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CONSUMPTION-LEISURE CHOICE WITH STOCHASTIC INCOME FLOW

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Abstract. This paper investigates the portfolio selection problem with flexible labor choice and stochastic income flow where the unit wage flow is governed by a stochastic process. The agent optimally chooses consumption, investment, and labor supply. We derive the closed-form solution by applying a martingale method even with the stochastic income flow.

1. Introduction

We study the optimal consumption, leisure, and investment decisions problem when the income flow is stochastic. The agent obtains a utility from consumption and leisure. The leisure rate can be considered as a value in terms of the unit of time. Due to the time limit, it always has its maximum and the remaining time for leisure is regarded as the working hours. Then, by multiplying the wage, rate we can obtain the labor income. Thus, the agent determines the labor supply endogenously. In our study, we suppose the wage rate is stochastic.

This paper is related to the literature on portfolio selection problems with flexible labor supply. [2] study an optimal portfolio selection problem with flexible labor supply and [4] and [3] extend the model with voluntary retirement. [5] also consider a labor-leisure choice problem

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with a CES(constant elasticity of substitution) utility function. [7] and [8] also consider a flexible labor supply with additional realistic feature like insurance and pension funds. None of these studies, however, consider the stochastic income flow. This paper also relates to the literature on portfolio choice with a stochastic income such as [1], [6], and others. They impose that the labor market has the same risk source as a financial market but the labor income is given exogenously. In this paper, we endogenize the labor income by optimally choosing the leisure rate.

This paper is organized as follows. Section 2 describes the baseline model including stochastic wage rate and Section 3 define the problem with flexible labor supply. Section 4 and 5 provide the solution and optimal controls in closed-form. Section 6 concludes.

2. Basic Model

In continuous time financial market, there are two kinds of assets, which are a risky asset and a riskless asset. The risky asset S_t is unfolded by

$$\frac{dS_t}{S_t} = \mu_s dt + \sigma_s dB_t,$$

and the riskless asset S_t^0 has a constant interest rate r. The process is B_t is the standard Brownian motion under the probability space $(\mathcal{F}, \Omega, \mathbb{P})$. Let us denote the consumption, leisure rates, investment amount at time t by c_t, l_t and π_t respectively. We suppose that c_t, l_t, π_t are $\mathcal{F}_t$-adapted processes which satisfy the following conditions.

$$\int_0^\infty c_t dt < \infty \text{ a.s.}, \quad \int_0^\infty l_t dt < \infty \text{ a.s.}, \quad \int_0^\infty \pi_t^2 dt < \infty \text{ a.s.}$$

We also assume that l_t has its maximum value (or time) as L. Then the remaining amount of leisure l_t can be considered by a rate for working, which means that $(L - l_t)$ represents a rate of labor supply. If we denote the unit labor wage by w_t, then the labor income is defined by $w_t(L - l_t)$. In this paper, we suppose w_t is governed by the following stochastic process,

$$\frac{dw_t}{w_t} = \mu_w dt + \sigma_w dB_t, \qquad w_0 = w.$$

Note that the uncertainties of S_t and w_t are same and it implies that the risk sources of financial and labor markets are perfectly correlated. This assumption makes the problem tractable and is a key factor to obtain the explicit solution in the following sections. We can derive the dynamic wealth process X_t as follows.

(2.1)

$$dX_t = (rX_t + \pi_t(\mu_s - r) - c_t + w_t(L - l_t))\,dt + \sigma_s \pi_t dB_t, \quad X_0 = x.$$

The agent obtains a utility from consumption and leisure and faces a Cobb-Douglas utility function, which is given by

$$u(c_t, l_t) = \frac{\left(c_t^{\alpha} l_t^{1-\alpha}\right)^{1-\gamma}}{\alpha(1-\gamma)},$$

where α represents the elasticity of consumption and γ is the risk aversion. We assume that $0 < \alpha < 1, \gamma > 0, \gamma \neq 1$. If we define $\gamma_1 = 1 - \alpha(1-\gamma)$, the life-time expected utility value is written as

$$\mathbb{E}\left[\int_0^{\infty} e^{-\beta t} \frac{c_t^{1-\gamma_1} l_t^{\gamma_1 - \gamma}}{1-\gamma_1} dt\right],$$

where β is the discount factor of the agent.

