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ON THE VOLUMES OF SIMPLE FINSLER MANIFOLDS

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Abstract. We prove that any simple Finsler manifolds with the same distances between points of the boundary have the same Holmes-Thompson volume.

1. Introduction

A compact Finsler manifold (M, F) with boundary ∂M is *simple* if it is simply connected, any geodesic has no conjugate points and ∂M is strictly convex; that is, the second fundamental form of the boundary is positive definite at every boundary point. Such a manifold is diffeomorphic to a ball in $\mathbb{R}^n$. The *boundary rigidity problem* consists in determining a compact, Riemannian manifold with boundary, up to isometry, by knowing the boundary distance function between boundary points. Michel [14] conjectured that all simple manifolds are boundary rigid. This is known for simple domains of Euclidean space, simple domains of an open hemisphere, simple domains of symmetric spaces of negative curvature (see [10]). Recently Pestov and Uhlmann [15] proved a conjecture due to Michel in the two-dimensional Riemannian case.

However in [4] Colbois, Newberger, and Verovic have a negative answer in Finsler case for boundary rigidity, and hence the rigidity problem in Finsler geometry requires more scrutiny (cf. [1, 3, 5, 8, 11, 16]). The author and Yim [12] proved boundary rigidity for simple subdomains with vanishing S-curvature on Minkowskian space. In order to study the analogous problems in the Finslerian case we ask the following question (see [2, Conjecture C]).

Question. Let (M, F_0) and (M, F_1) be simple Finsler manifolds with the same boundary ∂M. If $\mathrm{dist}_{F_1}(p, q) \geq \mathrm{dist}_{F_0}(p, q)$ for all $p, q \in \partial M$, then $\mathrm{vol}_{F_1}(M) \geq \mathrm{vol}_{F_0}(M)$.

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The volume minimizing property is related to the notion of filling volume introduced in [7]. In short, the filling volume of a manifold equipped with a distance function is the greatest lower bound for the volume of a Finsler space spanning the given manifold as a boundary and inducing boundary distances no less than given distances on it. Our main theorem below a little improves Ivanov's result ([9, Main Theorem]) into Finsler manifolds with dimension greater than two.

THEOREM 1.1. *Any simple Finsler manifolds with same distances between points of the boundary have the same Holmes-Thompson volume.*

Similar Finsler volume results in the *reversible* case were obtained recently by Koehler [13].

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2. Preliminaries

In this section, we shall recall some well-known facts about Finsler geometry. See [17], for more details. Let M be an n-dimensional smooth manifold and TM denote its tangent bundle. A *Finsler structure* on a manifold M is a map $F : TM \to [0, \infty)$ which has the following properties

- Regularity: F is smooth on $\widetilde{TM} := TM \setminus \{0\}$;
- Positive homogeneity: $F(x, \lambda y) = \lambda F(x, y)$, for all $\lambda > 0$, $y \in T_xM$;
- Strongly convexity: The fundamental quadratic form

$$g_{ij}(x, y)dx^i \otimes dx^j, g_{ij}(x, y) := \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}(x, y)$$

  is positive definite for all $(x, y) \in \widetilde{TM}$.

A manifold M endowed with a Finsler structure will be called a Finsler manifold. Note that we never require reversibility and smoothness at the zero section. The Finsler structure F induces a distance function dist_F on M for which (M, dist_F) is a length space.

Let π^*TM denote the pull-back of the tangent bundle TM by $\pi : \widetilde{TM} \to M$. Denote vectors in π^*TM by $(v; w), v \in \widetilde{TM}, w \in T_{\pi(v)}M$. For the sake of simplicity, we denote by $\partial_i|_v = (v; \frac{\partial}{\partial x^i}|_x), v \in T_xM$, the natural local basis for π^*TM. The Finsler metric F defines the *fundamental* tensor g in π^*TM by $g_F(\partial_i|_v, \partial_j|_v) := g^v_{ij} := g_{ij}(x,y)$ where $v = y^i \frac{\partial}{\partial x^i}|_x$.

