JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **33**, No. 1, February 2020
http://dx.doi.org/10.14403/jcms.2020.33.1.19

EXISTENCE OF POSITIVE SOLUTIONS OF PREDATOR-PREY SYSTEMS WITH DEGENERATE DIFFUSION RATES

Kimun Ryu*

Abstract. We discuss the coexistence of positive solutions to certain strongly-coupled predator-prey elliptic systems under the homogeneous Dirichlet boundary conditions. The sufficient condition for the existence of positive solutions is expressed in terms of the spectral property of differential operators of nonlinear Schrödinger type which reflects the influence of the domain and nonlinearity in the system. Furthermore, applying the obtained results, we investigate the sufficient conditions for the existence of positive solutions of a predator-prey system with degenerate diffusion rates.

1. Introduction

In this paper, we investigate the existence of positive solution to the following strongly-coupled nonlinear elliptic system:

$$\tag{1.1} \begin{cases} -\Delta[\varphi(u,v)u] = uf(x,u,v) & \\ -\Delta[\psi(u,v)v] = vg(x,u,v) & \text{in } \Omega, \\ (u,v) = (0,0) & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. The functions φ, ψ, f, g satisfy certain conditions which will be imposed later. We say that this steady-state system has a positive solution (u, v) if $u(x) > 0$ and $v(x) > 0$ for all $x \in \Omega$. The existence of a positive solution (u, v) to this system is called a *positive coexistence*.

A great amount of researches have been achieved relating to the system (1.1) under various boundary conditions. In [4], Ghoreishi and

Received December 13, 2019; Accepted December 30, 2019.
2010 Mathematics Subject Classification: 35J60.
Key words and phrases: predator-prey systems, positive solutions, degenerate diffusion rates.
* This work was supported by the research grant of Cheongju University(2018.03.01. - 2020.02.28.).

Logan provided the sufficient conditions for the positive coexistence to the strongly-coupled elliptic system under Robin boundary conditions:

$$\begin{cases} -\Delta u = uf(x,u,v) & \\ -\Delta v = vg(x,u,v) & \text{in } \Omega, \end{cases}$$

using the method of decomposing operator and the index theory. The predator-pry and the competition interactions were considered.

In [6], Leung and Feng found positive solutions for the degenerate elliptic system between appropriate upper-lower solutions under certain conditions of f, g and ψ

$$\begin{cases} -\Delta\psi(u) = f(x,u,v) & \\ -\Delta\psi(v) = g(x,u,v) & \text{in } \Omega, \\ (u,v) = (0,0) & \text{on } \partial\Omega. \end{cases}$$

In [3], the authors investigated the *coexistence state* (i.e., $u(x)$ and $v(x)$ are nonnegative and nontrivial) for the system with degenerate diffusions under certain conditions of h, k, ψ and φ

$$\begin{cases} -\Delta\psi(U(x)) = U(x)h(x,U(x),V(x)) & \\ -\Delta\varphi(x,V(x)) = V(x)k(x,U(x),V(x)) & \text{in } \Omega, \\ U(x) = V(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

using the method of upper-lower solutions. In [12], Shigesada *et al* proposed the competition elliptic system with the linear diffusions and growth rates

$$(1.2)\quad \begin{cases} -\Delta[(\alpha_1 + \beta_{11}u + \beta_{12}v)u] = u(a_1 - b_{11}u - b_{12}v) & \\ -\Delta[(\alpha_2 + \beta_{21}u + \beta_{22}v)v] = v(a_2 - b_{21}u - b_{22}v) & \text{in } \Omega, \end{cases}$$

where α_i, β_{ij}, a_i and b_{ij} are positive constants.

In [10], Ruan considered the coupled competition elliptic system (1.2) under Dirichlet boundary conditions by using the index theory. Furthermore, he gave the result that the system has positive solutions when β_{12} and β_{21} are sufficiently large. In [7], Lou and Ni investigated the existence of non-constant solutions of (1.2) under Neumann boundary conditions employing the method of Lyapunov functional and degree theory. For more references for the elliptic system relating to the system (1.2), one can see [2], [5], [6], [8], [9], the references therein.

We are concerned with the positive coexistence of the predator-prey system (1.1). This paper is organized as follows. In Section 2, some variational property of the corresponding eigenvalue problem to our system and some known lemmas which are useful throughout this paper, are provided. In Section 3, we give sufficient conditions for the coexistence of positive solutions of system (1.1) for predator-prey interactions by using the method of the decomposing operator and the theory of fixed point index on positive cones in a Banach space. Also the existence of positive solution to scalar equation is investigated. In the final section, we apply the obtained results to a Lotka-Volterra predator-prey system with degenerate diffusions.

2. Known-Lemmas and Single Equation

In this section, we state some known results and investigate the existence and uniqueness of positive solution for a single equation.

For $a(x) > 0$ in $C^2(\overline{\Omega})$ and $b(x) \in L^\infty(\Omega)$,

$$\begin{cases} \Delta[a(x)u] + b(x)u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

has a principal eigenvalue corresponding to the unique positive principal eigenfunction(See [11]). We denote the principal eigenvalue of (2.1) by $\lambda_1(\Delta a(x) + b(x))$.

The following two lemmas can be found in [11].

Lemma 2.1. *Assume that* $\frac{b_1(x)}{a_1(x)} > \frac{b_2(x)}{a_2(x)}$*, where* $a_i(x) > 0$ *in* $C^2(\overline{\Omega})$ *and* $b_i(x) \in L^\infty(\Omega)$ *for* $i = 1, 2$.

(i) If $\lambda_1(\Delta a_1(x) + b_1(x)) \leq 0$*, then* $\lambda_1(\Delta a_2(x) + b_2(x)) < 0$.

(ii) If $\lambda_1(\Delta a_2(x) + b_2(x)) \geq 0$*, then* $\lambda_1(\Delta a_1(x) + b_1(x)) > 0$.

Lemma 2.2. *(i)* $\lambda_1(\Delta a(x) + b(x))$ *is increasing in* $b(x)$.

