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A NOTE ON WEAK EXPANSIVE HOMEOMORPHISMS ON A COMPACT METRIC SPACE

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Abstract. In this paper we introduce the notion of the expansivity for homeomorphisms on a compact metric space and study some properties concerning weak expansive homeomorphisms. Also, we give some examples to illustrate our results.

1. Introduction

Utz [8] introduced the original notion of various expansive properties for homeomorphisms on a compact metric space and proved some general properties related to asymptotic trajectories, the cardinality of the set of periodic points and the powers of expansive homeomorphisms. Many authors [3–7] have extensively developed the variant theory related to expansive properties in various directions, e.g., entropy-expansiveness, continuum-wise expansiveness and measure expansiveness, etc.

Achigar et al. [1] introduced the notions of orbit expansiveness and refinement expansiveness for homeomorphisms on a non-Hausdorff space and studied expansive dynamical systems from the viewpoint of general topology. Ahn and Kim [2] introduced the notion of weak measure expansiveness for homeomorphisms on a measurable compact metric space by using the finite δ-partition of a compact metric space. They showed that a diffeomorphism on a compact smooth manifold is C^1-stably weak measure expansive if and only if it is Ω-stable. Moreover, they showed that C^1-generically, if a diffeomorphism is weak measure expansive, then it satisfies both Axiom A and the no cycle condition.

In this paper we introduce the notion of weak expansive homeomorphisms on a compact metric space and study some properties related

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to the restriction, invariance and product factors in weak expansive systems. Also, we give some examples to illustrate the notion of weak expansivity.

2. Main results

In this section we investigate some properties of weak expansive homeomorphisms on a compact metric space. Also, we give some examples concerning our results.

We consider a dynamical system which is a pair (X, f), where X is a topological(or compact metric) space and $f : X \to X$ is a homeomorphism. We need to recall some notions and definitions for our main results. Let $\mathcal{U} = \{U_i \subset X \mid i \in \Lambda\}$ be a collection of nonempty subsets of X with $\bigcup_{i\in\Lambda} U_i = X$, and $x, y \in X$. Then we write $D(x, y) \leq \mathcal{U}$ if x and y belong to the same element of $\mathcal{U}$. Otherwise, we write $D(x, y) > \mathcal{U}$.

We recall the notion of a δ-partition of a compact metric space X.

Definition 2.1. We say that a finite collection $P = \{A_1, \ldots, A_k\}$ of nonempty subsets of X is a *δ-partition*$(\delta > 0)$ of X if

(i) $A_i's$ are disjoint, and $\cup_{i=1}^k A_i = X$;
(ii) $\mathrm{diam}(A_i) \leq \delta$ and $\mathrm{Int}(A_i) \neq \emptyset$ for all $i = 1, \cdots, k$.

Here $\mathrm{diam}(A)$ and $\mathrm{Int}(A)$ denote the diameter of A and interior of A, respectively.

Notice that if a cover $\mathcal{U}$ of X is a partition, then we rewrite $D(x, y) \leq \mathcal{U}$ as follows $y \in \mathcal{U}(x)$, where $\mathcal{U}(x)$ denotes the element of the partition $\mathcal{U}$ containing x.

We introduce the notion of weak expansiveness of homeomorphisms on a compact metric space.

Definition 2.2. We say that a homeomorphism $f : X \to X$ is *weakly expansive* if there exists a finite δ-partition P of X such that $\Gamma_P^f(x) = \{x\}$ for every $x \in X$, where $\Gamma_P^f(x)$ is given by

$$\begin{aligned} \Gamma_P^f(x) &= \{y \in X \mid D(f^n(x), f^n(y)) \leq P, \forall n \in \mathbb{Z}\} \\ &\text{or} \\ &= \{y \in X \mid f^n(y) \in P(f^n(x)), \forall n \in \mathbb{Z}\}, \end{aligned}$$

where $P(x)$ denotes the element of the partition P containing x.