Let $\omega = \sum_{i=1}^n \frac{\partial F}{\partial y^i}\, dx^i$ be the Hilbert 1-form on the unit tangent bundle SM of M. In local coordinates, we have the symplectic volume form

$$dV = \omega \wedge (d\omega)^{n-1}$$

on SM. Let X_ω be the Reeb field of the Hilbert 1-form ω. In particular we have the geodesic flow of Finsler metric, i.e., the flow with infinitesimal generator X_ω, consists of contact diffeomorphisms. The geodesic flow on SM is denoted by ϕ_t, and is given by $\phi_t(v) := \dot{\gamma_v}(t)$, where $\gamma_v(t)$ denotes the geodesic with initial point $\gamma_v(0)$ and initial vector $v = \dot{\gamma}_v(0)$. Since $L_{X_\omega}\omega = 0$, the symplectic volume form on SM is invariants under the geodesic flows of Finsler metric. The Holmes-Thompson volume $\mathrm{vol}_F(M)$ of an n-dimensional compact Finsler manifold is the symplectic volume of the unit tangent bundle divided by the volume of the Euclidean unit sphere of dimension $n-1$.

3. The geodesic conjugacy

The purpose of this section is to investigate the geodesic conjugacy maps between manifolds. Two complete Finsler manifolds $M_i = (M, F_i), i = 0, 1$, are said to have $\mathcal{C}^k$-*geodesic conjugacy* if there is a $\mathcal{C}^k$-homeomorphism

$$\Psi : SM_0 \to SM_1$$

such that

$$\phi_t^{M_1} \circ \Psi = \Psi \circ \phi_t^{M_0},$$

for all $t \in \mathbb{R}$, where $\phi_t^{M_i}$ represents the geodesic flows on $SM_i, i = 0, 1$. Any dynamical properties defined by the geodesic flow are the same for both manifolds, and it is tempting to ask if geodesic conjugacy manifolds must be isometric. In general the answer to the question above is no (see [6] for a counterexample).

Proposition 3.1. *Let F_0 and F_1 be two simple Finsler metrics on the compact manifold M with boundary ∂M such that $\mathrm{dist}_{F_0}(p,q) = \mathrm{dist}_{F_1}(p,q)$ for all $p, q \in \partial M$. Then there exists a diffeomorphism $\psi : M \to M$ such that $\psi|_{\partial M} = \mathrm{Id}$ and $F_0 = \psi^* F_1$ on $T_{\partial M} M$.*

Proof. Let $(x,y) \in T(\partial M)$ and take a curve $\gamma : [0, \epsilon) \to \partial M$ adapted to (x, y). Since γ takes values in ∂M for all $t \in [0, \epsilon)$ we have

$$\mathrm{dist}_{F_0}\big(x, \gamma(t)\big) = \mathrm{dist}_{F_1}\big(x, \gamma(t)\big).$$

It follows that

$$F_0(x,y) = \lim_{t\to 0^+} = \frac{\mathrm{dist}_{F_0}\big(x,\gamma(t)\big)}{t} = \lim_{t\to 0^+} = \frac{\mathrm{dist}_{F_1}\big(x,\gamma(t)\big)}{t} = F_1(x,y).$$

Thus by the polarization identity we see that already we have $F_0 = F_1$ on $T(\partial M)$. This is not good enough though; in general we will need to modify F_1 before we obtain the stronger statement of the proposition.