(ii) If $\lambda_1(\Delta a(x) + b(x)) > 0$*, then* $\lambda_1(\Delta a(x) + b(x))$ *is decreasing in* $a(x)$.

(iii) If $\lambda_1(\Delta a(x) + b(x)) < 0$*, then* $\lambda_1(\Delta a(x) + b(x))$ *is increasing in* $a(x)$.

Lemma 2.3. *Let* $a(x) > 0$ *in* $C^2(\overline{\Omega})$, $b(x) \in L^\infty(\Omega)$ *and* $u \geq 0$, $u \not\equiv 0$ *in* Ω *with* $\kappa\frac{\partial u}{\partial n} + \tau u = 0$ *on* $\partial\Omega$.

(i) If $0 \not\equiv (\Delta a(x) + b(x))u \geq 0$*, then* $\lambda_1(\Delta a(x) + b(x)) > 0$.

(ii) If $0 \not\equiv (\Delta a(x) + b(x))u \leq 0$*, then* $\lambda_1(\Delta a(x) + b(x)) < 0$.

(iii) If $(\Delta a(x) + b(x))u \equiv 0$*, then* $\lambda_1(\Delta a(x) + b(x)) = 0$.

Lemma 2.4. *Let* $a(x) > 0$ *in* $C^2(\overline{\Omega})$, $b(x) \in L^\infty(\Omega)$ *and* M *be a positive constant such that* $b(x) + Ma(x) > 0$ *for all* $x \in \overline{\Omega}$.

If $\lambda_1\,(\Delta a(x) + b(x)) > 0$*, then* $r\left[\frac{1}{a(x)}(-\Delta + M)^{-1}(b(x) + Ma(x))\right] > 1$.

NOTATION 2.5. (i) $\|\cdot\|_{L^m}$ denotes the usual L^m*−norm* in Ω for a positive integer m and $\|\cdot\|_\infty$ denotes the usual *sup-norm* in $C(\overline{\Omega})$.

(ii) $[[u_1,u_2]]$ is an *ordered interval* in $C^2(\Omega)$, i.e., $[[u_1,u_2]] := \{u \in C^2(\Omega) : u_1(x) \le u(x) \le u_2(x) \text{ for all } x \in \Omega\}$.

DEFINITION 2.6. $\varphi(x,\xi) \in \mathcal{G}_\xi$ if and only if $\varphi(x,\xi) \in C(\overline{\Omega}\times[0,\infty))$ which satisfies the followings:

(G1) $\varphi(x,0)$ is C^2-function in x with $\varphi(x,0) > 0$ for all $x \in \overline{\Omega}$;

(G2) $\varphi(x,\xi)$ is C^2-function in ξ with $\varphi_\xi(x,\xi) \ge 0$ for all $(x,\xi) \in \overline{\Omega}\times[0,\infty)$.

DEFINITION 2.7. $f(x,\xi) \in \mathcal{F}_\xi$ if and only if $f(x,\xi) \in C(\overline{\Omega}\times[0,\infty))$ satisfies the followings:

(F1) $f(x,\xi)$ is $C^\alpha-$function in x, where $0 < \alpha < 1$;

(F2) $f(x,\xi)$ is C^1-function in ξ with $f_\xi(x,\xi) < 0$ on $(x,\xi) \in \overline{\Omega}\times[0,\infty)$;

(F3) $f(x,\xi) < 0$ on $(x,\xi) \in \overline{\Omega}\times[C_0,\infty)$ for some positive constant C_0.

Now we consider the single equation for $\varphi \in \mathcal{G}_u$ and $\psi \in \mathcal{F}_u$:

$$\begin{cases} -\Delta[\varphi(x,u)u] = uf(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

where Ω is a bounded connected domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$.

REMARK 2.8. If $\varphi(x,u) \in \mathcal{G}_u$, then the map $G_\varphi(u) = \varphi(x,u)u$ has a continuous inverse in u since $\frac{\partial G_\varphi}{\partial u} = \varphi_u(x,u)u + \varphi(x,u) > 0$ for all $(x,u) \in \overline{\Omega}\times[0,\infty)$. Define this inverse map as $G_\varphi^{-1}(u)$. Additionally, we can see that $\frac{\partial}{\partial u}(G_\varphi^{-1}(u)) > 0$ for all $(x,u) \in \overline{\Omega}\times[0,\infty)$ and $G_\varphi^{-1}(u) \in C^2(\overline{\Omega})$ if $u \in C^2(\overline{\Omega})$.

DEFINITION 2.9. A function $u(x)$ is called a *solution* of (2.2) if $\varphi(x,u)u \in C^{2,\alpha}(\overline{\Omega})$, where $0 < \alpha < 1$ and $u(x)$ satisfies (2.2).

DEFINITION 2.10. A function $\hat{u}(x)$ is called an *upper solution* of (2.2) if $\varphi(x,\hat{u})\hat{u} \in C^{2,\alpha}(\overline{\Omega})$ and $\hat{u}(x)$ satisfies the following conditions:

$$-\Delta[\varphi(x,\hat{u})\hat{u}] \ge \hat{u}f(x,\hat{u}) \text{ in } \Omega, \quad \hat{u} \ge 0 \text{ on } \partial\Omega. \tag{2.3}$$

Similarly, we define a *lower solution* of (2.2) by reversing the inequalities in (2.3).

Definition 2.11. Let X be a nonempty subset of some ordered set Y. A fixed point x of a map $f : X \to Y$ is called *maximal* if every fixed point y of f in X satisfies $x \geq y$.

For $\varphi \in \mathcal{G}_u$ and $f \in \mathcal{F}_u$, define an operator $F : [[0, \hat{u}]] \to C(\Omega)$ as $F = G_\varphi^{-1} \circ H$. Here H is a positive monotone increasing compact map given by $Hu := (-\Delta + M)^{-1}[(f(x,u) + M\varphi(x,u))u]$, where M is a positive constant large enough so that $(f(x,u) + M\varphi(x,u))u$ is monotone increasing with respect to u. The existence of such constant M follows from the assumption $\varphi \in \mathcal{G}_u$. Then the operator F is also a positive monotone increasing compact map. We may observe that u is a solution of (2.2) if and only if u is a fixed point of F.