3. Problem

To apply a martingale method, we transform the dynamic wealth process (2.1) into a static form. Let us define the market price of risk, exponential martingale, and pricing kernel by

$$\theta \triangleq \frac{\mu_s - r}{\sigma_s}, \qquad \eta_t = e^{-\frac{1}{2}\theta^2 t - \theta B_t}, \qquad H_t \triangleq e^{-rt}\eta_t$$

From a Girsanov theorem, there exists a risk-neutral equivalent martingale measure $\widetilde{P}$ under which $\widetilde{B}_t = B_t - \theta t$ is the standard Brownian motion. Then, the dynamic wealth process is rewritten as

$$dX_t = (rX_t - c_t + w_t(L - l_t))dt + \sigma_s \pi_t d\widetilde{B}_t,$$

and by recovering the physical measure using Bayes' rule, we can obtain the following static budget constraint.

$$\mathbb{E}\left[\int_0^{\infty} H_t(c_t + w_t l_t)dt\right] \leq x + \mathbb{E}\left[\int_0^{\infty} H_t w_t L dt\right].$$

Note that the second term in the right-hand side of the static budget represents the present value of the future income stream when the agent provides a maximal labor supply. It is calculated by

$$\mathbb{E}\left[\int_0^{\infty} H_t w_t L dt\right] = \frac{wL}{r_w},$$

where $r_w \triangleq r - \mu_w + \theta\sigma_w$, which is assumed to be positive.

Now, we can provide the primal problem as follows.

PROBLEM 3.1. The agent wants to maximize the life-time expected utility by optimally choosing the consumption, leisure, investment. In words, the value function is defined by

$$V(x,w) = \max_{c_t,l_t,\pi_t} \mathbb{E}\left[\int_0^\infty e^{-\beta t}\frac{c_t^{1-\gamma_1} l_t^{\gamma_1-\gamma}}{1-\gamma_1}dt\right],$$

subject to

$$\mathbb{E}\left[\int_0^\infty H_t(c_t + w_t l_t)dt\right] \le x + \frac{wL}{r_w}.$$

4. Solution

For a given $\lambda > 0$, let us define the Lagrangian by

$$\mathcal{L} = \max_{c_t,l_t}\left\{\mathbb{E}\left[\int_0^\infty e^{-\beta t}\left(\frac{c_t^{1-\gamma_1} l_t^{\gamma_1-\gamma}}{1-\gamma_1} - \lambda e^{\beta t} H_t(c_t + w_t l_t)\right)dt\right]\right\} + \lambda\left(x + \frac{wL}{r_w}\right). \tag{4.1}$$

Since $0 < l_t^* \le L$, we have to separate into the two cases where $l_t^* < L$ and $l_t^* = L$. For $l_t^* < L$, from the first order conditions we have the optimal consumption and leisure rates as follows.

$$c_t^* = \left(\frac{\gamma_1-\gamma}{w_t(1-\gamma_1)}\right)^{\frac{\gamma_1-\gamma}{\gamma}} y_t^{-\frac{1}{\gamma}}, \qquad l_t^* = \left(\frac{\gamma_1-\gamma}{w_t(1-\gamma_1)}\right)^{\frac{\gamma_1}{\gamma}} y_t^{-\frac{1}{\gamma}},$$

where $y_t = \lambda e^{\beta t} H_t$. By substituting into (4.1), the integrand in the expectation becomes

$$e^{-\beta t}\left(\frac{c_t^{*1-\gamma_1} l_t^{*\gamma_1-\gamma}}{1-\gamma_1} - \lambda e^{\beta t} H_t(c_t^* + w_t l_t^*)\right) = e^{-\beta t}\frac{\gamma}{1-\gamma_1}\left(\frac{\gamma_1-\gamma}{w_t(1-\gamma_1)}\right)^{\frac{\gamma_1-\gamma}{\gamma}} y_t^{1-\frac{1}{\gamma}}.$$