Let $\mathfrak{n}_{F_0}(x)$ denote the unit inward pointing normal with respect to $g_{F_0}^{\mathfrak{n}_{F_0}(x)}$, and define the boundary exponential map

$$\exp_{\partial(M,F_0)} : \partial M \times \{t \geq 0\} \to M, (x,t) \mapsto \exp_x\big(t \cdot \mathfrak{n}_{F_0}(x)\big),$$

which maps a neighborhood of $\partial M \times \{0\}$ diffeomorphically onto a neighborhood of ∂M. Now define

$$\psi := \exp_{\partial(M,F_1)} \circ \exp^{-1}_{\partial(M,F_0)}.$$

Then on some collar neighborhood U of ∂M, it is a diffeomorphism. It can be shown (although this requires a bit of effort) that it is possible to extend smoothly across all of M. Assume this is done. Then we claim $\psi : M \to M$ satisfies the requirements of the proposition. Indeed, $\psi|_{\partial M} = \mathrm{Id}$, and moreover given $x \in \partial M$, if γ_{F_0} is the unique F_0-geodesic adapted to $\big(x, \mathfrak{n}_{F_0}(x)\big)$ and similarly γ_{F_1} is the unique F_1-geodesic adapted to $\big(x, \mathfrak{n}_{F_1}(x)\big)$ then $\psi(\gamma_{F_0}) = \gamma_{F_1}$. Hence by differentiating we have $d_x\psi(\mathfrak{n}_{F_0}) = \mathfrak{n}_{F_1}$. Observe if $x \in \partial M$ and $y \in T_x(\partial M)$ then

$$\begin{aligned} g_{\psi^* F_1}\big(y, \mathfrak{n}_{F_0}(x)\big) &= g_{F_1}\big(d_x\psi(y), d_x\psi(\mathfrak{n}_{F_0}(x))\big) \\ &= g_{F_1}\big(y, \mathfrak{n}_{F_1}(x)\big) = 0, \end{aligned}$$

since $d_x\psi|_{T_x(\partial M)} = \mathrm{Id}$, and thus $\psi^* F_1$ has unit inward normal vector field equal to $\mathfrak{n}_{F_0}(x)$. Next, for $x \in \partial M$ we have the decomposition

$$T_x M = T_x(\partial M) \oplus \mathbb{R} \cdot \mathfrak{n}_{F_0}(x),$$

since $T_x(\partial M)$ is a codimension one vector subspace of $T_x M$ and $0 \neq \mathfrak{n}_{F_0}(x) \in T_x M \setminus T_x(\partial M)$. Finally, since F_1 (and hence $\psi^* F_1$, since $d_x\psi$ is the identity on $T_x(\partial M)$) and F_0 agree on $T(\partial M)$, it follows that $F_0 = \psi^* F_1$ on $T_{\partial M} M$, as we wanted to show. $\square$

As an immediate consequence of this proposition, we have:

Lemma 3.2. *For $M_i = (M, F_i), i = 0, 1$, as above, M have $\mathcal{C}^\infty$-geodesic conjugacy Ψ with $\Psi|_{S_{\partial M} M} = \mathrm{Id}$.*

Proof. By proposition 3.1 we may assume $F_0 = F_1$ on $T_{\partial M}M$. Since all geodesics hit the boundary, we can label them by their initial vector $v \in S_{\partial M}M$. Given $v \in \widetilde{TM}$, there exist a unique geodesic γ_v adapted to v; moreover γ_v is maximally defined on a finite interval $[\tau_-(v), \tau_+(v)]$ with $\gamma_v\big(\tau_\pm(v)\big) \in \partial M$. Define Ψ from SM_0 to SM_1 by

$$\Psi(v) := \phi^{M_1}_{-\tau_-(v)} \circ \phi^{M_0}_{\tau_-(v)}(v), v \in SM_0.$$

Since $\tau_-|_{S_{\partial M}M} = 0$, we certainly have

$$\phi^{M_1}_{-\tau_-(v)} \circ \phi^{M_0}_{\tau_-(v)} = \mathrm{Id}, v \in S_{\partial M}M.$$

