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THE PRICING OF VULNERABLE OPTIONS UNDER A CONSTANT ELASTICITY OF VARIANCE MODEL

Junhui U*, Donghyun Kim**, and Ji-Hun Yoon***

Abstract. This paper suggests the price of vulnerable European option under a constant elasticity of variance model by using asymptotic analysis technique and obtains the approximated solution of the option price. Finally, we illustrate an accuracy of the vulnerable option price so that the approximate solution is well-defined.

1. Introduction

The Black–Scholes model suggested Black and Scholes (1973) reflected well the financial market until the global financial crisis of 2007–2009 but, since then, because this model cannot fully account for the situation in a complex market, the significance of non-constant volatility and credit risk factors has come to the fore.

In this context, the constant elasticity of variance (CEV) model can complement the limitations of Black–Scholes model. The CEV model not only proves a good representation of the curve of implied volatility but also captures the price changes of four strategic commodities, namely copper, coal, gold, and crude oil stated in Geman and Shih (2008). The CEV model was first proposed by Cox and Ross (1976) as an alternative to the geometric Brownian motion, to model underlying asset prices. Since then, several improvements have been proposed to this model, for instance, a pricing and hedging of barrier options and lookback options by Davydov and Linetsky (2001). Further, Wong and Zhao (2010) investigated American options using the CEV model by

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***The corresponding author.

using the Laplace–Carson transform. Peng and Peng (2010) studied arithmetic Asian option pricing under the CEV model with the binomial tree method. The option price under the CEV model is obtained through the asymptotic expansion of Park and Kim (2011). A detailed description of the CEV model is provided in section 2.1.

After the recent global financial crisis, which began as a sub-prime mortgage crisis, the importance of risk and credit risk has come to the fore in finance. In particular, several financial studies have focused on the derivatives on credit. The financial and derivative markets worldwide are growing rapidly, and the importance of credit risk and risk continues to be high. Hence, we are at an essential stage in the study of option pricing for credit risk. A vulnerable option is a type of option that represents credit risk. The value of a vulnerable option is determined by the simultaneous consideration of the value of the underlying asset and the market value of the option writer. Johnson and Stulz (1987) proposed the pricing of European-style vulnerable options and provided several examples before the crisis. Klein (1996) proposed an analytic solution for the pricing of vulnerable Black–Scholes options, considering the correlation between the underlying asset of the option and the credit risk of the counterparty; further, in their model, the option writer has other liabilities. Hung and Liu (2005) studied the pricing of vulnerable options under an incomplete market. Yoon and Kim (2015) derived European-style vulnerable options under constant, as well as stochastic interest rates and also studied Jeon et al. (2017) the pricing of vulnerable path-dependent options, such as vulnerable barrier, vulnerable double barrier, and vulnerable lookback options using double Mellin transforms.

This paper studies European-style vulnerable options under CEV and presents an analytic solution. In section 2, we briefly review the CEV model. Here, the partial differential equation (PDE) is induced from the stochastic differential equation to obtain the price of the vulnerable option under the considered model. The solution is then calculated through an approximate analytical method. In section 3, we provide the conclusion.

2. Pricing vulnerable option under a CEV model

Here, we investigate the price of European-style vulnerable options under the CEV model. First, we briefly review the model in subsection 2.1. Second, in subsection 2.2, we describe the model for underlying assets with credit risks and CEV. Using the Feynman–Kac formula, we induce the PDE for the pricing option. Third, in subsection 2.3, we use asymptotic analysis to obtain the solution of the PDE. Finally, we calculate the analytical solution for the vulnerable option price under the CEV model.

2.1. Review of the CEV model

Recently, the study of underlying asset prices and their volatility has become a major concern in financial mathematics research. One of the main topics is the CEV model, which is a stochastic volatility model for capturing the stochastic volatility circumstance and *leverage effects*. The model is widely used by practitioners in the financial industry for modeling stocks and financial products. For instance, Wang et al. (2014) apply the model to study the optimal investment strategy and personal optimal portfolio. A class of CEV models can be described by a stochastic differential equation, as follows:

$$dX_t = \mu X_t dt + \sigma X_t^{\theta/2} dW_t.$$

In its differential form, the equation is a special case of drift term $\mu(t, X_t) = \mu\, X_t$ and diffusion term $\sigma(t, X_t) = \sigma\, X_t^{\theta/2}$, as a general class of stochastic differential equation $dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t$, where W_t is the Brownian motion. Parameter θ controls the relationship between volatility and underlying asset price and is characteristic to the CEV model. CEV models are classified according to elasticity parameter $\theta > 0$ as follows. (1) θ is less than 2. This case was first illustrated by Cox and Ross (1976), where the volatility of underlying asset prices is a decreasing function of underlying asset prices and the leverage effect can be observed, as the effect of increasing volatility when the underlying asset price decreases. (2) θ is equal to 2. The stochastic differential equation becomes the classical Black–Scholes model; therefore, the Black–Scholes model is particular case of a general class of the CEV models. (3) θ is greater than 2. This case was first introduced by Emanuel and Macbeth (1982) by expanding Coxs research, where the volatility of the underlying asset price is an increasing function of the underlying asset price and the reverse leverage effect can be observed.

