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# THE EVENTUAL PSEUDO ORBIT TRACING PROPERTY FOR HOMEOMORPHISMS

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Abstract. In this paper, we prove that a distal homeomorphism $f$ of a compact connected metric space $X$ does not have the eventual pseudo orbit tracing property.

## 1. Introduction

The eventual pseudo orbit tracing property was recently introduced by [2]. It is a general concept of the originally pseudo orbit tracing property (see [2]). It's interesting to see what has a property in dynamical systems (see [4, 5]). A hyperbolic system has the pseudo orbit tracing property and so it has the eventual pseudo orbit tracing property. Also, an interesting topic is that the system does not have some properties. Aoki [1] showed that if a distal homeomorphism $f$ of a compact metric space $X$ then $f$ does not have the originally pseudo orbit tracing property. It is a motivation of this paper. In the paper, we consider the eventual pseudo orbit tracing property for distal homeomorphisms.

## 2. Basic notions and Main theorem

Let $X$ be a compact metric space with metric $d$ and let $f : X \to X$ be a homeomorphism. For any $\delta > 0$, we say that a sequence $\{x_i : i \in \mathbb{Z}\}$ is called a $\delta$ *pseudo orbit* of $f$ if $d(f(x_i), x_{i+1}) < \delta$ for all $i \in \mathbb{Z}$. A homeomorphism $f$ has the *pseudo orbit tracing property* if for every $\epsilon > 0$, there is a $\delta > 0$ such that for any $\delta$ pseudo orbit $\{x_i : i \in \mathbb{Z}\}$ we can take a point $z \in X$ satisfying $d(f^j(z), x_j) < \epsilon$ for all $j \in \mathbb{Z}$. The

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following is a general notion of the above concept which was suggested in [2].

Definition 2.1. *Let $f : X \to X$ be a homeomorphism. We say that $f$ has the eventual pseudo orbit tracing property if for every $\epsilon > 0$ there is a $\delta > 0$ such that for any $\delta$ pseudo orbit $\{x_i : i \in \mathbb{Z}\}$ we can take a point $z \in X$ and a positive $k \in \mathbb{Z}$ satisfying (i) $d(f^{k+j}(z), x_{k+j}) < \epsilon$ for all $j \geq 0$, and (ii) $d(f^{-k-j}(z), x_{-k-j}) < \epsilon$ for all $j \geq 0$.*

Note that if $f$ has the eventual pseudo orbit tracing property if and only if $f^i$ has the eventual pseudo orbit tracing property, for all $i \in \mathbb{Z} \setminus \{0\}$.

For a point $x \in X$, we denote $Orb(x) = \{f^n(x) : n \in \mathbb{Z}\}$ which is called *orbit* of $x$. A homeomorphism $f$ is said to be *minimal* if $\overline{Orb(x)} = X$, for every $x \in X$, where $\overline{A}$ is the closure of $A$. A homeomorphism $f$ is said to be *distal* if for any $x, y \in X$, $\inf\{d(f^n(x), f^n(y)) : n \in \mathbb{Z}\} = 0$ then $x = y$. We say that $X$ is *nontrivial* if $X$ is not one point. Aoki [1] proved that a minimal homeomorphism $f : X \to X$ of a compact metric space $X$, if $X$ is connected and nontrivial, and if $f$ is distal then $f$ does not have the pseudo orbit tracing property. Let $S$ be a subset of $\mathbb{Z}$. Then $S$ is *syndetic* if there is a finite $T$ subset of $\mathbb{Z}$ such that $\mathbb{Z} = S + T$. For a point $x \in X$ and a neighborhood $U$ of $x$, $x$ is called *almost periodic point* if $\{n \in \mathbb{Z} : f^n(x) \in U\}$ is a syndetic. A homeomorphism $f$ is said to be *uniformly almost periodic* if for any neighborhood $U(x)$ of $x \in X$, there is a syndetic set $S$ such that $f^n(x) \in U(x)$, for $n \in S$. According to [3], it was proved that if a homeomorphism $f$ is uniformly almost periodic then $f$ is distal. Also, he proved that if a homeomorphism $f$ is distal then every point $x \in X$ is an almost periodic point. Aoki [1] proved that if a homeomorphism $f$ is distal then for each $x \in X$, $\overline{Orb(x)} = X$. About the results, we prove the following.

