1}{p}}}| - \epsilon)^+ \\
= &\sum_{n=1}^{\infty} n^{r-2} \int_0^{\infty} P(\sup_{k\geq n} |\frac{\sum_{i=1}^{k} X_i}{k^{\frac{1}{p}}}| - \epsilon > t) dt \\
= &\sum_{m=1}^{\infty} \sum_{n=2^{m-1}}^{2^m-1} n^{r-2} \int_0^{\infty} P(\sup_{k\geq n} |\frac{\sum_{i=1}^{k} X_i}{k^{\frac{1}{p}}}| > \epsilon + t) dt \\
= &\sum_{m=1}^{\infty} \int_0^{\infty} P(\sup_{k\geq 2^{m-1}} |\frac{\sum_{i=1}^{k} X_i}{k^{\frac{1}{p}}}| > \epsilon + t) dt \sum_{n=2^{m-1}}^{2^m-1} 2^{m(r-2)} \\
\leq &\ C\sum_{m=1}^{\infty} 2^{m(r-1)} \int_0^{\infty} P(\sup_{k\geq 2^{m-1}} |\frac{\sum_{i=1}^{k} X_i}{k^{\frac{1}{p}}}| > \epsilon + t) dt \\
= &\ C\sum_{m=1}^{\infty} 2^{m(r-1)} \int_0^{\infty} P(\sup_{l\geq m} \max_{2^{l-1}\leq k\leq 2^l} |\frac{\sum_{i=1}^{k} X_i}{k^{\frac{1}{p}}}| > \epsilon + t) dt \\
\leq &\ C\sum_{m=1}^{\infty} 2^{m(r-1)} \sum_{l=m}^{\infty} \int_0^{\infty} P(\max_{1\leq k\leq 2^l} |\sum_{i=1}^{k} X_i| > (\epsilon + t) 2^{(l-1)/p}) dt \\
\leq &\ C2^{2-r} \sum_{l=1}^{\infty} \int_0^{\infty} P(\max_{1\leq k\leq 2^l} |\sum_{i=1}^{k} X_i| > (\epsilon + t) 2^{(l-1)/p}) dt \sum_{m=1}^{l} 2^{m(r-1)} \\
\leq &\ C2^{2-r} \sum_{l=1}^{\infty} 2^{l(r-1)} \int_0^{\infty} P(\max_{1\leq k\leq 2^l} |\sum_{i=1}^{k} X_i| > (\epsilon + t) 2^{(l-1)/p}) dt \\
&(\text{let } t_1 = 2^{(l-1)/p} t) \\
= &\ C2^{2-r+\frac{1}{p}} \sum_{l=1}^{\infty} 2^{l(r-1-\frac{1}{p})} \int_0^{\infty} P(\max_{1\leq k\leq 2^l} |\sum_{i=1}^{k} X_i| > \epsilon 2^{\frac{l-1}{p}} + t_1) dt_1 \\
&\text{let } \epsilon = \epsilon_1 2^{\frac{2}{p}} \\
\leq &\ C\sum_{l=1}^{\infty} \sum_{n=2^l}^{2^{(l+1)}-1} 2^{(l+1)(r-2-\frac{1}{p})} \int_0^{\infty} P(\max_{1\leq k\leq n} |\sum_{i=1}^{k} X_i| > \epsilon_1 2^{\frac{l+1}{p}} + t_1) dt_1 \\
\leq &\ C\sum_{l=1}^{\infty} \sum_{n=2^l}^{2^{(l+1)}-1} n^{r-2-\frac{1}{p}} \int_0^{\infty} P(\max_{1\leq k\leq n} |\sum_{i=1}^{k} X_i| > \epsilon_1 n^{\frac{1}{p}} + t_1) dt_1
\end{aligned}$$

$$(3.10) \quad = C\sum_{n=1}^{\infty} n^{r-2-\frac{1}{p}} E(\max_{1\leq k\leq n} |\sum_{i=1}^{k} X_i| - \epsilon_1 n^{\frac{1}{p}})^+ < \infty \text{ by (3.2).}$$

Hence, the proof is complete. □

We consider case $pr = 1$ and $0 < p < 1$.

THEOREM 3.3. *Let $\{X_n, n \geq 1\}$ be a mean zero sequence of m-PNQD random variables which is weakly bounded by a random variable X. If*

$$E(|X|(\log(1+|X|))^2) < \infty, \tag{3.11}$$

then for all $\epsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-2} E(\max_{1\leq k\leq n} |\sum_{i=1}^{k} X_i| - \epsilon n^{\frac{1}{p}})^+ < \infty \tag{3.12}$$

holds.