Now it remains only to check that Ψ is actually a time preserving geodesic conjugacy, and for this it is enough to check on SM. Given $v \in SM_0$ and $t \in [\tau_-(v), \tau_+(v)]$, observe firstly that

$$\tau_-\big(\phi^{M_0}_t(v)\big) = \tau_-(v) - t,$$

and thus we have

$$\begin{aligned}
\Psi\big(\phi^{M_0}_t(v)\big) &= \phi^{M_1}_{-\tau_-(\phi^{M_0}_t(v))} \circ \phi^{M_0}_{\tau_-(\phi^{M_0}_t(v))}\big(\phi^{M_0}_t(v)\big) \\
&= \phi^{M_1}_{t-\tau_-(v)} \circ \phi^{M_0}_{\tau_-(v)}(v) \\
&= \phi^{M_1}_t \circ \big\{\phi^{M_1}_{-\tau_-(v)} \circ \phi^{M_0}_{\tau_-(v)}(v)\big\} \\
&= \phi^{M_1}_t\big(\Psi(v)\big).
\end{aligned}$$

□

This lemma will give us a way of showing that certain manifolds with boundary are boundary rigid by using rigidity results for geodesic flows on closed manifolds. We will return to this later on, but we finish this paper by showing that the existence of a geodesic conjugacy implies equality of volumes.

4. Proof of main theorem

In this section we prove our main theorem. Most of the ideas in this section were known to Gromov (see [7, section 5.5]) in different settings. However some modification are needed and so we will present them in our setting. The notion of geodesic conjugacy came up in Gromov's work on the Riemannian manifolds in [7]. The geodesic conjugacy of Croke-Kleiner's proof implies that it can be extended to Finsler manifolds with little modification as follows (see [6, Lemma 2.1]).

THEOREM 4.1. *Let* (M, F_0) *and* (M, F_1) *be simple Finsler manifolds with the same boundary* ∂M. *If* $\mathrm{dist}_{F_0}(p,q) = \mathrm{dist}_{F_1}(p,q)$ *for all* $p, q \in \partial M$, *then* $\mathrm{vol}_{F_0}(M) = \mathrm{vol}_{F_1}(M)$.

Proof. By Lemma 3.2 we have $\mathcal{C}^\infty$-geodesic conjugacy $\Psi : SM_0 \to SM_1$ with $\Psi|_{S_{\partial M}M} = \mathrm{Id}$. Let ω_i be the Hilbert 1-forms on (M, F_i) and X_{ω_0} be the Reeb field of ω_0. By Cartan's identity we have

$$i_{X_{\omega_0}} d\omega_0 = L_{X_{\omega_0}}\omega_0 - d i_{X_{\omega_0}}\omega_0 = 0$$

and

$$i_{X_{\omega_0}} d(\Psi^*\omega_1) = L_{X_{\omega_0}}(\Psi^*\omega_1) - d i_{X_{\omega_0}}(\Psi^*\omega_1) = 0.$$

Let

$$\omega_t := (1-t)\cdot\omega_0 + t\cdot(\Psi^*\omega_1)$$

for $0 \le t \le 1$, $X := X_{\omega_0}$, and $\dot\omega_t := \frac{d}{dt}\omega_t = -\omega_0 + \Psi^*\omega_1$. Then clearly $i_X\dot\omega_t = 0$ and $i_X d\omega_t = 0$; in particular $\dot\omega_t|_{\partial SM} = 0$. We will show that

$$\frac{d}{dt}\int_{SM} \omega_t \wedge (d\omega_t)^{n-1} = 0,$$

whence the result follows. Indeed,

$$\frac{d}{dt}\int_{SM} \omega_t\wedge(d\omega_t)^{n-1} = \int_{SM}\dot\omega_t\wedge(d\omega_t)^{n-1} + (n-1)\int_{SM}\omega_t\wedge(d\dot\omega_t)\wedge(d\omega_t)^{n-2}. \tag{4.1}$$