2.2. Underlying model

Under probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathcal{F}$ is a sigma field, we can define a filtration $\mathcal{F}_t$ generated by Brownian motion $\{W_t : t \geq 0\}$. T is the maturity time, and let X_t be the value of the asset underlying the option at time $t \geq 0$, μ_x the constant drift rate of the underlying assets, and σ_x its constant volatility. Further, let Y_t be the market value of the assets of the option writer with respect to current time $t \geq 0$ with μ_y and σ_y the constant drift rate and volatility, respectively. The difference of the stochastic differential equation between the CEV model and the fundamental model with geometric Brownian motion is that elasticity parameter θ is present in the diffusion part. Then, the dynamics of the X_t and Y_t are given by stochastic differential equations:

$$\begin{aligned} dX_t =& \mu_x X_t dt + \sigma_x X_t^{\theta/2} dW_t^x, \\ dY_t =& \mu_y Y_t dt + \sigma_y Y_t dW_t^y, \end{aligned}$$

where W_t^x and W_t^y are Brownian motions so that $d\langle W_t^x, W_t^y \rangle_t = \rho dt$. Then, by using Girsanovs theorem, the given dynamic model transforms into the stochastic differential equations as follows:

$$\begin{aligned} dX_t =& r X_t dt + \sigma_x X_t^{\theta/2} dW_t^{x*}, \\ dY_t =& r Y_t dt + \sigma_y Y_t dW_t^{y*}, \end{aligned}$$

under a martingale measure(risk-neutral measure) $\mathbb{P}^*$ equivalent to $\mathbb{P}$, where r is the risk-free interest rate and W_t^{x*} and W_t^{y*} are the transformed Brownian motions of W_t^x and W_t^y, respectively, with $d\langle W_t^{x*}, W_t^{y*} \rangle_t = \rho dt$.

To express the given stochastic differential equations as PDEs, let us consider the value of options at maturity T, that is, the payoff function. The payoff of a vulnerable call option considered in this paper is given by Klein (1996) as follows:

$$h(X_T, Y_T) = (X_T - K)^+ \left(1_{\{Y_T \geq D^*\}} + 1_{\{Y_T < D^*\}} \frac{(1-\alpha)Y_T}{D} \right),$$

where K is the strike price of the options and D^* is a fixed default boundary value. A loss of credit happens if the market value of the option writers asset Y_T at maturity T is below D^*. D is the value of the total liabilities of the option writer given by D^* plus the liability owing to the possibility of a counterparty maintaining operation even if V_T is less than D^*. α is the deadweight cost related to the bankruptcy or reorganization process of the firm, expressed as a percentage of the value of the assets of the option writer. If Y_T is greater than or equal to default boundary D^*, the entire claim is paid out. Otherwise, default occurs and only proportion $\frac{(1-\alpha)Y_T}{D}$ of the nominal claim is paid out, where ratio $\frac{Y_T}{D}$ represents the value of the option writer's assets available to pay the claim. In this situation, the no-arbitrage price of a European-style vulnerable call option with payoff function h is given by:

$$P(t,x,y) = \mathbb{E}^*[e^{-r(T-t)}h(X_T,Y_T)|X_t = x, Y_t = y],$$

where $\mathbb{E}^*$ denotes expectation under the martingale measure. By the Feynman–Kac formula, price $P(t,x,y)$ is the solution of the following PDE:

(2.1)

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma_x^2 x^\theta \frac{\partial^2 P}{\partial x^2} + \frac{1}{2}\sigma_y^2 y^2 \frac{\partial^2 P}{\partial y^2} + \rho\sigma_x\sigma_y x^{\theta/2} y \frac{\partial^2 P}{\partial x \partial y} + rx\frac{\partial P}{\partial x} + ry\frac{\partial P}{\partial y} - rP = 0$$

with terminal condition $P(T,x,y) = h(x,y)$.

2.3. Model formulation

We use the following operators to express the PDE for convenience:

(2.2)

$$\mathcal{L}P(t,x,y) = 0, \quad t \le T,$$

$$\mathcal{L} = \frac{\partial}{\partial t} + \frac{1}{2}\sigma_x^2 x^2 \frac{\partial^2}{\partial x^2} + \frac{1}{2}\sigma_y y^2 \frac{\partial^2}{\partial y^2} + \rho\sigma_x\sigma_y xy \frac{\partial^2}{\partial x \partial y} + rx\frac{\partial}{\partial x} + ry\frac{\partial}{\partial y} - rI,$$

where I is the identity operator. To solve the PDE, assume that price $P(t,x,y)$ is the solution to the PDE and has an asymptotic expansion with respect to δ, $P(t,x,y) = \sum_{n\ge 0} P_n(t,x,y)\delta^n$ for $0 < \delta \ll 1$.