EXAMPLE 2.3. *Let $X = \{x_1, \ldots, x_n\}$ be a finite set with the discrete topology and $f : X \to X$ be a homeomorphism. Since there is a δ-partition $P = \{\{x_1\}, ...\{x_n\}\}$ of X such that $\Gamma_P^f(x) = \{x\}$ for every $x \in X$, f is weakly expansive.*

EXAMPLE 2.4. *[2, Example 2.3] Let $X = S^1$ be the circle of $\mathbb{R}^2$ and $h_\alpha : S^1 \to S^1$ be an irrational rotation given by $h_\alpha(x) = x + \alpha$ for an irrational number $\alpha \in \mathbb{R}$. Then h_α is weakly expansive.*

EXAMPLE 2.5. *The identity map $Id_X : X \to X$ of an infinite compact metric space X is not weakly expansive.*

Proof. Taking $\delta = \operatorname{diam}(X)$, let $P = \{A_1, \cdots, A_k\}$ be any finite δ-partition of X. Then there is $x \in X$ and an infinite set $A_i \in P$ containing x such that $\{x\} \neq \Gamma_P^{Id_X}(x) = P(x) = A_i$ for some $i = 1, \cdots, k$. Hence the identity map Id_X is not weakly expansive. □

PROPOSITION 2.6. *If X is an infinite set with the cofinite topology, then are no weakly expansive homeomorphisms on X.*

Proof. Suppose that there is a weakly expansive homeomorphism $f : X \to X$. Then exists a finite δ-partition $P = \{A_i\}_{i=1}^k$ of X such that $\Gamma_P^f(x) = \{x\}$ for every $x \in X$. In view of a finite δ-partition P, there are nonemtpy finite subsets B_i of X such that

$$\emptyset \neq X \setminus B_i = \operatorname{Int}(A_i) \subset A_i \subset X, \ i = 1, \cdots, k.$$

In case $k \geq 2$, say, $k = 2$, if $A_1 = X \setminus B_1$, then A_2 should be finite. Thus $\operatorname{Int}(A_2) = \emptyset$. This is absurd. Hence there exists a unique finite δ-partition $P = \{X\}$ of X such that

$$\begin{aligned} \Gamma_P^f(x) &= \{y \in X | D(f^n(x), f^n(y)) \leq P, \forall n \in \mathbb{Z}\} \\ &= X \end{aligned}$$

for every $x \in X$. This contradicts the weak expansiveness of f. □

Since the cofinite topology on a nonempty finite set coincides with the discrete topology, we can easily obtain the following result by Example 2.3.

COROLLARY 2.7. *If X is a finite set with the cofinite topology, then any homeomorphism $f : X \to X$ is weakly expansive.*

In light of Proposition 2.6, for the definition of weak expansiveness for homeomorphisms on a space X, it is natural to assume that the space X is a compact metric space.

Achigar et al. [1] studied expansive dynamical systems from the viewpoint of general topology via the notions of orbit and refinement expansiveness for homeomorphisms on topological spaces.

Definition 2.8. [1] We say that a homeomorphism $f : X \to X$ is *orbit expansive* if there exists a finite open cover $\mathcal{U}$ of X such that $D(f^n(x), f^n(y)) \leq \mathcal{U}$ for all $n \in \mathbb{Z}$ thus implies $x = y$. In this case we call $\mathcal{U}$ an *o-expansive cover.*

Theorem 2.9. *Let X be a compact metric space. If a homeomorphism $f : X \to X$ is orbit expansive, then f is weakly expansive.*

Proof. If $\mathcal{U} = \{U_1, \ldots, U_k\}$ is an o-expansive open cover of X for f. Then we can construct a finite δ-partition $P = \{A_i\}_{i=1}^k$ as follows

$$A_i = U_i \setminus \cup_{j=1}^{i-1} U_j, \quad i = 1, \cdots, k,$$

where $\cup_{j=1}^{0} U_j = \emptyset$. We can take $\delta = \mathrm{diam}(X)$ to get $\mathrm{diam}(A_i) \leq \delta$ for every $A_i \in P$ and $\mathrm{int}(A_i) \neq \emptyset$ for each $i = 1, \cdots, k$. Then P is a finite δ-partition of X. From $\Gamma_P^f(x) \subset \Gamma_{\mathcal{U}}^f(x)$ for every $x \in X$, f is weakly expansive. □

Notice that the weakly expansive irrational rotation $h_\alpha : S^1 \to S^1$ given by Example 2.4 is not orbital expansive. Thus we see that the converse of Theorem 2.9 is not true.