We give the existence and uniqueness theorem for (2.2) through the following theorem.

Theorem 2.12. *For $\varphi(x,u) \in \mathcal{G}_u$ and $f(x,u) \in \mathcal{F}_u$, consider the problem (2.2).*

(i) If $\lambda_1(\Delta\varphi(x,0) + f(x,0)) \leq 0$, then (2.2) has no positive solution.

(ii) If $\lambda_1(\Delta\varphi(x,0) + f(x,0)) > 0$, then (2.2) has a unique positive solution.

Proof. (i) If $u(x)$ is a positive solution of (2.2), then $\lambda_1(\Delta\varphi(x,u) + f(x,u)) = 0$ by Lemma 2.3 (iii). If we take a positive constant P such that $f(x,0) + P > 0$ for all $x \in \overline{\Omega}$, then, we have $\frac{f(x,u)}{\varphi(x,u)} < \frac{f(x,u)+P}{\varphi(x,u)}$, and so $\lambda_1(\Delta\varphi(x,u) + f(x,u) + P) > 0$ by Lemma 2.1 (ii). Thus Lemma 2.2 (i) and (ii) imply $\lambda_1(\Delta\varphi(x,u)+f(x,u)+P) < \lambda_1(\Delta\varphi(x,0)+f(x,0)+P)$. From this fact, we can derive $0 = \lambda_1(\Delta\varphi(x,u)+f(x,u)) < \lambda_1(\Delta\varphi(x,0)+f(x,0))$.

(ii) First we prove the existence of positive solution of (2.2). If we construct an upper solution $\hat{u}(x)$ of (2.2), then adding $M\varphi(x,\hat{u})\hat{u}$ and applying $G_\varphi^{-1} \circ (-\Delta + M)^{-1}$ both sides, we have $F(\hat{u}) \leq \hat{u}$. Also note that $\overline{u} = 0$ is a solution of (2.2), and so we have $F'(\overline{u}) = F'(0) = \frac{1}{\varphi(x,0)}(-\Delta + M)^{-1}(f(x,0) + M\varphi(x,0))$ by the calculation. So Lemma 2.4 with the given assumption concludes that $r(F'(0)) > 1$. Finally applying Theorem 7.6 in [1], we can conclude that there is a positive maximal solution $u \gg 0$ in $[[0, \hat{u}]]$.

To construct an upper solution $\hat{u}(x)$ of (2.2), let $G_\varphi^{-1}(u)$ be the continuous inverse of the map $G_\varphi(u) = \varphi(x,u)u$ in u which is defined in

Remark 2.8. Denote $\hat{u} = G_\varphi^{-1}(C)$, where C is a sufficiently large positive constant such that $f(x, G_\varphi^{-1}(C)) \leq 0$ for all $x \in \overline{\Omega}$. The existence of such constant $C > 0$ follows from the fact that $\frac{\partial}{\partial u}(G_\varphi^{-1}(u)) > 0$ for all $u \geq 0$ and the assumption $f \in \mathcal{F}_u$ and $\varphi \in \mathcal{G}_u$. Since $-\Delta[\varphi(x, G_\varphi^{-1}(C))G_\varphi^{-1}(C)] = -\Delta C = 0$, we have

$$\begin{cases} -\Delta[\varphi(x, G_\varphi^{-1}(C))G_\varphi^{-1}(C)] \geq G_\varphi^{-1}(C) f(x, G_\varphi^{-1}(C)) & \text{in } \Omega, \\ G_\varphi^{-1}(C) \geq 0 & \text{on } \partial\Omega, \end{cases}$$

and so $\hat{u} = G_\varphi^{-1}(C)$ is a positive upper solution of (2.2).

Now we show the uniqueness of the positive solution of (2.2). Let u_m be the maximal solution of (2.2) and u_1 be another positive solution of (2.2). Then $u_1 \leq u_m$ by the maximality of u_m. By Lemma 2.3 (iii), we must have $\lambda_1(\Delta\varphi(x, u_1) + f(x, u_1)) = \lambda_1(\Delta\varphi(x, u_m) + f(x, u_m)) = 0$. Contrariwise, assume that $u_1 < u_m$. If we take a positive constant P such that $\frac{f(x,u)+P}{\varphi(x,u)}$ is strictly increasing with respect to u in $[[0, \hat{u}]]$, then we have $\frac{f(x,u_1)+P}{\varphi(x,u_1)} > \frac{f(x,u_m)+P}{\varphi(x,u_m)}$, and thus we can obtain $\lambda_1(\Delta\varphi(x, u_1) + f(x, u_1)) > 0$ by the same way as in the proof of (i), which derives a contradiction. This completes the proof. □

Lemma 2.13. *Any positive solution $u(x)$ of (2.2) with the same assumptions as in Theorem 2.12 has an a priori bound.*

Proof. Assume $\varphi(x, u)u$ attains its maximum at $x = x_0$ over $\overline{\Omega}$, i.e., $\max_{x \in \overline{\Omega}}\{\varphi(x, u)u\} = \varphi(x_0, u(x_0))u(x_0)$. Since $u = 0$ on $\partial\Omega$, $\varphi(x, u)u = 0$ on $\partial\Omega$, and so $x_0 \in \Omega$. Thus we have $-\Delta[\varphi(x, u(x_0))u(x_0)] \geq 0$, which implies $f(x_0, u(x_0)) \geq 0$. From the assumption $f \in \mathcal{F}_u$, we can find a positive constant C_0 such that $f(x, C_0) < 0$ for all $x \in \overline{\Omega}$. Since $f(x_0, u(x_0)) \geq 0 \geq f(x_0, C_0)$, we can get $u(x_0) \leq C_0$. Thus we have $\varphi(x, u)u \leq \max_{x \in \overline{\Omega}}\{\varphi(x, u)u\} \leq \varphi(x_0, C_0)C_0$, and so $u(x) \leq G_\varphi^{-1}(\varphi(x_0, C_0)C_0)$ for all $x \in \overline{\Omega}$, this completes the proof. □

Thanks to Theorem 2.12, for every $(\varphi, f) \in \mathcal{G}_u \times \mathcal{F}_u$, we can define the map $T : \mathcal{G}_u \times \mathcal{F}_u \to C^{2,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$ as

$$T(\varphi, f) := \begin{cases} u_v & \text{if } \lambda_1(\Delta\varphi(x, 0) + f(x, 0)) > 0, \\ 0 & \text{otherwise}, \end{cases}$$

where u_v is the unique positive solution of (2.2).