**Theorem A.** *Let $X$ be a compact connected metric space. If a homeomoprhism $f : X \to X$ is distal and nontrivial, then $f$ does not have the eventual pseudo orbit tracing property.*

## 3. Proof of Theorem A

A point $x \in X$ is said to be *non-wandering point* if for any neighborhood $U$ of $x$ there is $n \in \mathbb{Z}$ such that $f^n(U) \cap U \neq \emptyset$. Denote $\Omega(f)$ by the set of all non-wandering points of $f$.

Lemma 3.1. *Let $f : X \to X$ have the eventual pseudo orbit tracing property. Then for any $\epsilon > 0$ and any $x \in \Omega(f)$, there are $z \in X$ and a positive $k \in \mathbb{Z}$ such that $\overline{Orb_{f^k}(z)} \subset B(x, \epsilon)$.*

**Proof.** Let $\epsilon/4 > 0$ be given and $0 < \delta < \epsilon$ be the number of the eventual pseudo orbit tracing property for $f$. For $x \in \Omega(f)$ and any neighborhood $B(\delta/2, x)$, there is $k > 0$ such that $f^k(B(\delta/2, x)) \cap B(\delta/2, x) \neq \emptyset$. Since $f^k(B(\delta/2, x)) \cap B(\delta/2, x) \neq \emptyset$, one can take $y \in f^k(B(\delta/2, x)) \cap B(\delta/2, x) \neq \emptyset$ such that $y, f^k(y) \in B(\delta/2, x)$. Then we construct a sequence of points of $\{w_i\}_{i\in\mathbb{Z}}$ such that $w_{nk+i} = f^i(y)$ for all $n \in \mathbb{Z}$ and $0 \le i < k$. It is clear that

$\{w_i\}_{i\in\mathbb{Z}} = \{\ldots, f^{-1}(y), y, \ldots, f^{k-1}(y), f^k(y), \ldots\}$ is a $\delta$ pseudo orbit of $f$. Since $f$ has the eventual pseudo orbit tracing property, there are a point $z \in X$ and $l \in \mathbb{Z}$ such that

$$d(f^{l+i}(z), w_{l+i}) < \epsilon/4 \text{ for all } |i| \ge l.$$

Then one can take $i_1 \in \mathbb{Z}$ such that $l + i_1 = n_1 k$ for some $n_1 \in \mathbb{Z}$. Thus one can see that $d(f^{n_1 k}(z), w_{n_1 k}) < \epsilon/4$ and so $d(f^{n_1 k}(z), y) < \epsilon/3$. By $d(f^{n_1 k}(z), y) < \epsilon/4$, we have

$$d(f^{jn_1 k}(z), y) < \epsilon/4 \text{ for all } j \in \mathbb{Z}.$$

Set $jn_1 = m \in \mathbb{Z}$. Since $d(f^{jn_1 k}(z), y) = d(f^{mk}(z), y) < \epsilon/4$, for all $m \in \mathbb{Z}$ we have $d(f^{mk}(z), x) \le d(f^{mk}(z), y) + d(y, x) < \epsilon/4 + \delta/2 < 3\epsilon/4$, and so $\overline{Orb_{f^k}(z)} \subset B(\epsilon, x)$. □

The following result is a motivation of [1] which concerning with the eventual pseudo orbit tracing property.

Theorem 3.2. *Let $X$ be a compact, connected and nontrivial. If a homeomorphism $f : X \to X$ is minimal then $f$ does not have the eventual pseudo orbit tracing property.*

**Proof.** Let $\text{diam}X = L$. Take $\epsilon = L/3$. Then we assume that $f$ has the eventual pseudo orbit tracing property. According to Lemma 3.1, for some $x \in X$, one can find $z \in X$ and an integer $k > 0$ such that $Orb_{f^k}(z) \subset B(\epsilon, x)$. Since $X$ is connected, we have $Orb_{f^k}(z) = Orb_f(z) = X$. Thus we have $\text{diam}X \le 2\epsilon$. Since $\epsilon = \text{diam}X/3$, this is a contradiction. □

Remark 3.3. *If a homeomorphism $f : X \to X$ has the eventual pseudo orbit tracing property then $f^k$ has the eventual pseudo orbit tracing property for any positive integer $k > 0$.*

The following was proved by Aoki [1, Lemma 3].