THEOREM 2.1. *Suppose that price function $P(t,x,y)$ has an asymptotic expansion $P(t,x,y) = \sum_{n\ge 0} P_n(t,x,y)\delta^n$ for $0 < \delta \ll 1$. Then, we have a system of PDEs as follows:*

(2.3)

$$\mathcal{L}P_0 = 0, \quad P_0(T,x,y) = h(x,y),$$

(2.4)

$$\mathcal{L}P_1 = \frac{1}{2}\sigma_x^2 x^2 (\log x)\frac{\partial^2 P_0}{\partial x^2} + \frac{1}{2}\rho\sigma_x\sigma_y xy(\log x)\frac{\partial^2 P_0}{\partial x \partial y}, \quad P_1(T,x,y) = 0,$$

(2.5)

$$\cdots$$

(2.6)

$$\begin{aligned}\mathcal{L}P_n = &\frac{1}{2}\sigma_x^2 x^2 \sum_{k=0}^{n-1} \frac{(-1)^{n-k+1}(\log x)^{n-k}}{(n-k)!}\frac{\partial^2 P_k}{\partial x^2} \\ &+ \rho\sigma_x\sigma_y xy \sum_{k=0}^{n-1} \frac{(-1)^{n-k+1}(\log x)^{n-k}}{2^{n-k}(n-k)!}\frac{\partial^2 P_k}{\partial x \partial y}, \quad P_n(T,x,y) = 0.\end{aligned}$$

Proof. Assume that parameter θ can be expressed as $\theta = 2-\delta$ for $0 < \delta \ll 1$. The reason for this assumption is that the elasticity parameter θ tends to be less than 2 in many financial problems based on Park and Kim (2011) and are determined within a small neighborhood of 2. Then, using the Taylor series expansion for x^θ and $x^{\theta/2}$ with respect to δ, gives us:

$$\begin{aligned} x^{2-\delta} =& x^2 \sum_{n\geq 0} \frac{(-1)^n}{n\,!}\delta^n (\log x)^n, \\ x^{\frac{1}{2}(2-\delta)} =& x \sum_{n\geq 0} \frac{(-1)^n}{2^n\,n\,!}\delta^n (\log x)^n. \end{aligned}$$

PDE (2.1) can be rewritten by the asymptotic expansion of $P(t,x,y)$ and Taylor series:

$$\begin{aligned} &\underbrace{\frac{\partial}{\partial t}\sum_{n\geq 0} P_n(t,x,y)\delta^n}_{\mathcal{T}(1)} + \underbrace{\frac{1}{2}\sigma_x^2 x^2 \sum_{n\geq 0}\left(\frac{(-1)^n}{n\,!}\delta^n(\log x)^n\right)\frac{\partial^2}{\partial x^2}\sum_{n\geq 0} P_n(t,x,y)\delta^n}_{\mathcal{T}(2)} \\ &+ \underbrace{\frac{1}{2}\sigma_y^2 y^2 \frac{\partial^2}{\partial y^2}\sum_{n\geq 0} P_n(t,x,y)\delta^n}_{\mathcal{T}(3)} \\ &+ \underbrace{\rho\sigma_x\sigma_y xy \sum_{n\geq 0}\left(\frac{(-1)^n}{2^n\,n\,!}\delta^n(\log x)^n\right)\frac{\partial^2}{\partial x\partial y}\sum_{n\geq 0} P_n(t,x,y)\delta^n}_{\mathcal{T}(4)} \\ &+ \underbrace{r\left(x\frac{\partial}{\partial x}\sum_{n\geq 0} P_n(t,x,y)\delta^n + y\frac{\partial}{\partial y}\sum_{n\geq 0} P_n(t,x,y)\delta^n - \sum_{n\geq 0} P_n(t,x,y)\delta^n\right)}_{\mathcal{T}(5)} = 0. \end{aligned}$$

Here, the coefficient on δ with degree 0 is:

$$\frac{\partial P_0}{\partial t} + \frac{1}{2}\sigma_x^2 x^2 \frac{\partial^2 P_0}{\partial x^2} + \frac{1}{2}\sigma_y^2 y^2 \frac{\partial^2 P_0}{\partial y^2} + \rho\sigma_x\sigma_y xy \frac{\partial^2 P_0}{\partial x\partial y} + rx\frac{\partial P_0}{\partial x} + ry\frac{\partial P_0}{\partial y} - rP_0 = 0.$$

Or, using operator $\mathcal{L}$ in (2.2), we can rewrite the above PDE as:

$$\mathcal{L}P_0 = 0.$$

Further, the coefficients on δ with degree n greater than or equal to 1 are calculated as follows:

$$\mathcal{T}(1) = \frac{\partial}{\partial t}\sum_{n\geq 0} P_n(t,x,y)\delta^n \to \frac{\partial}{\partial t}P_n(t,x,y),$$