When $l_t^* = L$, the optimal consumption rate is given by

$$c_t^* = L^{\frac{\gamma_1-\gamma}{\gamma_1}} y_t^{-\frac{1}{\gamma_1}}.$$

Thus, the Lagrangian in (4.1) can be rewritten as

$$\mathcal{L} = \mathbb{E}\left[\int_0^\infty e^{-\beta t}\left\{\frac{\gamma}{1-\gamma_1}\left(\frac{\gamma_1-\gamma}{w_t(1-\gamma_1)}\right)^{\frac{\gamma_1-\gamma}{\gamma}} y_t^{1-\frac{1}{\gamma}}\mathbf{1}_{\{\tilde{y}<y_t\}}\right.\right. \tag{4.2}$$
$$\left.\left.+\left(\frac{\gamma_1}{1-\gamma_1}L^{\frac{\gamma_1-\gamma}{\gamma_1}}y_t^{1-\frac{1}{\gamma_1}} - w_t L y_t\right)\mathbf{1}_{\{0<y_t\le\tilde{y}\}}\right\}dt\right] + \lambda\left(x+\frac{wL}{r_w}\right),$$

where $\mathbf{1}_D$ is the indicator function which has 1 if D is true and 0 otherwise. The boundary $\tilde{y}$ is determined by the following condition.

$$l_t^* = \left(\frac{\gamma_1-\gamma}{w_t(1-\gamma_1)}\right)^{\frac{\gamma_1}{\gamma}}\tilde{y}^{-\frac{1}{\gamma}} = L, \quad \text{or} \quad \tilde{y} = \left(\frac{\gamma_1-\gamma}{w_t(1-\gamma_1)}\right)^{\gamma_1} L^{-\gamma}.$$

Note that since w_t is a stochastic process, the expectation in Lagrangian (4.2) would be the function of w_t and y_t. However, we can make it as a function of one variable, which is the key part in this paper. We summarize the result in the following lemma.

LEMMA 4.1. *The Lagrangian* (4.1) *can be written as*

$$\mathcal{L} = w^{1-\gamma_1}\mathbb{E}^{\mathbb{Q}}\left[\int_0^\infty e^{-\hat{\beta} t}\left\{\frac{\gamma}{1-\gamma_1}\left(\frac{\gamma_1-\gamma}{w_t(1-\gamma_1)}\right)^{\frac{\gamma_1-\gamma}{\gamma}} z_t^{1-\frac{1}{\gamma}}\mathbf{1}_{\{\tilde{z}<z_t\}}\right.\right. \tag{4.3}$$
$$\left.\left.+\left(\frac{\gamma_1}{1-\gamma_1}L^{\frac{\gamma_1-\gamma}{\gamma_1}}z_t^{1-\frac{1}{\gamma_1}} - L z_t\right)\mathbf{1}_{\{0<z_t\le\tilde{z}\}}\right\}dt\right] + \lambda\left(x+\frac{wL}{r_w}\right),$$

where $z_t = y_t w_t^{\gamma_1}$,

$$\hat{\beta} = \beta - (1-\gamma_1)\mu_w + \frac{1}{2}\gamma_1(1-\gamma_1)\sigma_w^2,$$

and $\mathbb{E}^{\mathbb{Q}}[\cdot]$ *is the expectation under the new measure where* $B_t^Q = B_t + (1-\gamma_1)\sigma_w t$ *is the standard Brownian motion.*