LEMMA 3.4. *If a homeomorphism $f$ is distal then we have $\overline{Orb_f(x)} = X$, for all $x \in X$.*

Let $\mathcal{B}$ be the family of Borel sets of $X$. Then we define a Boreal probability measure $\mu$ on $X$ with $\mu(X) = 1$. A measure $\mu$ on $X$ is called *$f$-invariant* if $\mu(f(B)) = \mu(B)$ for all $B \in \mathcal{B}$. Note that if $X$ is compact then there exists $f$-invariant measures on $X$ by Krylov and Bololioubov's theorem.

LEMMA 3.5. *If $X$ is distal, then for all $z \in X$, there is a $f$-invariant Probability measure $\mu$ on $\overline{Orb_f(z)}$ such that $\mu(U) > 0$, where $U$ is a non-empty open set in $\overline{Orb_f(z)}$.*

**Proof.** Since $X$ is distal, according to Lemma 3.4, we have $\overline{Orb_f(z)} = X$ for all $z \in X$. Then for every non-empty open set $U \subset \overline{Orb_f(z)}$, we have $\mu(U) > 0$. $\square$

**Proof of Theorem A** Suppose that $f$ has the eventual pseudo orbit tracing property. Let $\epsilon = \text{diam}X/16$. Then there is $0 < \delta < \epsilon$ such that every $\delta$ pseudo orbit $\{z_i\}_{i\in\mathbb{Z}}$ is $\epsilon$-traced by some point of $X$. According to Lemma 3.1, for any $a \in \Omega(f)$ there are $b \in X$ and $k > 0$ such that $\overline{Orb_{f^k}(b)} \subset B(\epsilon, a)$. For simplicity, we assume that $g = f^k$. Note that $f$ is distal then $f^i$ is distal for all $i \in \mathbb{Z} \setminus \{0\}$. Then we know that $g$ is distal. Since $X$ is connected and compact, there are a points $x_1, x_2, \ldots, x_m \in X$ such that (i) $x_1 = b$, (ii) $d(x_i, x_{i+1}) < \delta/2$ for $1 \le i \le m-1$, and (iii) $\bigcup_{i=1}^m B(\delta, x_i) = X$. Since $g$ is distal, for any point $x \in X$ is almost periodic. Hence, for the points of $\{x_1, x_2, \ldots, x_m\}$, there is $l_i \in \mathbb{N}$ such that $d(x_i, g^{l_i}(x_i)) < \delta/2$. From that, we construct a $\delta$ pseudo orbit $\{z_i\}_{i\in\mathbb{Z}} \subset X$ as follows (see [1]):

- $z_i = g^{-i}(x_1) = g^{-i}(b)$ for $i < 0$,
- $z_i = g^i(x_1) = g^i(b)$ for $0 \le i \le l_1 - 1$,
- $z_{l_1+i} = g^i(x_2)$ for $0 \le i \le l_2 - 1$,

$\cdots$

- $z_{l_1+l_2+\cdots+l_{m-1}+i} = g^i(x_m)$ for $0 \le i \le l_m - 1$,
- $z_{l_1+l_2+\cdots+l_m+i} = g^i(x_{m-1})$ for $0 \le i \le l_{m-1} - 1$,

$\cdots$

- $z_{l_1+2l_2+\cdots+2l_{m-1}+l_m+i} = g^i(x_1) = g^i(b)$ for $i \ge 0$.