Proof. First, as in the proof of Theorem 3.1 we define X_{ni}, $X_{ni}^{'}$ and $\tilde{X_{ni}}$, respectively. By letting $a = n^{\frac{1}{p}}$ and $q = 2$ in Lemma 2.4 we obtain that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{-2} E(\max_{1\leq k\leq n} |\sum_{i=1}^{k} X_i| - \epsilon n^{\frac{1}{p}})^+ \\
\leq\ & (\frac{1}{\epsilon^2}+1) \sum_{n=1}^{\infty} n^{-2-\frac{1}{p}} E(\max_{1\leq k\leq n} |\sum_{i=1}^{k} \tilde{X_{ni}}|^2) \\
& + \sum_{n=1}^{\infty} n^{-2} E(\max_{1\leq k\leq n} |\sum_{i=1}^{k} X_{ni}^{'}|) + \sum_{n=1}^{\infty} n^{-2} (\max_{1\leq k\leq n} |\sum_{i=1}^{k} EX_{ni}|) \\
=\ & : J_1 + J_2 + J_3.
\end{aligned} \tag{3.13}$$

For J_1, we have

$$
\begin{aligned}
J_1 &\le C(\frac{1}{\epsilon^2}+1)\sum_{n=1}^{\infty} n^{-2-\frac{1}{p}}(\log n)^2 \sum_{i=1}^{n} E|\tilde{X_{ni}}|^2 \\
&\le C\sum_{n=1}^{\infty} n^{-2-\frac{1}{p}}(\log n)^2 \sum_{i=1}^{n} E|X_{ni}|^2 \\
&\le C\sum_{n=1}^{\infty} n^{-2-\frac{1}{p}}(\log n)^2 \sum_{i=1}^{n} \{E|X_i|^2 I(|X_i| \le n^{\frac{1}{p}}) + n^{\frac{2}{p}} EI(|X| > n^{\frac{1}{p}})\} \\
&\le C\sum_{n=1}^{\infty} n^{-1-\frac{1}{p}}(\log n)^2 \sum_{i=1}^{n} E(|X|^2 I(|X_i| \le n^{\frac{1}{p}})) \\
&\quad +C\sum_{n=1}^{\infty} n^{-1+\frac{1}{p}}(\log n)^2 P(|X| > n^{\frac{1}{p}}) \\
&=: J_{11}+J_{12}. \qquad (3.14)
\end{aligned}
$$

For J_{11}, by $E|X|(\log(1+|X|))^2 < \infty$ we have

$$
\begin{aligned}
J_{11} &= C\sum_{n=1}^{\infty} n^{-1-\frac{1}{p}}(\log n)^2 \sum_{m=1}^{n} E(|X|^2 I((m-1)^{\frac{1}{p}} < |X| \le m^{\frac{1}{p}})) \\
&= C\sum_{m=1}^{\infty} E(|X|^2 I((m-1)^{\frac{1}{p}} < |X| \le m^{\frac{1}{p}})) \sum_{n=m}^{\infty} n^{-1-\frac{1}{p}}(\log n)^2 \\
&\le C\sum_{m=1}^{\infty} m^{-\frac{1}{p}}(\log(1+m))^2 E(|X|^2 I((m-1)^{\frac{1}{p}} < |X| \le m^{\frac{1}{p}})) \\
&\le CE(|X|(\log((1+|X|)))^2) < \infty. \qquad (3.15)
\end{aligned}
$$

For J_{12} we have

$$
\begin{aligned}
J_{12} &\leq C\sum_{n=1}^{\infty} n^{-1}(\log n)^2 E(|X|I(|X|>n^{\frac{1}{p}})) \\
&= C\sum_{n=1}^{\infty} n^{-1}(\log n)^2 \sum_{m=n}^{\infty} E(|X|I(m^{\frac{1}{p}}<|X|<(m+1)^{\frac{1}{p}})) \\
&= C\sum_{m=1}^{\infty} E(|X|I(m^{\frac{1}{p}}<|X|<(m+1)^{\frac{1}{p}}))\sum_{n=1}^{m} n^{-1}(\log n)^2 \\
&\leq C\sum_{m=1}^{\infty} (\log(1+m))^2 E(|X|I(m<|X|^p<(m+1))) \\
&\leq CE(|X|(\log(1+|X|))^2)<\infty. \qquad (3.16)
\end{aligned}
$$

For J_2 we obtain that

$$
\begin{aligned}
J_{2} &\leq C\sum_{n=1}^{\infty} n^{-1} E(|X|I(|X|>n^{\frac{1}{p}})) \\
&= C\sum_{n=1}^{\infty} n^{-1} \sum_{m=n}^{\infty} E(|X|I(m<|X|^p\leq m+1)) \\
&= C\sum_{m=1}^{\infty} E(|X|I(m<|X|^p\leq m+1))\sum_{n=1}^{m} n^{-1} \\
&\leq C\sum_{m=1}^{\infty} \log(1+m) E(|X|I(m<|X|^p\leq m+1)) \\
&\leq CE(|X|\log(1+|X|)) \\
&< \infty. \qquad (3.17)
\end{aligned}
$$

For J_3, it is easy to see that

$$
\begin{aligned}
J_{3} &\leq C\sum_{n=1}^{\infty} n^{-2} \sum_{i=1}^{n} E(|X_i|I(|X_i|>n^{\frac{1}{p}})) \\
&\leq C\sum_{n=1}^{\infty} n^{-1} E(|X|I(|X|>n^{\frac{1}{p}})) \\
&\leq CE(|X|\log(1+|X|))<\infty \text{ by (3.17)}. \qquad (3.18)
\end{aligned}
$$

Hence the proof is complete. □

Next, we consider the Marcinkiewicz-Zygmund-type strong law of large numbers for m-PNQD sequence satisfying (1.4).