Theorem 2.10. *Suppose that a homeomorphism $f : X \to X$ is weakly expansive with a finite δ-partition P of X. Let Y be a subset of a compact metric space X such that $f(Y) = Y$. Furthermore, suppose that $P_Y = \{C \mid C = A \cap Y \neq \emptyset, A \in P\}$ is a finite δ-partition P_Y of Y. Then $f|_Y : Y \to Y$ is weakly expansive.*

Proof. It is easy to show that $\Gamma_{P_Y}^{f|_Y}(y) \subset \Gamma_P^f(y)$ for every $y \in Y$. In view of the weak expansiveness of f, we have

$$\Gamma_{P_Y}^{f|_Y}(y) = \{y\}$$

for every $y \in Y$. □

Notice that the converse of Theorem 2.10 does not hold in general.

Remark 2.11. If P is a finite δ-partition of X and $Y \subset X$, then P_Y given by the assumption of Theorem 2.10 need not be a finite δ-partition of Y. Say, let $Y = (\frac{1}{4}, \frac{1}{2}] \subset X = [0, 1]$. Then $P = \{[0, \frac{1}{2}), [\frac{1}{2}, 1]\}$ is a finite δ-partition of X, but $P_Y = \{[0, \frac{1}{2}) \cap Y = (\frac{1}{4}, \frac{1}{2}), [\frac{1}{2}, 1] \cap Y = \{\frac{1}{2}\}\}$ and $\mathrm{Int}(\{\frac{1}{2}\}) = \emptyset$. So P_Y is not a finite δ-partition of Y.

We obtain the following result that is adapted from [2, Theorem 2.7].

Proposition 2.12. *Let X, Y be compact metric spaces and a homeomorphism $f : X \to X$ be weakly expansive. If a map $\phi : X \to Y$ is homeomorphic, then the map $\phi \circ f \circ \phi^{-1} : Y \to Y$ is weakly expansive.*

Proof. From the weak expansivity of f, there is a finite δ_1-partition $P = \{A_i\}_{i=1}^k$ of X such that $\Gamma_P^f(x) = \{x\}$ for every $x \in X$. Let $\delta_2 = \max_{1\le i\le k}\{\text{diam}\phi(A_i)\}$ and $\phi(P) = \{\phi(A_i)\}_{i=1}^k$. Since ϕ is a homeomorphism and $\emptyset \neq \text{int}(A_i) \subset A_i$ for every $i = 1, \cdots, k$, we have

$$\emptyset \neq \phi(\text{Int}(A_i)) = \text{Int}(\phi(\text{Int}(A_i))) \subset \text{Int}(\phi(A_i)),\ i = 1, \cdots, k.$$

Thus $\phi(P) = \{\phi(A_i)\}_{i=1}^k$ is a finite δ_2-partition of Y.

On the other hand, the rest of proof can be proved in a similar argument as the proof of [2, Theorem 2.7]. That is, there is a finite δ_2-partition $\phi(P)$ of Y such that $\Gamma_{\phi(P)}^{\phi\circ f\circ\phi^{-1}}(z) = \{z\}$ for every $z \in Y$. This completes the proof. □

Theorem 2.13. *Let $X_1 \times X_2$ be a product metric compact space of compact metric spaces X_1 and X_2 and $f_i : X_i \to X_i$ be a homeomorphism for each $i = 1, 2$. If each homeomorphism $f_i (i = 1, 2)$ is weakly expansive, then the homeomorphism $f_1 \times f_2 : X_1 \times X_2 \to X_1 \times X_2$ given by $(f_1 \times f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2))$ is also weakly expansive.*

Proof. Suppose that f_i is weakly expansive for each $i = 1, 2$. Then, for each $i = 1, 2$, there is a finite δ_i-partition $P_i = \{A_{j_i}^i | 1 \le j_i \le m_i\}$ of X_i such that $\Gamma_{P_i}^{f_i}(x_i) = \{x_i\}$ for every $x_i \in X_i$, respectively. Define a partition $P = P_1 \times P_2$ of the product space $X_1 \times X_2$ by

$$P = P_1 \times P_2 = \{A_{j_1}^1 \times A_{j_2}^2 \mid A_{j_i}^i \in P_i, 1 \le j_i \le m_i, i = 1, 2\}.$$