It is clear that the sequence $\{z_i\}_{\in\mathbb{Z}}$ is a $\delta$ pseudo orbit and is $\delta$ dense in $X$. Since $g$ has the eventual pseudo orbit tracing property, there are $w \in X$ and $L \in \mathbb{Z}$ such that $d(g^{-L-i}(w), z_{-L-i}) < \epsilon$ for all $i \geq 0$ and $d(g^{L+i}(w), z_{L+i}) < \epsilon$ for all $i \geq 0$. Thus we have

$d(g^{-L-i}(w), g^{-L-i}(x_i)) = d(g^{-L-i}(w), g^{-L-i}(b)) < \epsilon$ for $i \geq 0$ and (i) if $L = k$ then $d(g^{k+i}(w), z_{k+i}) = d(g^{k+i}(w), g^i(b)) < \epsilon$ for $i \geq 0$, and (ii) if $L \neq k$ then one can take $L_1 \in \mathbb{Z}$ such that $k = L + L_1$, and so $d(g^{L+L_1+i}(w), z_{L+L_1+i}) = d(g^{L+L_1+i}(w), g^i(b)) < \epsilon$ for $i \geq 0$, where $k = l_1 + l_m + 2\sum_{i=2}^{m-1} l_i$.

Then we have $g^{-L-i}(w) \in B(\epsilon, g^{-i}(b)) \subset B(\epsilon, \overline{Orb_g(b)})$ for $i > 0$ and $g^{k+i}(w) \in B(\epsilon, g^i(b)) \subset B(\epsilon, \overline{Orb_g(b)})$ for $i \geq 0$, where $B(\epsilon, \overline{Orb_g(y)}) = \bigcup_{x \in \overline{Orb_g(b)}} B(\epsilon, x)$. Therefore, one can see that

$$g^{-L}\overline{Orb_g^-(w)} = g^{-L}\overline{\{g^i(w) : i < 0\}} \subset B(\epsilon, \overline{Orb_g(b)}) \text{ and}$$

$$g^{L+L_1}\overline{Orb_g^+(w)} = g^{L+L_1}\overline{\{g^i(w) : i \geq 0\}} \subset B(\epsilon, \overline{Orb_g(b)}).$$

Since $\overline{\{g^i(w) : i < 0\}} \cup \overline{\{g^i(w) : i \geq 0\}} = \overline{\{g^i(b) : i \in \mathbb{Z}\}}$, according to Baire's theorem we have that either $\overline{Orb_g^-(w)}$ or $\overline{Orb_g^+(w)}$ has non-empty interior in $\overline{\{g^i(b) : i \in \mathbb{Z}\}}$. Since $g : \overline{Orb_g(w)} \to \overline{Orb_g(w)}$ is a homeomorphism, there is a $g$-invariant measure $\mu$ on $\overline{Orb_g(w)}$. By Lemma 3.4 and Lemma 3.5, for any non-empty open set $U$, we have $\mu(U) > 0$.

Then as in the proof of [1], if $\overline{Orb_g^-(w)}$ contains interior points in $\overline{Orb_g(w)}$ then $g^{-1}\overline{Orb_g^-(w)} = \overline{Orb_g^-(w)}$ and so $g^{-j}\overline{Orb_g^-(w)} = \overline{Orb_g^-(w)}$ for all $j \geq 1$. If $\overline{Orb_g^+(w)}$ contains interior points in $\overline{Orb_g(w)}$ then $g\overline{Orb_g^+(w)} = \overline{Orb_g^+(w)}$ and so $g^j\overline{Orb_g^+(w)} = \overline{Orb_g^+(w)}$ for all $j \geq 1$. From the above cases, we have $\overline{Orb_g(w)} = \overline{Orb_g^-(w)}$ or $\overline{Orb_g(w)} = \overline{Orb_g^+(w)}$. Thus we have $\overline{Orb_g(w)} = B(\epsilon, \overline{Orb_g(b)})$. As $\overline{Orb_g(b)} \subset B(\epsilon, a)$, one can see that $\overline{Orb_g(w)} \subset B(\epsilon, \overline{Orb_g(b)}) \subset B(2\epsilon, a)$. Since $\{z_i\}_{i\in\mathbb{Z}}$ is $\delta$ dense in $X$ and $g$ has the eventual pseudo orbit tracing property, we have $\max\{d(\overline{Orb_g(w)}, X) : x \in X\} \leq 2\epsilon$ and so $X = B(2\epsilon, \overline{Orb_g(w)}) = B(4\epsilon, a)$. Then we know that $\mathrm{diam}X < 8\epsilon$. Since $16\epsilon = \mathrm{diam}X$, this is a contradiction. □

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