Lemma 2.14. *The mapping T is continuous in sense of $\mathcal{G}_u \times \mathcal{F}_u \to C^{2,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$.*

Proof. Assume that $(\varphi_n, f_n) \to (\varphi, f)$ in $\mathcal{G}_u \times \mathcal{F}_u$. Then we have two possibilities.

Case 1 : $T(\varphi, f) \not\equiv 0$, i.e., $\lambda_1(\Delta(\varphi(x,0) + f(x,0)) > 0$.
By the variational property of the eigenvalue problem, we can see $\lambda_1(\Delta\varphi_n(x,0) + f_n(x,0)) > 0$ for sufficiently large n. Let u_{v_n} be the positive solution of

$$-\Delta[\varphi_n(x,u)u] = uf_n(x,u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

By Agmon, Douglis and Nirenberg inequality, we have

$$\|\varphi_n(x,u_{v_n})u_{v_n}\|_{W^{2,m}} \le C\big(\|\varphi_n(x,u_{v_n})u_{v_n}\|_{L^m} + \|u_{v_n}f_n(x,u_{v_n})\|_{L^m}\big)$$

for some positive constant C depending on $m \ge 2$. From this estimate and Sobolev imbedding theorem, we conclude that $\|\varphi_n(x,u_{v_n})u_{v_n}\|_{C^{1,\alpha}} < \infty$, and thus $\|u_{v_n}\|_{C^{1,\alpha}} < \infty$ since the map $G_{\varphi_n}(x,\xi) = \varphi_n(x,\xi)\xi$ has a continuous inverse which is C^2-function in ξ. Also we can easily check that $\|u_{v_n}\|_{C^{2,\alpha}} < \infty$ by Schauder estimate. Therefore there exists a subsequence, denoted it again by u_{v_n}, which converges to a positive function $\widetilde{u} \in C^{2,\alpha}(\overline{\Omega})$. Furthermore, since the positive solution of the equation, $-\Delta[\varphi(x,u)u] = uf(x,u)$ in Ω and $u = 0$ on $\partial\Omega$, is unique, we have $\widetilde{u} \equiv u_v$, and therefore $u_{v_n} \to u_v$ in $C^{2,\alpha}(\overline{\Omega})$.

Case 2 : $T(\varphi, f) \equiv 0$.
In this case, $u_v \equiv 0$ and we can similarly prove that $u_{v_n} \to u_v \equiv 0$ in $C^{2,\alpha}(\overline{\Omega})$. □

Let E be a Banach space and let F be a strongly positive nonlinear compact operator on E such that $F(0) = 0$. The following lemma can be found in [1].

LEMMA 2.15. *Assume $F'(0)$ exists with $r(F'(0)) > 1$, where $F'(0)$ is a Fréchet derivative of F at $u = 0$. If the solution to the equation $u = \theta Fu$ has an a priori bound for all $\theta \in (0,1]$, then F has a positive fixed point u such that $Fu = u$ in the positive cone of E.*

3. Positive solutions of (1.1)

In this section, we give sufficient conditions for the positive coexistence of the system (1.1) with three different interactions between two species by using the method of decomposing operator.

Denote the trivial nonnegative nonzero solutions of our system (1.1), if they exist when one of the species is absent, by $(u_0, 0)$ and $(0, v_0)$. These solutions are usually called *semi-trivial* solutions. By virtue of

Theorem 2.12, one can give sufficient conditions for the existence of semi-trivial solutions, for example, if $\varphi(x,u,0) \in \mathcal{G}_u$ and $f(x,u,0) \in \mathcal{F}_u$, then the sufficient condition of the existence for $(u_0, 0)$ can be considered as $\lambda_1(\Delta\varphi(x,0,0) + f(x,0,0)) > 0$.

REMARK 3.1. Throughout this section, $\varphi \in \mathcal{G}_u$, $f \in \mathcal{F}_u$ and $\psi \in \mathcal{G}_v$, $f \in \mathcal{F}_v$ mean that $\varphi(u,v) \in \mathcal{G}_u$, $f(x,u,v) \in \mathcal{F}_u$ for a fixed $v \geq 0$ and $\psi(u,v) \in \mathcal{G}_v$, $f(x,u,v) \in \mathcal{F}_v$ for a fixed $u \geq 0$.

Now we impose the following conditions for the function g so that (1.1) may express the predator-prey interactions when $f \in \mathcal{F}_u \cap \mathcal{F}_v$ and $\varphi, \psi \in \mathcal{G}_u \cap \mathcal{G}_v$.

DEFINITION 3.2. $g(x,u,v) \in \mathcal{F}_g$ if and only if $g(x,u,v)$ satisfies the followings:

(F1*) $g(x,u,v) \in C(\overline{\Omega} \times [0,\infty) \times [0,\infty))$ and $C^\alpha-$function in x, where $0 < \alpha < 1$;

(F2*) $g(x,u,v)$ is C^1-function in u and v with $g_u > 0$ and $g_v < 0$;

(F3*) for each $M > 0$, there exists a constant $C_2 = C_2(M) > 0$ such that $g(x,u,v) < 0$ on $(x,u) \in \overline{\Omega} \times (0, M]$ when $v > C_2$.

