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Correction to "On prime near-rings with generalized (σ, τ)-derivations, Kyungpook Math. J., 45(2005), 249–254"

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Abstract. In the proof of Theorem 3 on p.253 in [4], both right and left distributivity are assumed simultaneously which makes the proof invalid. We give a corrected proof for this theorem by introducing an extension of Lemma 2.2 in [2].

1. Introduction

Over the last few decades lots of work has been done on commutativity of prime rings with derivations. It is natural to look for comparable results on near rings (see for example [6]). Historically speaking, the study of derivations of prime near-rings was initiated by H. E. Bell and G. Mason in 1987 [3]. An analogue of Posner's result on prime near-rings was obtained by Beidar in [1]. Results concerning prime near-rings with derivations have since been generalized in several ways. Throughout the paper N is a zero-symmetric left near-ring with a multiplicative center Z. For $x, y \in N$, the symbol $[x, y]$ will denote the commutator $xy - yx$, while the symbol

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(x, y) will denote the additive group commutator $x + y - x - y$. Our main aim in this paper is to give a correction for Theorem 3 on p.253 in [4].

The proof of Theorem 3 of [4], stated as Theorem 2.12 in this paper, is incorrect because of the assumption of both left and right distributivity. In Section 3, we first introduce Lemma 3.1, which is an extension of [2, Lemma 2.2], then give a correction for Theorem 2.12.

2. Preliminaries

We first recall some of the background which will be used throughout this work.

Definition 2.1. A *left near-ring* is a set N together with two binary operations $+$ and . such that $(N, +)$ is a group (not necessarily abelian) and $(N, .)$ is a semigroup satisfying the distributive law $x(y + z) = xy + xz$ for all $x, y, z \in N$.

Definition 2.2. A near-ring N is said to be *zero-symmetric* if $0.x = 0$ $\forall a \in N$.

Definition 2.3. A left near-ring N is called a *prime near-ring* if $a, b \in N$ and $aNb = 0$ imply that $a = 0$ or $b = 0$.

Definition 2.4. A *derivation* on a near-ring N is defined to be an additive endomorphism satisfying the "product rule" $d(xy) = xd(y) + d(x)y$ for all $x, y \in N$.

Definition 2.5. Let σ, τ be two endomorphisms of a near-ring N. An additive mapping $d : N \to N$ is called a (σ, τ)*-derivation* if $d(xy) = \sigma(x)d(y) + d(x)\tau(y)$ for all $x, y \in N$.

Definition 2.6. Let N be a near-ring, and d a derivation of N. An additive mapping $f : N \to N$ is said to be *left generalized* (σ, τ)*-derivation* of N associated with d if

$$f(xy) = d(x)\tau(y) + \sigma(x)f(y)\ \forall x, y \in N.$$

f is said to be a *right generalized* (σ, τ)*-derivation* of N associated with d if

$$f(xy) = f(x)\tau(y) + \sigma(x)d(y)\ \forall x, y \in N.$$

f is said to be a *generalized* (σ, τ)*-derivation* associated with d if it is both left and right generalized (σ, τ)-derivation associated with d.

We now recall some properties about near-rings with derivation and generalized (σ, τ)-derivation.

Lemma 2.7.([4, Lemma 1])

(i) *Let f be a right generalized (σ, τ)-derivation of near-ring N associated with d. Then $f(xy) = \sigma(x)d(y) + f(x)\tau(y)$ $\forall x, y \in N$.*

(ii) *Let f be a left generalized (σ, τ)-derivation of near-ring N associated with d. Then $f(xy) = \sigma(x)f(y) + d(x)\tau(y)$ $\forall x, y \in N$.*

Lemma 2.8.([4, Lemma 3]) *Let N be a prime near-ring, f a generalized (σ,τ)-derivation of N associated with d, and $a \in N$.*

(i) *If $af(N) = 0$, then $a = 0$.*

(ii) *If $f(N)\tau(a) = 0$, then $a = 0$.*

Lemma 2.9.([5, Lemma 3]) *Let N be a prime near-ring, d a non-zero (σ,τ)-derivation of N, and $a \in N$.*

(i) *If $d(N)\sigma(a) = 0$, then $a = 0$.*

(ii) *If $ad(N) = 0$, then $a = 0$.*

Lemma 2.10.([4, Lemma 2])

(i) *Let f be a right generalized (σ,τ)-derivation of near-ring N associated with d. Then*

$$(f(x)\tau(y) + \sigma(x)d(y))\tau(z) = f(x)\tau(y)\tau(z) + \sigma(x)d(y)\tau(z),$$

for all $x, y \in N$.