$$\begin{aligned}\mathcal{T}(2) &= \frac{1}{2}\sigma_x^2 x^2 \sum_{n\geq 0}\left(\frac{(-1)^n}{n\,!}\delta^n(\log x)^n\right)\frac{\partial^2}{\partial x^2}\sum_{n\geq 0} P_n(t,x,y)\delta^n \\ &= \frac{1}{2}\sigma_x^2 x^2 \sum_{n\geq 0}\left(\sum_{k=0}^{n}\frac{(-1)^{n-k}(\log x)^{n-k}}{(n-k)\,!}\frac{\partial^2}{\partial x^2}P_k(t,x,y)\right)\delta^n \\ &\to \frac{1}{2}\sigma_x^2 x^2 \sum_{k=0}^{n}\frac{(-1)^{n-k}(\log x)^{n-k}}{(n-k)\,!}\frac{\partial^2}{\partial x^2}P_k(t,x,y),\end{aligned}$$

$$\begin{aligned}\mathcal{T}(3) &= \frac{1}{2}\sigma_y^2 y^2 \frac{\partial^2}{\partial y^2}\sum_{n\geq 0} P_n(t,x,y)\delta^n \\ &\to \frac{1}{2}\sigma_y^2 y^2 \frac{\partial^2}{\partial y^2}P_n(t,x,y),\end{aligned}$$

$$\begin{aligned}\mathcal{T}(4) &= \rho\sigma_x\sigma_y xy \sum_{n\geq 0}\left(\frac{(-1)^n}{2^n\, n\,!}\delta^n(\log x)^n\right)\frac{\partial^2}{\partial x\partial y}\sum_{n\geq 0} P_n(t,x,y)\delta^n \\ &= \rho\sigma_x\sigma_y xy \sum_{n\geq 0}\left(\sum_{k=0}^{n}\frac{(-1)^{n-k}(\log x)^{n-k}}{(n-k)\,!}\left(\frac{1}{2}\right)^{n-k}\frac{\partial^2}{\partial x\partial y}P_k(t,x,y)\right)\delta^n \\ &\to \rho\sigma_x\sigma_y xy \sum_{k=0}^{n}\frac{(-1)^{n-k}(\log x)^{n-k}}{(n-k)\,!}\left(\frac{1}{2}\right)^{n-k}\frac{\partial^2}{\partial x\partial y}P_k(t,x,y),\end{aligned}$$

$$\begin{aligned}\mathcal{T}(5) &= r\left(x\frac{\partial}{\partial x}\sum_{n\geq 0} P_n(t,x,y)\delta^n y\frac{\partial}{\partial y}\sum_{n\geq 0} P_n(t,x,y)\delta^n - \sum_{n\geq 0} P_n(t,x,y)\delta^n\right) \\ &= r\sum_{n\geq 0}\left(x\frac{\partial}{\partial x}P_n(t,x,y) + y\frac{\partial}{\partial y}P_n(t,x,y) - P_n(t,x,y)\right)\delta^n \\ &\to r\left(x\frac{\partial}{\partial x}P_n(t,x,y) + y\frac{\partial}{\partial y}P_n(t,x,y) - P_n(t,x,y)\right).\end{aligned}$$

Therefore, the sum of the five coefficients on δ with degree n is expressed as:

$$\begin{aligned}\mathcal{L}P_n =&\frac{1}{2}\sigma_x^2 x^2 \sum_{k=0}^{n-1} \frac{(-1)^{n-k+1}(\log x)^{n-k}}{(n-k)\,!}\frac{\partial^2}{\partial x^2}P_k(t,x,y)\\ &+ \rho\sigma_x\sigma_y xy \sum_{k=0}^{n-1} \frac{(-1)^{n-k+1}(\log x)^{n-k}}{2^{n-k}(n-k)\,!}\frac{\partial^2}{\partial x \partial y}P_k(t,x,y).\end{aligned} \tag{2.7}$$

Finally, from the asymptotic expansion $P(t,x,y) = \sum_{n\geq 0} P_n(t,x,y)\delta^n$ and the final condition $P(T,x,y) = h(x,y)$, we have the final condition corresponding to each PDE stated in (2.3)-(2.6). □

We denote right-hand side of (2.7) as $g_n(t,x,y)$ in the following. The following Lemma 2.1 and Theorem 2.2 provide the analytic solution for PDE (2.2).

LEMMA 2.2. *Suppose that function $P_0(t,x,y)$ is a solution of PDE (2.3) with the terminal condition $P_0(T,x,y) = h(x,y)$. Then, function $P_0(t,x,y)$ is given by:*

$$\begin{aligned}P_0(t,x,y) =&x\Phi_2(a_1,a_2,\rho) - e^{-r(T-t)}K\Phi_2(b_1,b_2,\rho)\\ &- \frac{(1-\alpha)y}{D}\{xe^{(r+\rho\sigma_x\sigma_y)(T-t)}\Phi_2(c_1,c_2,\rho) - K\Phi_2(d_1,d_2,\rho)\},\end{aligned}$$