LEMMA 3.3. *Assume that* $\varphi, \psi \in \mathcal{G}_u \cap \in \mathcal{G}_v$, $f \in \mathcal{F}_u, \mathcal{F}_v$ *and* $g \in \mathcal{F}_g$. *Then any nonnegative solution* (u,v) *of the system* (1.1) *has an a priori bound.*

Proof. Consider the problem:

$$\begin{cases} -\Delta[\varphi(u,v)u] = uf(x,u,v) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $v \in C^2(\overline{\Omega})$. Let $G_{\varphi_0}^{-1}(u)$ be the continuous inverse of the map $G_{\varphi_0}(u) = \varphi(u,0)u$ in u. The existence of such inverse map follows from the assumptions $\varphi \in \mathcal{G}_u, \mathcal{G}_v$. Since $f \in \mathcal{F}_u, \mathcal{F}_v$, we can find positive constants C_1 and C_3 such that $f(x, C_1, 0) < 0$ and $f(x, 0, C_3) < 0$ for all $x \in \overline{\Omega}$. If we assume that $\varphi(x,u,v)u$ attains its maximum at x_1 over $\overline{\Omega}$, then we can show that $f(x_1, u(x_1), v(x_1)) \geq 0$ as in Lemma 2.13. Since $f(x_1, u(x_1), 0) \geq f(x_1, u(x_1), v(x_1)) \geq 0 \geq f(x_1, C_1, 0)$ and $f(x_1, 0, v(x_1)) \geq f(x_1, u(x_1), v(x_1)) \geq 0 \geq f(x_1, 0, C_3)$, we can get $u(x_1) \leq C_1$ and $v(x_1) \leq C_3$. Thus we have $\max_{x\in\overline{\Omega}}\{\varphi(u,v)u\} \leq \varphi(C_1,C_3)C_1$, and so $\varphi(u,0)u \leq \max_{x\in\overline{\Omega}}\{\varphi(u,0)u\} \leq \max_{x\in\overline{\Omega}}\{\varphi(u,v)u\} \leq \varphi(C_1,C_3)C_1$. Finally we can conclude that

$$u(x) \leq G_{\varphi_0}^{-1}(\varphi(C_1,C_3)C_1)$$

for all $x \in \overline{\Omega}$.

Consider the problem

$$\begin{cases} -\Delta[\psi(u,v)v] = vg(x,u,v) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $u \le M = \|G_{\varphi_0}^{-1}(\varphi(C_1,C_3)C_1)\|_\infty$ and $u \in C^2(\overline{\Omega})$. Since $g(x,u,v) \in \mathcal{F}_g$, there exists a positive constant C_2 such that $g(x,u,v) < 0$ on $(x,u) \in \overline{\Omega} \times (0,M]$ when $v > C_2$. If we assume that $\psi(u,v)u$ attains its maximum at x_2 over $\overline{\Omega}$, then we can also show that $\psi(0,v)v \le \psi(M,C_2)C_2$ by the similar manner, and so $v(x) \le G_{\psi_0}^{-1}(\psi(M,C_2)C_2)$, where $G_{\psi_0}^{-1}(v)$ is the continuous inverse of the map $G_{\psi_0}(v) = \psi(0,v)v$ in v. □

Remark 3.4. The interesting fact in Lemma 3.3 is that an *a priori* bound for the system (1.1) with self-cross diffusions is affected by the self-cross diffusions and the growth rates at the same time. However, if we consider the system with self-diffusions only, then we can see that an *a priori* bound depends only on the growth rates.

Notation 3.5. Throughout this paper, let Q and R be *a priori* bounds for u and v, respectively. That is to say, the nonnegative solutions u and v of (1.1) satisfy $u(x) \le Q$ and $v(x) \le R$.

Let $\varphi \in \mathcal{G}_u, \mathcal{G}_v$ and $f \in \mathcal{F}_u, \mathcal{F}_v$. By virtue of Theorem 2.12, for every $v \in C^2(\overline{\Omega})$, we can define the map $S : C^2(\overline{\Omega}) \to C^{2,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$ as

$$Sv = \begin{cases} u_v & \text{if } \lambda_1(\Delta\varphi(0,v) + f(x,0,v)) > 0, \\ 0 & \text{otherwise,} \end{cases}$$

where u_v is the unique positive solution of (2.2). Then we can see that the mapping S is continuous in sense of $C^2(\overline{\Omega}) \to C^{2,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$ by Lemma 2.14.

For $\psi \in \mathcal{G}_u, \mathcal{G}_v$, $g \in \mathcal{F}_g$ and $v \in E = C^2(\overline{\Omega})$, consider the problem:

$$\begin{cases} -\Delta[\psi(Sv,v)v] = vg(x,Sv,v) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Then we can define an operator $F : E \to E$ as $w = Fv$ through the equation:

$$\begin{cases} -\Delta[\psi(Sv,w)w] + M\psi(Sv,w)w = vg(x,Sv,v) + M\psi(Sv,v)v & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Define another strongly positive increasing compact operator $K : E \to E$ as $Kv = (-\Delta + M)^{-1}[(g(x,Sv,v) + M\psi(Sv,v))v]$, where M is a positive constant sufficiently large so that $g(x,Sv,v) + M\psi(Sv,v)$ is positive for all $x \in \overline{\Omega}$. Since $\psi \in \mathcal{G}_u, \mathcal{G}_v$, the existence of M follows. Then we can see that $F = G_\psi^{-1} \circ K$, where $G_\psi^{-1}(w)$ is the continuous inverse of the map $G_\psi(w) = \psi(Sv, w)w$ in w and F is a strongly positive increasing compact operator. Observe that v is a fixed point of F if and only if v is a solution of (3.1). Also the operator F has a positive fixed point v if and only if (Sv, v) is a nonnegative solution of (1.1).

THEOREM 3.6. *Consider the system (1.1) with* $\varphi, \psi \in \mathcal{G}_u, \mathcal{G}_v$, $f \in \mathcal{F}_u, \mathcal{F}_v$ *and* $g \in \mathcal{F}_g$.