Proof. The expectation part in Lagrangian (4.2) can be rewritten as

$$\begin{aligned}
&\mathbb{E}\left[\int_0^\infty e^{-\beta t}\left\{\frac{\gamma}{1-\gamma_1}\left(\frac{\gamma_1-\gamma}{w_t(1-\gamma_1)}\right)^{\frac{\gamma_1-\gamma}{\gamma}}(y_t w_t^{\gamma_1})^{1-\frac{1}{\gamma}}\, w_t^{1-\gamma_1}\mathbf{1}_{\{\tilde{y}<y_t\}}\right.\right.\\
&\qquad\left.\left.+\left(\frac{\gamma_1}{1-\gamma_1}L^{\frac{\gamma_1-\gamma}{\gamma_1}}(y_t w_t^{\gamma_1})^{1-\frac{1}{\gamma_1}}-L y_t w_t^{\gamma_1}\right)w_t^{1-\gamma_1}\mathbf{1}_{\{0<y_t\le\tilde{y}\}}\right\}dt\right]\\
&=\mathbb{E}\left[\int_0^\infty w_t^{1-\gamma_1}e^{-\beta t}\left\{\frac{\gamma}{1-\gamma_1}\left(\frac{\gamma_1-\gamma}{1-\gamma_1}\right)^{\frac{\gamma_1-\gamma}{\gamma}} z_t^{1-\frac{1}{\gamma}}\mathbf{1}_{\{\tilde{z}<z_t\}}\right.\right.\\
&\qquad\left.\left.+\left(\frac{\gamma_1}{1-\gamma_1}L^{\frac{\gamma_1-\gamma}{\gamma_1}}z_t^{1-\frac{1}{\gamma_1}}-L z_t\right)\mathbf{1}_{\{0<z_t\le\tilde{z}\}}\right\}dt\right],
\end{aligned}$$

where $\tilde{z}=\tilde{y}w_t^{\gamma_1}=\left(\frac{\gamma_1-\gamma}{1-\gamma_1}\right)^{\gamma_1}L^{-\gamma}$. Note that $w_t=we^{(\mu_w-\frac{1}{2}\sigma_w^2)t+\sigma_w B_t}$. If we define an exponential martingale by $\tilde{\eta}_t=e^{-\frac{1}{2}(1-\gamma_1)^2\sigma_w^2 t+(1-\gamma_1)\sigma_w B_t}$, then we have

$$w_t^{1-\gamma_1}=\tilde{\eta}_t w^{1-\gamma_1}e^{(1-\gamma_1)(\mu_w-\frac{1}{2}\sigma_w^2)t+\frac{1}{2}(1-\gamma_1)^2\sigma_w^2 t}.$$

Thus, by Girsanov theorem, the above expectation can be represented under the new measure $\mathbb{Q}$

$$\begin{aligned}
w^{1-\gamma_1}\mathbb{E}^{\mathbb{Q}}&\left[\int_0^\infty e^{-\hat{\beta}t}\left\{\frac{\gamma}{1-\gamma_1}\left(\frac{\gamma_1-\gamma}{1-\gamma_1}\right)^{\frac{\gamma_1-\gamma}{\gamma}} z_t^{1-\frac{1}{\gamma}}\mathbf{1}_{\{\tilde{z}<z_t\}}\right.\right.\\
&\qquad\left.\left.+\left(\frac{\gamma_1}{1-\gamma_1}L^{\frac{\gamma_1-\gamma}{\gamma_1}}z_t^{1-\frac{1}{\gamma_1}}-L z_t\right)\mathbf{1}_{\{0<z_t\le\tilde{z}\}}\right\}dt\right],
\end{aligned}$$

Note that $B_t+(1-\gamma_1)\sigma_w t$ is the standard Brownian motion under the measure $\mathbb{Q}$ and by rearrange the powers we obtain $\hat{\beta}$. □

Let us suppose $\hat{\beta}>0$ and define the expectation value of the right-hand side in Lagrangian (4.3) by $\varphi(z)$ where $z=\lambda w^{\gamma_1}$. Then from the duality relation, we can recover the primal value function in Problem 3.1 by the following theorem.

THEOREM 4.2. *The value function in Problem 3.1 can be obtained from*

$$V(x)=\inf_{\lambda}\left\{w^{1-\gamma_1}\varphi(\lambda w^{\gamma_1})+\lambda\left(x+\frac{wL}{r_w}\right)\right\}.$$

Proof. This is the direct consequence of Legendre transformation inverse formula. See Section 3.8 of Karatzas and Shreve . □