Since $i_X\big(\dot\omega_t\wedge(d\omega_t)^{n-1}\big) = 0$ and $\dot\omega_t\wedge(d\omega_t)^{n-1}$ is a top dimensional form it follows that $\dot\omega_t\wedge(d\omega_t)^{n-1} = 0$. Next, we note that

$$\begin{aligned} d\big(\omega_t\wedge\dot\omega_t\wedge(d\omega_t)^{n-2}\big) &= \dot\omega_t\wedge(d\omega_t)^{n-1} - \omega_t\wedge d\dot\omega_t\wedge(d\omega_t)^{n-2} \\ &+ \omega_t\wedge\dot\omega_t\wedge d\big((d\omega_t)^{n-2}\big) \\ &= -\omega_t\wedge d\dot\omega_t\wedge(d\omega_t)^{n-2}. \end{aligned}$$

This shows that the first integral in (4.1) is zero, and the second is the integral of an exact form, and thus by Stokes' theorem is equal to

$$-(n-1)\int_{\partial SM}\omega_t\wedge\dot\omega_t\wedge(d\omega_t)^{n-2},$$

which is certainly zero since $\dot\omega_t|_{\partial SM} = 0$. □

References

[1] D. Burago and D. Chen, *Finsler perturbation with nondense geodesics with irrational directions*, preprint, available at arXiv:math/1902.10546 (2019).

[2] D. Burago and S. Ivanov, *On asymptotic volume of Finsler tori, minimal surfaces in normed spaces, and symplectic filling volume*, Ann. of Math., **156** (2002), no. 2, 891–914.

[3] ———, *Boundary distance, lens maps and entropy of geodesic flows of Finsler metrics*, Geom. Topol., **20** (2016), 469–490.

[4] B. Colbois, F. Newberger, and P. Verovic, *Some smooth Finsler deformations of hyperbolic surfaces*, Ann. Global Anal. Geom., **35** (2009), 191–226.

[5] M. Cossarini, *Discrete surfaces with length and area and minimal fillings of the circle*, thesis, IMPA, (2018).

[6] C. B. Croke and B. Kleiner, *Conjugacy and rigidity for manifolds with a parallel vector field*, J. Differential Geom., **39** (1994), 659–680.

[7] M. Gromov, *Filling Riemannian manifolds*, J. Differential Geom., **18** (1983), 1–148.

[8] M. V. De Hoop, J. Ilmavirta, M. Lassas, and T Saksala, *Inverse problem for compact Finsler manifolds with the boundary distance map*, preprint, available at arXiv:math/1901.03902 (2019).

[9] S. Ivanov, *Filling minimality of Finslerian 2-discs*, Proc. Steklov Inst. Math., **273** (2011), 176–190.

[10] ———, *Volume comparison via boundary distances*, Bhatia, Rajendra (ed.) et al., Proceedings of the international congress of mathematicians (ICM 2010), Hyderabad, India, World Scientific, 769–784, 2011.

[11] ———, *Local monotonicity of Riemannian and Finsler volume with respect to boundary distances*, Geom. Dedicata, **164** (2013), 83–96.

[12] C.-W. Kim and J.-W. Yim, *Rigidity of noncompact Finsler manifolds*, Geom. Dedicata, **81** (2000), 245–259.

[13] H. Koehler, *On filling minimality of simple Finsler manifolds*, Rev. Mat. Iberoam., **30** (2014), 331–348.

[14] R. Michel, *Sur la rigidité imposée par la longueur des géodésiques*, Invent. Math., **65** (1981), 71–83.

[15] L. Pestov and G. Uhlmann, *Two dimensional compact simple Riemannian manifolds are boundary distance rigid*, Ann. of Math., **161** (2005), no. 2, 1093–1110.

[16] S. Sabourau, *Strong deformation retraction of the space of Zoll Finsler projective planes,* J. Symplectic Geom., **17** (2019), 443–476.

[17] Y.-B. Shen and Z. Shen, *Introduction to modern Finsler geometry*, World Scientific, 2006.

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