(i) If $\lambda_1(\Delta\varphi(0,0) + f(x,0,0)) \leq 0$, *then (1.1) has no positive solution; and in addition, if* $\lambda_1(\Delta\psi(0,0) + g(x,0,0)) \leq 0$, *then (1.1) has no nonnegative nonzero solution.*

(ii) Assume $\lambda_1(\Delta\psi(0,0)+g(x,0,0)) \leq 0$ *and* $\lambda_1(\Delta\varphi(0,0)+f(x,0,0)) > 0$. *If* $\lambda_1(\Delta\psi(u_0,0) + g(x,u_0,0)) > 0$, *then (1.1) has a positive solution.*

Proof. (i) Suppose that (u, v) is a positive solution of the system (1.1). Then u satisfies $-\Delta[\varphi(u,v)u] = uf(x,u,v)$ in Ω and $u = 0$ on $\partial\Omega$. By Lemma 2.3 (iii), we have $\lambda_1(\Delta\varphi(u,v) + f(x,u,v)) = 0$. By the same way in the proof of Theorem 2.12 (i), we can derive a contradiction.

Next suppose that (u, v) is a nonnegative nonzero solution and $\lambda_1(\Delta\varphi(0,0) + f(x,0,0))$, $\lambda_1(\Delta\psi(0,0) + g(x,0,0)) \leq 0$. Without loss of generality, assume that $u \not\equiv 0$ and $v \equiv 0$. Then u satisfies $-\Delta[\varphi(u,0)u] = uf(x,u,0)$ in Ω and $u = 0$ on $\partial\Omega$, and so we can have $0 = \lambda_1(\Delta\varphi(u,0) + f(x,u,0)) < \lambda_1(\Delta\varphi(0,0)+f(x,0,0))$, which also derives a contradiction.

(ii) To apply Lemma 2.15, denote $F_\theta = \theta F$. For $\theta \in (0,1]$, assume that $F_\theta(v_\theta) = v_\theta$, i.e., v_θ is a solution to the equation $F_\theta(v) = v$. Then we have

$$-\Delta[\psi(Sv_\theta, \frac{v_\theta}{\theta})v_\theta] = \theta v_\theta g(x, Sv_\theta, v_\theta) + M\theta[\psi(Sv_\theta, v_\theta) - \frac{1}{\theta}\psi(Sv_\theta, \frac{v_\theta}{\theta})]v_\theta.$$

Since $0 \leq Sv_\theta \leq Q$ by Lemma 3.3, there exists a sufficiently large constant $K > 0$ such that $g(x, Sv_\theta, G_{\psi_\theta}^{-1}(K)) < 0$ from the hypothesis (F3*), where $G_{\psi_\theta}^{-1}(v)$ is the inverse of the map $G_{\psi_\theta}(v) = \psi(Sv_\theta, \frac{v}{\theta})\frac{v}{\theta}$. Then we have

$$-\Delta[\psi(Sv_\theta, \frac{1}{\theta}G_{\psi_\theta}^{-1}(K))\frac{1}{\theta}G_{\psi_\theta}^{-1}(K)] = -\Delta K = 0 \text{ in } \Omega,$$

also since $\psi(Sv_\theta, v) - \frac{1}{\theta}\psi(Sv_\theta, \frac{v}{\theta}) \leq 0$ for all $v \geq 0$, we can see that

$$\begin{cases} -\Delta[\psi(Sv_\theta, \frac{1}{\theta}G_{\psi_\theta}^{-1}(K))G_{\psi_\theta}^{-1}(K)] \geq \theta G_{\psi_\theta}^{-1}(K)g(x, Sv_\theta, G_{\psi_\theta}^{-1}(K)) & \\ \quad +M\theta[\psi(Sv_\theta, G_{\psi_\theta}^{-1}(K)) - \frac{1}{\theta}\psi(Sv_\theta, \frac{1}{\theta}G_{\psi_\theta}^{-1}(K))]G_{\psi_\theta}^{-1}(K) & \text{in } \Omega, \\ G_{\psi_\theta}^{-1}(K) \geq 0 & \text{on } \partial\Omega, \end{cases}$$

and thus $G_{\psi_\theta}^{-1}(K)$ is an upper solution to the following equation:

$$\begin{cases} -\Delta[\psi(Sv_\theta, \frac{v}{\theta})v] = \theta v g(x, Sv_\theta, v) + M\theta[\psi(Sv_\theta, v) - \frac{1}{\theta}\psi(Sv_\theta, \frac{v}{\theta})]v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Therefore we have $v_\theta \leq G_{\psi_\theta}^{-1}(K)$. Thus v_θ has an *a priori* bound. Also since $\lambda_1(\Delta\psi(u_0, 0) + g(x, u_0, 0)) > 0$ from the assumption, we must have

$$r(F'(0)) = r\big(\frac{1}{\psi(u_0, 0)}(-\Delta + M)^{-1}[g(x, u_0, 0) + M\psi(u_0, 0)]\big) > 1$$

by Lemma 2.4. Consequently, we can conclude that F has a positive fixed point v in E by Lemma 2.15.

To complete the proof, we need $u = Sv > 0$. If $Sv \equiv 0$, then $v > 0$ satisfies $-\Delta[\psi(0, v)v] = vg(x, 0, v)$ in Ω and $v = 0$ on $\partial\Omega$, and so $v \equiv v_0 > 0$ by the uniqueness of v_0, which is a contradiction since $\lambda_1(\Delta\psi(0, 0) + g(x, 0, 0)) \leq 0$ from the assumption. Hence (Sv, v) is a positive solution to our system (1.1). □

4. Predator-prey systems with degenerate diffusion rates

In this section, we apply the obtained results to the slightly modified Lotka-Volterra problem

(4.1)

$$\begin{cases} -\Delta[(\epsilon_1 + u^{m-1}(x))u(x)] = (a(x) - b(x)u^p(x) - c(x)v^q(x))u(x) & \\ -\Delta[(\epsilon_2 + v^{n-1}(x))v(x)] = (e(x) + h(x)u^r(x) - k(x)v^s(x))v(x) & \text{in } \Omega, \\ (u, v) = (0, 0) & \text{on } \partial\Omega, \end{cases}$$

where $m, n > 1$, $\epsilon_1, \epsilon_2, p, q, r, s > 0$, $c(x), h(x) \in C^\alpha(\Omega)$ and $a(x), b(x), e(x), k(x) \in C^\alpha(\Omega)$ are nonnegative functions with $b(x), k(x) > 0$ for all $x \in \Omega$. For a function $r(x) \in C^\alpha(\Omega)$, we define

$$\overline{r} = \sup\{r(x) : x \in \overline{\Omega}\} \text{ and } \underline{r} = \inf\{r(x) : x \in \overline{\Omega}\}.$$

Lemma 4.1. *If* $\min\{\underline{a} - \overline{c}\big((\overline{e}\underline{b}^{\frac{r}{p}} + \overline{h}\overline{a}^{\frac{r}{p}})/(\underline{b}^{\frac{r}{p}}\underline{k})\big)^{\frac{q}{s}}, \underline{e}\} > 0$, *then (4.1) has a positive solution for sufficiently small* ϵ_1, ϵ_2.