(ii) *Let f be a generalized (σ,τ)-derivation of near-ring N associated with d. Then*

$$(d(x)\tau(y) + \sigma(x)f(y)\tau(z)) = d(x)\tau(y)\tau(z) + \sigma(x)f(y)\tau(z),$$

for all $x, y \in N$.

Theorem 2.11.([4, Theorem 2]) *Let N be a prime near-ring with a non-zero generalized (σ,τ)-derivation f associated with d. If $f(N) \subset Z$ then $(N,+)$ is abelian. Moreover, if N is 2-torsion free then N is a commutative ring.*

Theorem 2.12.([4, Theorem 3]) *Let N be a prime near-ring with a non-zero generalized (σ,τ)-derivation f associated with non-zero (σ,τ)-derivation d such that $\tau f = f\tau, f\sigma = \sigma f, \sigma d = d\sigma$, and $\tau d = d\tau$. If $[f(N), f(N)] = 0$ then $(N,+)$ is abelian. Moreover, if N is 2-torsion free, then N is commutative.*

3. Corrected Proof of Theorem 2.12

We first give the following auxiliary lemma.

Lemma 3.1. *Let N be a prime near-ring and f be a generalized (σ,τ)-derivation associated with non-zero (σ,τ)-derivation d such that $\sigma f = f\sigma$, $\tau f = f\tau$ and $\sigma d = d\sigma$, $\tau d = d\tau$. If $d(f(N)) = \{0\}$, then $f(d(N)) = \{0\}$.*

Proof. Assume that $d(f(x)) = 0$. It follows for all $x, y \in N$,

$$0 = d(f(xy)) = d(f(x)\tau(y)) + d[\sigma(x)d(y)) = d(f(x)\tau(y)) + d(\sigma(x)d(y));$$

$$0 = \sigma(f(x))d(\tau(y)) + \sigma^2(x)d^2(y) + d(\sigma(x))\tau(d(y)). \tag{3.1}$$

Applying d to (3.1), we get

$$0 = d(\sigma(f(x))d(\tau(y))) + d(\sigma^2(x)d^2(y)) + d(d(\sigma(x))\tau(d(y))).$$

$$\begin{aligned} 0 &= \sigma^2(f(x))d^2(\tau(y)) + \sigma^3(x)d^3(y) \\ &\quad + d(\sigma^2(x))\tau(d^2(y)) + d(\sigma^2(x))\tau(d^2(y)) + d^2(\sigma(x))\tau^2(d(y)). \end{aligned} \tag{3.2}$$

Substituting $d(y)$ for y in (3.1) gives

$$0 = \sigma(f(x))d^2(\tau(y)) + \sigma^2(x)d^3(y) + d(\sigma(x))\tau(d^2(y)). \tag{3.3}$$

Replacing x by $\sigma(x)$ in (3.3), we get

$$0 = \sigma^2(f(x))d^2(\tau(y)) + \sigma^3(x)d^3(y) + d(\sigma^2(x))\tau(d^2(y)). \tag{3.4}$$

Hence (3.2) yields

$$0 = d(\sigma^2(x))\tau(d^2(y)) + d^2(\sigma(x))\tau^2(d(y)). \tag{3.5}$$

Replacing x by $d(x)$ in (3.1) we obtain

$$0 = \sigma(f(d(x)))d(\tau(y)) + \sigma^2(d(x))d^2(y) + d^2(\sigma(x))\tau(d(y)). \tag{3.6}$$

Taking $\tau(y)$ for y in (3.6) gives

$$0 = \sigma(f(d(x)))d(\tau^2(y)) + \sigma^2(d(x))d^2(\tau(y)) + d^2(\sigma(x))\tau^2(d(y)). \tag{3.7}$$

By using (3.5) we have

$$0 = \sigma(f(d(x)))d(\tau^2(y)). \tag{3.8}$$

Replacing x by $\sigma^-1(x)$ in (3.8), we get

$$f(d(x))d(\tau^2(y)) = 0.$$

$\forall x, y \in N$. Thus, $f(d(x)) = 0\ \forall x \in N$ by using Lemma 2.9 (ii). This finishes the proof. □

The preceding lemma will now be used to establish the correction.