Notice that the dynamics of $z_t = y_t w_t^{\gamma_1}$ is given by

$$
\begin{aligned}
\frac{dz_t}{z_t} &= \left(\beta - r - \frac{1}{2}\theta^2 + \gamma_1\mu_w - \frac{1}{2}\gamma_1\sigma_w^2 + \frac{1}{2}(\gamma_1\sigma_w - \theta)^2\right) dt \\
&\quad + (\gamma_1\sigma_w - \theta)dB_t \\
&= \left(\beta - r + \gamma_1\mu_w - \theta\sigma_w + \frac{1}{2}\gamma_1(1-\gamma_1)\sigma_w^2\right) dt + (\gamma_1\sigma_w - \theta)dB_t^Q \\
&\triangleq \mu_z dt + \sigma_z dB_t^Q
\end{aligned}
$$

If we define the following auxiliary function

$$
\begin{aligned}
\phi(t, z_t) = \mathbb{E}^{\mathbb{Q}} \Bigg[\int_t^\infty e^{-\hat{\beta}(s-t)} \Bigg\{ & \frac{\gamma}{1-\gamma_1}\left(\frac{\gamma_1 - \gamma}{1-\gamma_1}\right)^{\frac{\gamma_1-\gamma}{\gamma}} z_s^{1-\frac{1}{\gamma}} \mathbf{1}_{\{\tilde{z} < z_s\}} \\
&+ \left(\frac{\gamma_1}{1-\gamma_1} L^{\frac{\gamma_1-\gamma}{\gamma_1}} z_s^{1-\frac{1}{\gamma_1}} - L z_s\right) \mathbf{1}_{\{0 < z_s \le \tilde{z}\}} \Bigg\} ds \Bigg],
\end{aligned}
$$

then by Feynman-Kac's formula, $\phi(t, z_t)$ satisfy the following partial differential equation.

$$
\begin{cases}
\mathcal{D}\phi(t,z) + e^{-\hat{\beta}t}\frac{\gamma}{1-\gamma_1}\left(\frac{\gamma_1-\gamma}{1-\gamma_1}\right)^{\frac{\gamma_1-\gamma}{\gamma}} z^{1-\frac{1}{\gamma}} = 0, & \tilde{z} < z, \\
\mathcal{D}\phi(t,z) + e^{-\hat{\beta}t}\left(\frac{\gamma_1}{1-\gamma_1} L^{\frac{\gamma_1-\gamma}{\gamma}} z^{1-\frac{1}{\gamma_1}} - Lz\right) = 0, & 0 < z < \tilde{z},
\end{cases}
$$

where the differential operator is given by $\mathcal{D} = \frac{\partial}{\partial t} + \mu_z \frac{\partial}{\partial z} + \frac{1}{2}\sigma_z^2 \frac{\partial^2}{\partial z^2}$. Obviously, we have $\phi(t,z) = e^{-\hat{\beta}t}\varphi(z)$ and $\varphi(z)$ is the solution to the following ordinary differential equation.

(4.4)

$$
\begin{cases}
-\hat{\beta}\varphi(z) + \mu_z z\varphi'(z) + \frac{1}{2}\sigma_z^2 z^2\varphi''(z) + \frac{\gamma}{1-\gamma_1}\left(\frac{\gamma_1-\gamma}{1-\gamma_1}\right)^{\frac{\gamma_1-\gamma}{\gamma}} z^{1-\frac{1}{\gamma}} = 0, & \tilde{z} < z, \\
-\hat{\beta}\varphi(z) + \mu_z z\varphi'(z) + \frac{1}{2}\sigma_z^2 z^2\varphi''(z) + \frac{\gamma_1}{1-\gamma_1} L^{\frac{\gamma_1-\gamma}{\gamma}} z^{1-\frac{1}{\gamma_1}} - Lz = 0, & 0 < z \le \tilde{z},
\end{cases}
$$