where

$$\begin{aligned}a_1 &= \frac{\log(x/K) + \left(r + \sigma_x^2/2\right)(T-t)}{\sigma_x\sqrt{T-t}},\\ b_1 &= \frac{\log(x/K) + \left(r - \sigma_x^2/2\right)(T-t)}{\sigma_x\sqrt{T-t}},\\ c_1 &= \frac{\log(x/K) + \left(r + \sigma_x^2/2 + \rho\sigma_x\sigma_y\right)(T-t)}{\sigma_x\sqrt{T-t}},\\ d_1 &= \frac{\log(x/K) + \left(r - \sigma_x^2/2 + \rho\sigma_x\sigma_y\right)(T-t)}{\sigma_x\sqrt{T-t}},\end{aligned}$$

$$a_2 = \frac{\log(y/D^*) + \left(r - \sigma_y^2/2 + \rho\sigma_x\sigma_y\right)(T-t)}{\sigma_y\sqrt{T-t}},$$
$$b_2 = \frac{\log(y/D^*) + \left(r - \sigma_y^2/2 +\right)(T-t)}{\sigma_y\sqrt{T-t}},$$
$$c_2 = \frac{\log(y/D^*) + \left(r + \sigma_y^2/2 + \rho\sigma_x\sigma_y\right)(T-t)}{\sigma_y\sqrt{T-t}},$$
$$d_2 = \frac{\log(y/D^*) + \left(r + \sigma_y^2/2\right)(T-t)}{\sigma_y\sqrt{T-t}}.$$

and $\Phi_2(n_1, n_{2,}, \rho)$ *is the cumulative bivariate normal distribution defined by:*

$$\Phi_2(n_1, n_2, \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{n_1}\int_{-\infty}^{n_2} \exp\left(-\frac{1}{2(1-\rho^2)}(u^2 - 2\rho uv + v^2)\right) du\, dv.$$

Proof. See Yoon and Kim (2015) for a detailed proof. □

THEOREM 2.3. *For each* $n \geq 1$*:*

$$P_n(t, x, y) = e^{\alpha p(y) + \beta q(x,y) + \gamma(T-t)} \Lambda_n(\tau, p^*, q^*),$$

where

$$\Lambda_n(\tau, p^*, q^*) = \int_{\varsigma=0}^{\tau}\int_{\zeta=-\infty}^{\infty}\int_{\xi=-\infty}^{\infty} \mathbb{H}(\tau-\varsigma, p^*-\zeta, q^*-\xi) G_n(\varsigma, \zeta, \xi)\, d\xi d\zeta d\varsigma,$$
$$\mathbb{H}(\varsigma, \zeta, \xi) = \frac{1}{4\pi\varsigma}\exp\left(-\frac{\zeta^2+\xi^2}{4\varsigma}\right),$$
$$G_n(\varsigma, \zeta, \xi) = -e^{\alpha\sqrt{b}\zeta + \beta\sqrt{d}\xi + \gamma\varsigma} g_n\left(T-\varsigma, \exp\left(\frac{\sigma_x\sqrt{b}}{\sigma_y^2}\zeta - \frac{\sqrt{d}}{\sigma_y}\xi\right), \exp\left(\frac{\sqrt{b}}{\rho\sigma_y}\zeta\right)\right),$$
$$p(y) = \rho\sigma_y \log y, \quad q(x, y) = \rho\sigma_x \log y - \sigma_y \log x, \quad \tau = T - t,$$
$$\alpha = -\frac{a}{2b}, \quad \beta = -\frac{c}{2d}, \quad \gamma = -\frac{a^2}{4b} - \frac{c^2}{4d} - r,$$
$$p^*(y) = \frac{p(y)}{\sqrt{b}}, \quad q^*(x, y) = \frac{q(x, y)}{\sqrt{d}},$$
$$a = r\rho\sigma_y - \frac{1}{2}\rho\sigma_y^3, \quad b = \frac{1}{2}\rho^2\sigma_y^4,$$
$$c = \frac{1}{2}\sigma_x^2\sigma_y - \frac{1}{2}\rho\sigma_x\sigma_y + r\rho\sigma_x, \quad d = \frac{1}{2}\sigma_x^2\sigma_y^2(1-\rho^2).$$

Proof. If we find a solution for PDE (2.6) for each positive integer n, the delta expansion of $\sum_{n\geq 0} P_n(t,x,y)\delta^n$ is completely determined. Therefore, $P(t,x,y)$ is also induced. To solve PDE (2.6) for each positive integer n, the transformation of variables is required as follows:

$$\begin{aligned} &p = A\log y, \qquad q = B\log y + C\log x, \qquad \tau = T - t, \\ &V_n(\tau,p,q) = P_n(t,x,y). \end{aligned}$$

Constants A, B, and C are positive numbers and will be calculated using the subsequent process. The chain rule induces that:

(2.8)

$$\begin{aligned} \frac{\partial P_n}{\partial t} &= -\frac{\partial V_n}{\partial \tau}, \\ \frac{\partial P_n}{\partial x} &= \frac{C}{x}\frac{\partial V_n}{\partial q}, \\ \frac{\partial^2 P_n}{\partial x^2} &= \frac{C}{x^2}\left(C\frac{\partial^2 V_n}{\partial q^2} - \frac{\partial V_n}{\partial q}\right), \\ \frac{\partial P_n}{\partial y} &= \frac{1}{y}\left(A\frac{\partial V_n}{\partial p} + B\frac{\partial V_n}{\partial q}\right), \\ \frac{\partial^2 P_n}{\partial y^2} &= \frac{1}{y^2}\left(A^2\frac{\partial^2 V_n}{\partial p^2} + B^2\frac{\partial^2 V_n}{\partial q^2} + 2AB\frac{\partial^2 V_n}{\partial p\,\partial q} - A\frac{\partial V_n}{\partial p} - B\frac{\partial V_n}{\partial q}\right), \\ \frac{\partial^2 P_n}{\partial x\,\partial y} &= \frac{C}{xy}\left(A\frac{\partial^2 V_n}{\partial p\,\partial q} + B\frac{\partial^2 V_n}{\partial q^2}\right). \end{aligned}$$

Substituting (2.8) into (2.7) drives the following PDE in terms of τ, p, and q:

$$\begin{aligned} &\mathcal{L}P_n(t,x,y) \\ &= \frac{\partial P_n}{\partial t} + \frac{1}{2}\sigma_x^2 x^2\frac{\partial^2 P_n}{\partial x^2} + \frac{1}{2}\sigma_y^2 y^2\frac{\partial^2 P_n}{\partial y^2} + \rho\sigma_x\sigma_y xy\frac{\partial^2 P_n}{\partial x\,\partial y} + rx\frac{\partial P_n}{\partial x} + ry\frac{\partial P_n}{\partial y} - rP_n \\ &= -\frac{\partial V_n}{\partial \tau} + \frac{1}{2}\sigma_x^2 C\left(C\frac{\partial^2 V_n}{\partial q^2} - \frac{\partial V_n}{\partial q}\right) \\ &\quad + \frac{1}{2}\sigma_y^2\left(A^2\frac{\partial^2 V_n}{\partial p^2} + B^2\frac{\partial^2 V_n}{\partial q^2} + 2AB\frac{\partial^2 V_n}{\partial p\,\partial q} - A\frac{\partial V_n}{\partial p} - B\frac{\partial V_n}{\partial q}\right) \\ &\quad + \rho\sigma_x\sigma_y C\left(A\frac{\partial^2 V_n}{\partial p\,\partial q} + B\frac{\partial^2 V_n}{\partial q^2}\right) + rC\frac{\partial V_n}{\partial q} + rA\frac{\partial V_n}{\partial p} + rB\frac{\partial V_n}{\partial q} - rV_n \end{aligned}$$

$$= -\frac{\partial V_n}{\partial \tau} + \underbrace{\left(rA - \frac{1}{2}\sigma_y^2 A\right)}_{a}\frac{\partial V_n}{\partial p} + \underbrace{\left(\frac{1}{2}\sigma_y^2 A^2\right)}_{b}\frac{\partial^2 V_n}{\partial p^2}$$
$$+ \underbrace{\left(-\frac{1}{2}\sigma_x^2 C - \frac{1}{2}\sigma_y^2 B + rC + rB\right)}_{c}\frac{\partial V_n}{\partial q}$$
$$+ \underbrace{\left(\frac{1}{2}\sigma_x^2 C^2 + \frac{1}{2}\sigma_y^2 B^2 + \rho\sigma_x\sigma_y BC\right)}_{d}\frac{\partial^2 V_n}{\partial q^2} + \underbrace{\left(\sigma_y^2 AB + \rho\sigma_x\sigma_y AC\right)}_{e}\frac{\partial^2 V_n}{\partial p\,\partial q} - rV_n$$
$$= g_n(t, x, y).$$

For convenience, we substitute the coefficients of the equation as above. That is,

$$a = rA - \frac{1}{2}\sigma_y^2 A, \quad b = \frac{1}{2}\sigma_y^2 A^2, \quad c = -\frac{1}{2}\sigma_x^2 C - \frac{1}{2}\sigma_y^2 B + rC + rB,$$
$$d = \frac{1}{2}\sigma_x^2 C^2 + \frac{1}{2}\sigma_y^2 B^2 + \rho\sigma_x\sigma_y BC, \quad e = \sigma_y^2 AB + \rho\sigma_x\sigma_y AC.$$

Then, we have:

(2.9)
$$-\frac{\partial V_n}{\partial \tau} + a\frac{\partial V_n}{\partial p} + b\frac{\partial^2 V_n}{\partial p^2} + c\frac{\partial V_n}{\partial q} + d\frac{\partial^2 V_n}{\partial q^2} + e\frac{\partial^2 V_n}{\partial p\,\partial q} - rV_n = g_n(t, x, y).$$

Defining $\Lambda_n(\tau, p, q)$ as $\Lambda_n(\tau, p, q) = e^{-(\alpha p + \beta q + \gamma\tau)} V_n(\tau, p, q)$ and applying the chain rule induce:

(2.10)
$$\begin{aligned}
\frac{\partial V_n}{\partial \tau} &= e^{\alpha p + \beta q + \gamma\tau}\left(\gamma\Lambda_n + \frac{\partial \Lambda_n}{\partial \tau}\right),\\
\frac{\partial V_n}{\partial p} &= e^{\alpha p + \beta q + \gamma\tau}\left(\alpha\Lambda_n + \frac{\partial \Lambda_n}{\partial p}\right),\\
\frac{\partial^2 V_n}{\partial p^2} &= e^{\alpha p + \beta q + \gamma\tau}\left(\alpha^2\Lambda_n + 2\alpha\frac{\partial \Lambda_n}{\partial p} + \frac{\partial^2 \Lambda_n}{\partial p^2}\right),\\
\frac{\partial V_n}{\partial q} &= e^{\alpha p + \beta q + \gamma\tau}\left(\beta\Lambda_n + \frac{\partial \Lambda_n}{\partial q}\right),\\
\frac{\partial^2 V_n}{\partial q^2} &= e^{\alpha p + \beta q + \gamma\tau}\left(\beta^2\Lambda_n + 2\beta\frac{\partial \Lambda_n}{\partial q} + \frac{\partial^2 \Lambda_n}{\partial q^2}\right),\\
\frac{\partial^2 V_n}{\partial p\,\partial q} &= e^{\alpha p + \beta q + \gamma\tau}\left(\alpha\beta\Lambda_n + \beta\frac{\partial \Lambda_n}{\partial p} + \alpha\frac{\partial \Lambda_n}{\partial q} + \frac{\partial^2 \Lambda_n}{\partial p\,\partial q}\right).
\end{aligned}$$

Substituting (2.10) into (2.9) induces:

$$
\begin{aligned}
&-\frac{\partial V_n}{\partial \tau}+a\frac{\partial V_n}{\partial p}+b\frac{\partial^2 V_n}{\partial p^2}+c\frac{\partial V_n}{\partial q}+d\frac{\partial^2 V_n}{\partial q^2}+e\frac{\partial^2 V_n}{\partial p\,\partial q}-rV_n\\
=&\,e^{\alpha p+\beta q+\gamma\tau}\left[-\left(\gamma\Lambda_n+\frac{\partial\Lambda_n}{\partial\tau}\right)+a\left(\alpha\Lambda_n+\frac{\partial\Lambda_n}{\partial p}\right)+b\left(\alpha^2\Lambda_n+2\alpha\frac{\partial\Lambda_n}{\partial p}+\frac{\partial^2\Lambda_n}{\partial p^2}\right)\right]\\
&+e^{\alpha p+\beta q+\gamma\tau}\left[c\left(\beta\Lambda_n+\frac{\partial\Lambda_n}{\partial q}\right)\right]\\
&+e^{\alpha p+\beta q+\gamma\tau}\left[d\left(\beta^2\Lambda_n+2\beta\frac{\partial\Lambda_n}{\partial q}+\frac{\partial^2\Lambda_n}{\partial q^2}\right)+e\left(\alpha\beta\Lambda_n+\beta\frac{\partial\Lambda_n}{\partial p}+\alpha\frac{\partial\Lambda_n}{\partial q}+\frac{\partial^2\Lambda_n}{\partial p\partial q}\right)\right]\\
=&\,g_n(t,x,y).
\end{aligned}
$$

Therefore, by multiplying both side by $e^{\alpha p+\beta q+\gamma\tau}$, it can be rewritten as follows:

$$
\begin{aligned}
&-\frac{\partial\Lambda_n}{\partial\tau}+(a+2\alpha b+\beta e)\frac{\partial\Lambda_n}{\partial p}+b\frac{\partial^2\Lambda_n}{\partial p^2}+(c+2\beta d+\alpha e)\frac{\partial\Lambda_n}{\partial q}+d\frac{\partial^2\Lambda_n}{\partial q^2}\\
&+e\frac{\partial^2\Lambda_n}{\partial p\partial q}+\left(-\gamma+\alpha a+\alpha^2 b+\beta c+\beta^2 d+\alpha\beta e-r\right)\Lambda_n=e^{-(\alpha p+\beta q+\gamma\tau)}g_n(t,x,y).
\end{aligned}
$$

For the above expression to be a two-dimensional heat equation, it is required the four coefficients on terms $\frac{\partial\Lambda}{\partial p}$, $\frac{\partial\Lambda}{\partial q}$,$\frac{\partial^2\Lambda_n}{\partial p\partial q}$, and Λ_n to be zero. Hence, the next system of equations is obtained:

$$
\begin{cases}
a+2\alpha b+\beta e=0,\\
c+2\beta d+\alpha e=0,\\
e=0,\\
-\gamma+\alpha a+\alpha^2 b+\beta c+\beta^2 d+\alpha\beta e-r=0.
\end{cases}
$$

Therefore, it follows that:

$$
\begin{cases}
a+2\alpha b=0,\\
c+2\beta d=0,\\
-\gamma+\alpha a+\alpha^2 b+\beta c+\beta^2 d-r=0.
\end{cases}
$$

Here, we can obtain necessary coefficients as follows:

$$
\alpha=-\frac{a}{2b},\qquad \beta=-\frac{c}{2d},\qquad \gamma=-\frac{a^2}{4b}-\frac{c^2}{4d}-r.
$$