Proof. First observe that $\varphi(u, v) = \epsilon_1 + u^{m-1}$, $\psi(u, v) = \epsilon_2 + v^{n-1}$, $f(x, u, v) = a(x) - b(x)u^p(x) - c(x)v^q(x)$, $g(x, u, v) = e(x) + h(x)u^r(x) - k(x)v^s(x)$ and we can easily check that the problem (4.1) satisfies the corresponding hypotheses for φ, ψ, f, g. Since $\underline{a}, \underline{e} > 0$ from the assumption, we have $\lambda_1(-\Delta) < (\underline{a}/\epsilon_1)$ and $\lambda_1(-\Delta) < (\underline{e}/\epsilon_2)$ for sufficiently small $\epsilon_1, \epsilon_2 > 0$, and therefore $\lambda_1(\Delta\varphi(0, 0) + f(x, 0, 0)) > 0$ and $\lambda_1(\Delta\psi(0, 0) + g(x, 0, 0)) > 0$ by Lemma 2.2 (ii). These imply that the semi-trivial solutions $(u_0, 0)$ and $(0, v_0)$ of system (4.1) exist and positive in each models.

As a special case of Lemma 2.13, $u_0 \leq (\overline{a}/\underline{b})^{\frac{1}{p}}$ and $v_0 \leq \big((\overline{e}\underline{b}^{\frac{r}{p}} + \overline{h}\overline{a}^{\frac{r}{p}})/(\underline{b}^{\frac{r}{p}}\underline{k})\big)^{\frac{1}{s}}$. From the assumption, we can derive

$$\lambda_1(-\Delta) < \{\underline{a} - \overline{c}\big((\overline{e}\underline{b}^{\frac{r}{p}} + \overline{h}\overline{a}^{\frac{r}{p}})/(\underline{b}^{\frac{r}{p}}\underline{k})\big)^{\frac{q}{s}}\}/\epsilon_1 \quad \text{and} \quad \lambda_1(-\Delta) < \underline{e}/\epsilon_2$$

for sufficiently small $\epsilon_1, \epsilon_2 > 0$, and so $\lambda_1(\Delta\epsilon_1 + a(x) - c(x)v_0^q) > 0$ and $\lambda_1(\Delta\epsilon_2 + e(x) + h(x)u_0^r) > 0$ by Lemma 2.2 (i). Thus Theorem 3.6 (ii) concludes the result. □

Observe that the problem (4.1) becomes the degenerate Lotka-Volterra problem when $\epsilon_1, \epsilon_2 \equiv 0$, that is,

$$(4.2) \quad \begin{cases} -\Delta u^m(x) = (a(x) - b(x)u^p(x) - c(x)v^q(x))u(x) & \\ -\Delta v^n(x) = (e(x) - h(x)u^r(x) - k(x)v^s(x))v(x) & \text{in } \Omega, \\ (u, v) = (0, 0) & \text{on } \partial\Omega. \end{cases}$$

Corollary 4.2. *Consider the degenerate Lotka-Volterra problem (4.2). Then the result in Lemma 4.1 remains true.*

Proof. Let $(u_{\epsilon_1}, v_{\epsilon_2})$ be the positive solution of (4.1), then u_{ϵ_1} and v_{ϵ_2} have *a priori* bounds $(\overline{a}/\underline{b})^{\frac{1}{p}}$ and $\big((\overline{e}\underline{b}^{\frac{r}{p}} + \overline{h}\overline{a}^{\frac{r}{p}})/(\underline{b}^{\frac{r}{p}}\underline{k})\big)^{\frac{1}{s}}$ as a special case of Lemma 2.13, respectively. By using Schauder estimate and the elliptic regularity theorem, we can show that $\{u_{\epsilon_1}\}$ and $\{v_{\epsilon_2}\}$ have convergent subsequences, denote them again by $\{u_{\epsilon_1}\}$ and $\{v_{\epsilon_2}\}$ as in Lemma 2.14 when $\epsilon_1, \epsilon_2 \downarrow 0$.

Now we construct a positive lower solution $\underline{u}$ which does not depend on sufficiently small ϵ_1, that is to say,

$$\begin{cases} -\Delta[(\epsilon_1 + \underline{u}^{m-1})\underline{u}] \leq (a(x) - b(x)\underline{u}^p - c(x)v^q)\underline{u} & \text{in } \Omega, \\ \underline{u} \leq 0 & \text{on } \partial\Omega, \end{cases}$$

where $0 \leq v \leq \big((\overline{e}\underline{b}^{\frac{r}{p}} + \overline{h}\overline{a}^{\frac{r}{p}})/(\underline{b}^{\frac{r}{p}}\underline{k})\big)^{\frac{1}{s}}$. To do this, consider the principal eigenvalue $\lambda_1(-\Delta) > 0$ and the corresponding eigenfunction $\phi > 0$. Define $\underline{u} = (\delta_1\phi)^{\frac{1}{m}}$ for a sufficiently small $\delta_1 > 0$ which is determined later. Then we have