Taking $\delta = \max\{\delta_i \mid i = 1, 2\}$, P is a finite δ-partition of $X_1 \times X_2$. We claim that there is a finite δ-partition $P = P_1 \times P_2$ of $X_1 \times X_2$ such $\Gamma_P^{f_1\times f_2}(x_1, x_2) = \Gamma_{P_1}^{f_1}(x_1) \times \Gamma_{P_2}^{f_2}(x_2)$ for every $(x_1, x_2) \in X_1 \times X_2$. Indeed, for every $(x_1, x_2) \in X_1 \times X_2$, we have

$$\begin{aligned} (y_1, y_2) \in\ & \Gamma_P^{f_1\times f_2}(x_1, x_2) \\ \Leftrightarrow\ & (f_1 \times f_2)^n(y_1, y_2) \in P((f_1 \times f_2)^n(x_1, x_2)),\ n \in \mathbb{Z} \\ \Leftrightarrow\ & (f_1^n(y_1), f_2^n(y_2) \in P_1((f_1^n(x_1)) \times P_2(f_2^n(x_2)),\ n \in \mathbb{Z} \\ \Leftrightarrow\ & y_1 \in \Gamma_{P_1}^{f_1}(x_1),\ y_2 \in \Gamma_{P_2}^{f_2}(x_2),\ n \in \mathbb{Z} \\ \Leftrightarrow\ & (y_1, y_2) \in \Gamma_{P_1}^{f_1}(x_1) \times \Gamma_{P_2}^{f_2}(x_2). \end{aligned}$$

Since f_1 and f_2 are weakly expansive, it follows from the claim that $\Gamma_P^{f_1\times f_2}(x_1,x_2)=\{(x_1,x_2)\}$ for every $(x_1,x_2)\in X_1\times X_2$. Hence $f_1\times f_2$ is weakly expansive. This completes the proof. □

For two finite partitions $P=\{A_1,\cdots,A_k\}$ and $Q=\{B_1,\cdots,B_l\}$ we denote $P\vee Q=\{A_i\cap B_i|1\le i\le k, 1\le j\le l\}$ and $f^{-n}(P)=\{f^{-n}(A_1),\cdots,f^{-n}(A_k)\}$.

We obtain the following result which is revised in the proof of Lemma 2.5 in [2].

THEOREM 2.14. *Let $f:X\to X$ be a homeomorphism and $k(\neq 0,1)$ be an integer. Then f is weakly expansive if and only if f^k is weakly expansive.*

Proof. Suppose that f is weakly expansive. Then there exists a finite δ-partition $P=\{A_1,\cdots,A_l\}$ of X such that $\Gamma_P^f(x)=\{x\}$ for every $x\in X$. Letting $Q=\vee_{i=0}^{k-1}f^{-i}(P)$, then Q is also a finite δ-partition P of X and is represented as follows

$$Q=\{\cap_{i=0}^{k-1}f^{-i}(A_{l_i})\mid A_{l_i}\in P, 1\le l_i\le l\}.$$

We claim that $\Gamma_Q^{f^k}(x)\subset\Gamma_P^f(x)$ for every $x\in X$. Indeed, if $y\in\Gamma_Q^{f^k}(x)$, then

$$\begin{aligned}
f^{nk}(y) &\in Q(f^{nk}(x)),\ \forall n\in\mathbb{Z}\\
&\Rightarrow \{f^{nk}(x),f^{nk}(y)\}\subset\cap_{i=0}^{k-1}f^{-i}(A_{n_i})),\ \forall n\in\mathbb{Z}\\
&\Rightarrow \{f^{nk+i}(x),f^{nk+i}(y)\}\subset A_{n_i},\ i=0,\cdots,k-1,\ \forall n\in\mathbb{Z}\\
&\Rightarrow f^{nk+i}(y)\in P(f^{nk+i}(x))),\ i=0,\cdots,k-1,\ \forall n\in\mathbb{Z}\\
&\Rightarrow f^m(y)\in P(f^m(x)),\ \forall m\in\mathbb{Z}\\
&\Rightarrow y\in\Gamma_P^f(x),
\end{aligned}$$

where $A_{n_i}\in P$ for $1\le n_i\le l$ and for each $m\in\mathbb{Z}$, there is $n\in\mathbb{Z}$ such that $m=nk+i$ for some $i=0,\cdots,k-1$. Since $\Gamma_Q^{f^k}(x)\subset\Gamma_P^f(x)$ for every $x\in X$, then there is a finite δ-partition Q of X for f^k such that $\Gamma_Q^{f^k}(x)=\{x\}$ for every $x\in X$. Hence f^k is weakly expansive.

Conversely, suppose that f^k is weak expansive. Then there is a finite δ-partition Q of X such $\Gamma_Q^{f^k}(x)=\{x\}$ for every $x\in X$. Also, we see that $\Gamma_Q^f(x)\subset\Gamma_Q^{f^k}(x)$ for every $x\in X$. Thus f is weakly expansive. This completes the proof. □

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