For a given equation to be a two-dimensional heat equation, coefficient $e=\sigma_y^2 AB+\rho\sigma_x\sigma_y AC$ must disappear. As such, we use nonzero constants B and C satisfying $e=0$ as $B=\rho\sigma_x$, and $C=-\sigma_y$. Further, since $p(y)=A\log y$ has nothing to do with nonzero constant A, we can then write A as $A = \rho\sigma_y$. Accordingly, we use the coefficients in Yang et al. (2014). For this reason, we can determine coefficients a, b, c, and d through $A = \rho\sigma_y$, $B = \rho\sigma_x$, and $C = -\sigma_y$ as follows:

$$a = r\rho\sigma_y - \frac{1}{2}\rho\sigma_y^3, \quad b = \frac{1}{2}\rho^2\sigma_y^4,$$
$$c = \frac{1}{2}\sigma_x^2\sigma_y - \frac{1}{2}\rho\sigma_x\sigma_y + r\rho\sigma_x, \quad d = \frac{1}{2}\sigma_x^2\sigma_y^2(1-\rho^2).$$

Now, replacing $p(y)$, $q(x, y)$, and $-e^{\alpha\sqrt{b}p^*(y)+\beta\sqrt{d}q^*(x,y)+\gamma\tau}g_n(t, x, y)$ with $p^*(y) = \frac{p(y)}{\sqrt{b}}$,$q^*(x, y) = \frac{q(x,y)}{\sqrt{d}}$ and $G_n(\tau, p^*, q^*)$, respectively, we obtain the heat equation with the initial condition:

$$\begin{cases} \dfrac{\partial\Lambda_n}{\partial\tau} = \dfrac{\partial^2\Lambda_n}{\partial p^{*2}} + \dfrac{\partial^2\Lambda_n}{\partial q^{*2}} + G_n(\tau, p^*, q^*), \\ \Lambda_n(0, p^*, q^*) = 0. \end{cases}$$

Using Duhamel's principle for a two-dimensional heat equation, we derive solution $\Lambda_n(\tau, p^*, q^*)$:

$$\Lambda_n(\tau, p^*, q^*) = \int_{\varsigma=0}^{\tau}\int_{\zeta=-\infty}^{\infty}\int_{\xi=-\infty}^{\infty} \mathbb{H}(\tau-\varsigma, p^*-\zeta, q^*-\xi)\, G_n(\varsigma, \zeta, \xi)\, d\xi\, d\zeta\, d\varsigma,$$

where heat kernel $\mathbb{H}(\varsigma, \zeta, \xi)$ for a two-dimensional heat equation and function $G_n(\varsigma, \zeta, \xi)$ are respectively given by:

$$\mathbb{H}(\varsigma, \zeta, \xi) = \frac{1}{4\pi\varsigma}\exp\left(-\frac{\zeta^2+\xi^2}{4\varsigma}\right),$$
$$\begin{aligned} G_n(\varsigma, \zeta, \xi) &= -e^{\alpha\sqrt{b}\zeta+\beta\sqrt{d}\xi+\gamma\varsigma}g_n(t(\tau), x(\sqrt{b}\zeta, \sqrt{d}\xi), y(\sqrt{b}\zeta)) \\ &= -e^{\alpha\sqrt{b}\zeta+\beta\sqrt{d}\xi+\gamma\varsigma}g_n\left(T-\varsigma, \exp\left(\frac{\sigma_x\sqrt{b}}{\sigma_y^2}\zeta - \frac{\sqrt{d}}{\sigma_y}\xi\right), \exp\left(\frac{\sqrt{b}}{\rho\sigma_y}\zeta\right)\right). \end{aligned}$$

Therefore, we obtain the desired results. □

Accuracy

The existence of asymptotic expansion of price function $P(t, x, y)$ is demonstrated from a similar method in Park and Kim (2011) : For each positive integer $M \geq 1$, there exists a positive constant K satisfying

$$\left| P(t, x, y) - \sum_{0\leq n\leq M} P_n(t, x, y)\delta^n \right| \leq K\delta^{M+1}$$

Since $0 < \delta \ll 1$, the existence of asymptotic expansion of $P(t, x, y)$ is guaranteed.

Proof. See Park and Kim [10]. □

3. Conclusions

In this paper, we derive the analytic solution for vulnerable European option pricing under the CEV model by using an asymptotic expansion in terms of elasticity parameter θ. The CEV model with credit risk is widely applied to model stocks, financial products and optimal portfolio in many studies and works of finance. To obtain a more accurate option price under that more research on vulnerable options with not only constant volatility but also stochastic volatility in the CEV model is required additionally.

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*
Department of Mathematics
Pusan National University
Busan 46241, Republic of Korea
E-mail: junhui.u.math@gmail.com

**
Department of Mathematics
Pusan National University
Busan 46241, Republic of Korea
E-mail: donghyunkim@pusan.ac.kr

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Department of Mathematics
Pusan National University
Busan 46241, Republic of Korea
E-mail: yssci99@pusan.ac.kr