$$\begin{aligned}
& \Delta[(\epsilon_1+\underline{u}^{m-1})\underline{u}]+(a(x)-b(x)\underline{u}^p-c(x)v^q)\underline{u}\\
= \;& \Delta[(\epsilon_1+(\delta_1\phi)^{\frac{m-1}{m}})(\delta_1\phi)^{\frac{1}{m}}]+(a(x)-b(x)(\delta_1\phi)^{\frac{p}{m}}-c(x)v^q)(\delta_1\phi)^{\frac{1}{m}}\\
\geq \;& \epsilon_1(\Delta(\delta_1\phi)^{\frac{1}{m}})-\lambda_1(-\Delta)\cdot(\delta_1\phi)+\big(\underline{a}-\overline{b}(\delta_1\phi)^{\frac{p}{m}}\\
& -\overline{c}\big((\overline{e}\underline{b}^{\frac{r}{p}}+\overline{h}\overline{a}^{\frac{r}{p}})/(\underline{b}^{\frac{r}{p}}\underline{k})\big)^{\frac{q}{s}}\big)(\delta_1\phi)^{\frac{1}{m}}\\
= \;& (\delta_1\phi)^{\frac{1}{m}}\{\epsilon_1\phi^{-\frac{1}{m}}(\Delta\phi^{\frac{1}{m}})-\lambda_1(-\Delta)\cdot(\delta_1\phi)^{\frac{m-1}{m}}+\underline{a}-\overline{b}(\delta_1\phi)^{\frac{p}{m}}\\
& -\overline{c}\big((\overline{e}\underline{b}^{\frac{r}{p}}+\overline{h}\overline{a}^{\frac{r}{p}})/(\underline{b}^{\frac{r}{p}}\underline{k})\big)^{\frac{q}{s}}\}\\
\geq \;& 0
\end{aligned}$$

for all $x\in\Omega$ since $\underline{a}-\overline{c}\big((\overline{e}\underline{b}^{\frac{r}{p}}-\underline{f}\overline{a}^{\frac{r}{p}})/(\underline{b}^{\frac{r}{p}}\underline{g})\big)^{\frac{q}{s}}>0$ and ϵ_1 is sufficiently small, if we choose a sufficiently small constant $\delta_1>0$ such that

$$-\lambda_1(-\Delta)\cdot(\delta_1\phi)^{\frac{m-1}{m}}+\underline{a}-\overline{b}(\delta_1\phi)^{\frac{p}{m}}-\overline{c}\big((\overline{e}\underline{b}^{\frac{r}{p}}+\overline{h}\overline{a}^{\frac{r}{p}})/(\underline{b}^{\frac{r}{p}}\underline{k})\big)^{\frac{q}{s}}>0$$

for all $x\in\Omega$. Observe that $\underline{u}=(\delta_1\phi)^{\frac{1}{m}}$ on $\partial\Omega$.
Similarly, for a sufficiently small $\epsilon_2>0$, we can verify that $\underline{v}=(\delta_2\phi_2)^{\frac{1}{n}}$ satisfies

$$\begin{cases} -\Delta[(\epsilon_2+\underline{v}^{n-1})\underline{v}]\leq(e(x)+h(x)u^r-k(x)\underline{v}^s)\underline{v} & \text{in } \Omega,\\ \underline{v}\leq 0 & \text{on } \partial\Omega,\end{cases}$$

where $0\leq u\leq(\overline{a}/\underline{b})^{\frac{1}{p}}$ and $\delta_2>0$ is a sufficiently small constant.

Consequently, we can conclude that the limit of positive solutions u_{ϵ_1} and v_{ϵ_2} of (4.1) become the positive solutions of (4.2) when $\epsilon_1,\epsilon_2\downarrow 0$. The positivity of these solutions of (4.2) follows from the fact that $u_{\epsilon_1}\geq(\delta_1\phi)^{\frac{1}{m}}$ and $v_{\epsilon_2}\geq(\delta_2\phi)^{\frac{1}{n}}$ when $\epsilon_1,\epsilon_2\downarrow 0$. □

Acknowledgement. The author thanks the anonymous referees for their valuable comments and suggestions to improve the content of this article.

References

[1] H. Amann, *Fixed point equations and nonlinear eigenvalue problems in ordered banach spaces*, SIAM Rev., **18** (1976), no. 4, 620–709.
[2] D. G. Aronson, A. Tesei, and H. Weinberger, *A density-dependent diffusion system with stable discontinuous stationary solutions*, Ann. Mat. Pura Appl., **152** (1988), no. 4, 259–280.
[3] A. Cañada and J. L. Gámez, *Elliptic systems with nonlinear diffusion in population dynamics*, Differential Equations Dynam. Systems **3** (1995), no. 2, 189–204.
[4] A. Ghoreishi and R. Logan, *Positive solutions of a class of biological models in a heterogeneous environment*, Bull. Austral. Math. Soc., **44** (1991), no. 1, 79–94.

[5] M. E. Gurtin, *Some mathematical models for population dynamics that lead to segregation*, Quart. Appl. Math., **32** (1974/75), 1–9.
[6] *A. Leung and G. Fan, Existence of positive solutions for elliptic systems—degenerate and nondegenerate ecological models*, J. Math. Anal. Appl., **151** (1990), no. 2, 512–531.
[7] Y. Lou and W. M. Ni, *Diffusion, self-diffusion and cross-diffusion*, J. Differential Equations, **131** (1996), no. 1, 79–131.
[8] Y. Lou and W. M. Ni, *Diffusion vs cross-diffusion: an elliptic approach*, J. Differential Equations **154** (1999), no. 1, 157–190.
[9] G. Rosen, *Effects of diffusion on the stability of the equilibrium in multi-species ecological systems*, Bull. Math. Biology **39** (1977), no. 3, 373–383.
[10] W. H. Ruan, *Positive steady-state solutions of a competing reaction-diffusion system with large cross-diffusion coefficients*, J. Math. Anal. Appl., **197** (1996), no. 2, 558–578.
[11] K. Ryu and I. Ahn, *Positive steady–states for two interacting species models with linear self–cross diffusions*, Discrete Contin. Dyn. Syst., **9** (2003), no. 4, 1049-1061.
[12] N. Shigesada, K. Kawasaki, and E. Teramoto, *Spatial segregation of interacting species*, J. Theoret. Biol. **79** (1979), no. 1, 83–99.

*
Department of Mathematics Education,
Cheongju University
Cheongju, Chungbuk 28503, Republic of Korea
E-mail: ryukm@cju.ac.kr