Corollary 3.4. *Let $r > 1$ and $1 < p < 2$. Let $\{X_n, n \geq 1\}$ be a sequence of zero mean m-PNQD random variables which is weakly bounded by a random variable X with$E|X|^p < \infty$. Under the conditions of Theorem 3.1 $n^{-\frac{1}{p}} \sum_{i=1}^{n} X_i \to 0$ a.s. holds.*

Proof. Under conditions of Theorem 3.1 it is true that for every $\epsilon > 0$

$$\begin{aligned} & \sum_{n=1}^{\infty} n^{r-2-\frac{1}{p}} E(\max_{1\leq k\leq n} |\sum_{i=1}^{k} X_i| - \epsilon n^{\frac{1}{p}})^{+} \\ \geq \ & \epsilon \sum_{n=1}^{\infty} n^{r-2} P(\max_{1\leq k\leq n} |\sum_{i=1}^{k} X_i| > 2\epsilon n^{\frac{1}{p}}). \end{aligned}$$

Hence, (3.2) implies

$$\sum_{n=1}^{\infty} n^{r-2} P(\max_{1\leq k\leq n} |\sum_{i=1}^{k} X_i| > \epsilon n^{\frac{1}{p}}) < \infty \tag{3.19}$$

for every $\epsilon > 0$.

Since $r > 1$ it can be seen by (3.19) that

$$\sum_{n=1}^{\infty} n^{-1} P(\max_{1\leq k\leq n} |\sum_{i=1}^{k} X_i| > \epsilon n^{\frac{1}{p}}) \leq \sum_{n=1}^{\infty} n^{r-2} P(\max_{1\leq k\leq n} |\sum_{i=1}^{k} X_i| > \epsilon n^{\frac{1}{p}}).$$

It is easy to obtain $n^{-\frac{1}{p}} \sum_{i=1}^{n} X_i \to 0$ $a.s.$, $n \to \infty$. □

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References

[1] L.E. Baum and M. Katz, *Convergence rates in the law of large numbers,* Trans. Amer. Math. Soc., **120** (1965), no. 1, 108-123.

[2] Y.S. Chow, *On the rate of moment complete convergence of sample sums and extremes,* Bull. Inst. Math. Acad. Sinica, **16** (1988), 177-2011.

[3] P. Erdös, *On a theorem of Hsu and Robbins*, Ann. Math. Stat. **20** (1949), no. 2, 286-291.

[4] M.L. Guo and D.J. Zhu, *Equivalent conditions of complete moment convergence of weighted sums for ρ^*-mixing sequence random variables*, Stat. Prob. Letts., **83** (2013), 13-20.

[5] P.L. Hsu and H. Robbins, *Complete convergence and the law of large numbers*, Proc. Natl. Acad. Sci. USA., **33** (1947), no. 2, 25-31.

[6] A. Kuczmaszewska, *On complete convergence in Marcinkiewicz-Zygmund type SLLN for negatively associated random variables,* Acta. Math. Hungar., **28** (2010), 116-130.

[7] A. Kuczmaszewska, A. Zbigniew, and Lagodowski., *Convergence rates in SLLN for some class of dependent random fields,* J. Math. Anal. Appl., **380** (2011), 571-584.

[8] E.L. Lehmann, *Some concepts of dependence,* Ann. Math. Stat., **37** (1966), 1137-1153.

[9] A.T. Shen, M.X. Xue, and A. Volodin, *Complete moment convergence for arrays of rowwise NSD random vaiables*, Stochastics, **88** (2016), no. 4, 606-621.

[10] S.H. Sung, *Moment inequalities and complete moment convergence,* J. Inequal. Appl., **2009** (2009), Article ID 271265 pages 14.

[11] X.J. Wang and S.H. Hu, *Complete convergence and complete moment convergence for martingale difference sequence*, Acta Mathematica Sinica, Engl. Ser., **30** (2014), 119-132.

[12] Y.F. Wu, M. Ordóñez Cabrera, and A. Volodin, *Complete convergence and complete moment convergence for arrays of rowwise END random variables*, Glasnik Matematicki, **49** (2014), no. 2, 447-466.

[13] Y, Wu, and A. Rosalsky, *Strong convergence for m-pairwise negatively quadrant dependent random varibles*, Glasnik Matematicki, **50** (2015), no. 1, 245-259.

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Division of Mathematics and Informational Statistics,
Wonkwang University, Jeonbuk, 570-749, Korea
E-mail: songhack@wonkwang.ac.kr