Proposition 4.3. *Let us denote the positive and negative real roots of the quadratic equation*

$$
\frac{1}{2}\sigma_z^2 n^2 + (\mu_z - \frac{1}{2}\sigma_z^2)n - \hat{\beta} = 0,
$$

by n_+ and n_-. Then the function $\varphi(z)$ which satisfies the system of ODEs in (4.4) *is obtained from*

$$
\varphi(z) = \begin{cases}
Az^{n_-} + \frac{\gamma}{K(1-\gamma_1)}\left(\frac{\gamma_1-\gamma}{1-\gamma_1}\right)^{\frac{\gamma_1-\gamma}{\gamma}} z^{1-\frac{1}{\gamma}}, & \tilde{z} < z, \\
Bz^{n_+} + \frac{\gamma_1}{K_1(1-\gamma_1)} L^{\frac{\gamma_1-\gamma}{\gamma}} z^{1-\frac{1}{\gamma_1}} - \frac{L}{r_w} z, & 0 < z \le \tilde{z},
\end{cases}
$$

where the constants K and K_1 are given by

$$K = r + \frac{\beta - r}{\gamma} + \frac{1-\gamma}{2\gamma^2}\theta^2 - \mu_w + \frac{\gamma_1}{\gamma}\mu_w + \frac{\gamma_1(1-\gamma_1)}{2\gamma}\sigma_w^2 - \frac{1-\gamma}{2}\sigma_w^2,$$

$$K_1 = \gamma_1 + \frac{\beta - r}{\gamma_1} - \frac{1-\gamma_1}{2\gamma_1^2}\theta^2$$

and the coefficients A and B are determined by

$$\begin{aligned} A = \frac{1}{n_+ - n_-}\Bigg\{ &-\frac{1-\gamma+\gamma n_+}{K(1-\gamma_1)}\left(\frac{\gamma_1-\gamma}{1-\gamma_1}\right)^{\frac{\gamma_1-\gamma}{\gamma}} \tilde{z}^{1-\frac{1}{\gamma}-n_-} \\ &+ \frac{1-\gamma_1+\gamma_1 n_+}{K_1(1-\gamma_1)} L^{\frac{\gamma_1-\gamma}{\gamma}} \tilde{z}^{1-\frac{1}{\gamma_1}-n_-} - \frac{L(1-n_+)}{r_w}\tilde{z}^{1-n_-} \Bigg\}, \\ B = \frac{1}{n_+ - n_-}\Bigg\{ &-\frac{1-\gamma+\gamma n_-}{K(1-\gamma_1)}\left(\frac{\gamma_1-\gamma}{1-\gamma_1}\right)^{\frac{\gamma_1-\gamma}{\gamma}} \tilde{z}^{1-\frac{1}{\gamma}-n_+} \\ &+ \frac{1-\gamma_1+\gamma_1 n_-}{K_1(1-\gamma_1)} L^{\frac{\gamma_1-\gamma}{\gamma}} \tilde{z}^{1-\frac{1}{\gamma_1}-n_+} - \frac{L(1-n_-)}{r_w}\tilde{z}^{1-n_+} \Bigg\}. \end{aligned}$$

Proof. For $\tilde{z} < z$, we have the particular solution as $\varphi_p(z) = C_1 z^{1-\frac{1}{\gamma}}$ and the homogeneous solution as $\varphi_h(z) = Az^{n_-} + A'z^{n_+}$. Due to the growth condition, $A' = 0$. Similarly, for $0 < z \le \tilde{z}$, the general solution is given by $\varphi(z) = Bz^{n_+} + C_2 z^{1-\frac{1}{\gamma_1}} + C_3 z$. The coefficients A and B are determined by the value-matching and smooth-pasting conditions at $z = \tilde{z}$. □

5. Optimal Controls

From the duality relation in Theorem 4.2, the optimal wealth process at time t is obtained from

$$X_t^* = -w_t\varphi'(z_t) - \frac{w_t L}{r_w}, \tag{5.1}$$

where $z_0 = \lambda w^{\gamma_1}$ satisfies the following algebraic equation $x/w = -\varphi'(\lambda w^{\gamma_1}) - \frac{L}{r_w}$. Moreover, we only consider the case where $\varphi(z)$ is strictly convex, which guarantees the one-to-one correspondence between z_t and X_t^*/w_t. Then we can summarize the optimal consumption, leisure, and investment as follows.

THEOREM 5.1. *Let us denote by $\tilde{x}$ by*

$$\tilde{x} = -n_- A\tilde{z}^{n_- -1} + \frac{1-\gamma}{K(1-\gamma_1)}\left(\frac{\gamma_1-\gamma}{1-\gamma_1}\right)^{\frac{\gamma_1-\gamma}{\gamma}} \tilde{z}^{-\frac{1}{\gamma}} - \frac{L}{r_w},$$

Then, the optimal consumption, leisure, and investments of Problem 3.1 are given by

$$c_t^*/w_t = \begin{cases} \left(\frac{\gamma_1-\gamma}{1-\gamma_1}\right)^{\frac{\gamma_1-\gamma}{\gamma}} (z_t^*)^{-\frac{1}{\gamma}}, & X_t^*/w_t < \tilde{x}, \\ L^{\frac{\gamma_1-\gamma}{\gamma_1}} (z_t^{**})^{-\frac{1}{\gamma_1}}, & \tilde{x} \le X_t^*/w_t, \end{cases}$$

$$l_t^* = \begin{cases} \left(\frac{\gamma_1-\gamma}{1-\gamma_1}\right)^{\frac{\gamma_1}{\gamma}} (z_t^*)^{-\frac{1}{\gamma}}, & X_t^*/w_t < \tilde{x}, \\ L, & \tilde{x} \le X_t^*/w_t, \end{cases}$$

$$\pi_t^*/w_t = \begin{cases} \frac{\gamma\sigma_w-\theta}{\sigma_s}\left(-n_-(n_- -1)Az_t^{*n_- -1} - \frac{1-\gamma}{\gamma K(1-\gamma_1)}\left(\frac{\gamma_1-\gamma}{1-\gamma_1}\right)^{\frac{\gamma_1-\gamma}{\gamma}} z_t^{*-\frac{1}{\gamma}}\right) & \\ \quad +\frac{\sigma_w}{\sigma_s}\left(-n_-Az_t^{*n_- -1} + \frac{1-\gamma}{K(1-\gamma_1)}\left(\frac{\gamma_1-\gamma}{1-\gamma_1}\right)^{\frac{\gamma_1-\gamma}{\gamma}} z_t^{*-\frac{1}{\gamma}} - \frac{L}{r_w}\right), & X_t^*/w_t < \tilde{x}, \\ \frac{\gamma\sigma_w-\theta}{\sigma_s}\left(-n_+(n_+ -1)Bz_t^{*n_+ -1} - \frac{1}{\gamma_1 K}L^{\frac{\gamma_1-\gamma}{\gamma}} z_t^{**-\frac{1}{\gamma_1}}\right) & \\ \quad +\frac{\sigma_w}{\sigma_s}\left(-n_+Bz_t^{**n_+ -1} + \frac{1}{K_1}L^{\frac{\gamma_1-\gamma}{\gamma}} z_t^{**-\frac{1}{\gamma_1}}\right), & \tilde{x} \le X_t^*/w_t, \end{cases}$$

where z_0^* and z_0^{**} are determined by the following algebraic equations.

$$x/w = -n_-Az_0^{*n_- -1} + \frac{1-\gamma}{K(1-\gamma_1)}\left(\frac{\gamma_1-\gamma}{1-\gamma_1}\right)^{\frac{\gamma_1-\gamma}{\gamma}} z_0^{*-\frac{1}{\gamma}} - \frac{L}{r_w},$$

and

$$x/w = -n_+Bz_0^{**n_+ -1} + \frac{1}{K_1}L^{\frac{\gamma_1-\gamma}{\gamma}} z_0^{**-\frac{1}{\gamma_1}}.$$

Proof. The optimal consumption and leisure are already given while we derive the Lagrangian (4.2). We can determine the optimal investment by comparing the diffusion terms of wealth dynamics. More specifically, the diffusion term of the wealth process in (5.1) is given by $-(\varphi'(z_t) - L/r_w)\sigma_w w_t - w_t\varphi''(z_t)\sigma_z z_t$. From the wealth dynamics in (2.1), this diffusion term should be same as $\sigma_s\pi_t$. □

Notice that the labor supply and labor income are obtained from $L - l_t^*$ and $w_t(L - l_t^*)$

6. Conclusion

We investigate the optimal consumption, leisure, and investment problem when the income flow for a unit labor supply is stochastic. By applying martingale method, we can derive the explicit solutions even when we have two state variables.

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