Proof of Theorem 2.12. We start with the same argument given in the proof of [4, Theorem 3]. If z and $z+z$ commute element wise with $f(N)$, then for any $x, y \in N$,

$$(z + z)(f(x) + f(y)) = (f(x) + f(y))(z + z).$$

So we get $zf(x) + zf(y) - zf(x) - zf(y) = 0$ and this gives

$$zf(x, y) = 0\ \forall x, y \in N. \tag{3.9}$$

Substituting $f(t)$, $t \in N$, for z in (9) we have

$$f(t)f((x, y)) = 0.$$

Since τ is an automorphsim of N we have $\tau(f(t))\tau(f(x, y)) = 0$. By using the assumption $\tau f = f\tau$, we get

$$f(\tau(t))\tau(f(x, y)) = 0 \ \forall x, y, t \in N.$$

By Lemma 2.8, we get $f(x, y) = 0 \ \forall x, y \in N$. Now for $w \in N$, we have

$$0 = f(wx, wy) = f(w(x, y)) = d(w)\tau(x, y) + \sigma(w)f(x, y).$$

and this gives

$$d(w)\tau(x, y) = 0. \tag{3.10}$$

Replacing w by wr in (10) and using Lemma 2.10 we get

$$d(w)\tau(r)\tau(x, y) + \sigma(w)d(r)\tau(x, y) = 0,$$

and this gives

$$d(w)N\tau(x, y) = 0 \ \forall x, y, w \in N.$$

Since N is a prime near-ring, $d \neq 0$, we get $(x, y) = 0$, so $x + y - x - y = 0$, and this gives $x + y = y + x$. Hence $(N, +)$ is abelian.

Now, assume that N is 2-torsion free. By the assumption $[f(N), f(N)] = 0$, we have

$$f(\tau(z))f(f(x)y) = f(f(x)y)f(\tau(z)) \ \forall x, y, z \in N.$$

Using $\tau f = f\tau$, $f\sigma = \sigma f$ and using Lemma 2.7, we obtain

$$\begin{aligned} f(\tau(z))d(f(x))\tau(y) &+ f(\tau(z))\sigma(f(x))f(y) \\ &= d(f(x))\tau(y)\tau(f(z)) + \sigma(f(x))f(y)\tau(f(z)). \end{aligned}$$

and this gives

$$f(\tau(z))d(f(x))\tau(y) = d(f(x))\tau(y)\tau(f(z)). \tag{3.11}$$

If we take yw instead of y in (11), then

$$f(\tau(z))d(f(x))\tau(y)\tau(w) = d(f(x))\tau(y)\tau(w)\tau(f(z)).$$

So we have

$$d(f(x))\tau(y)f(\tau(z))\tau(w) = d(f(x))\tau(y)\tau(w)\tau(f(z)) \ \forall x, y, w \in N.$$

Hence $d(f(x))N(f(\tau(z)), \tau(w)) = 0$. Since N is prime, we have

$$d(f(x)) = 0 \text{ or } f(N) \subseteq Z.$$

If $f(N) \subseteq Z$, then N is a commutative ring by Theorem 2.11. It remains to show that $d(f(x)) = 0$ is impossible.

Assume, by way of contradiction, that $d(f(x)) = 0$, then

$$0 = d(f(xy)) = d(d(x)\tau(y) + \sigma(x)f(y)) = d(d(x)\tau(y)) + d(\sigma(x)f(y)).$$

So, we have

$$d^2(x)\tau^2(y) + \sigma(d(x))d(\tau(y)) + d(\sigma(x))\tau(f(y)) = 0 \ \forall x, y \in N.$$

Replacing y by $\tau^{-1}(y)$ in the last equation, we get

$$d^2(x)\tau(y) + d(\sigma(x))d(y) + d(\sigma(x))(f(y)) = 0 \ \forall x, y \in N. \tag{3.12}$$

We claim that $f(d(x)y) = 0 \ \forall x, y \in N$. Indeed, by using the definition of f, we have

$$f(d(x)y) = d^2(x)\tau(y) + \sigma(d(x))f(y). \tag{3.13}$$

Also

$$f(d(x)y) = f(d(x))\tau(y) + \sigma(d(x))d(y).$$

By Lemma 3.1 we get

$$f(d(x)y) = \sigma(d(x))d(y). \tag{3.14}$$

Substituting (13) and (14) in (12), we get $2f(d(x)y) = 0$, but N is 2-torsion free which implies $f(d(x)y) = 0$. This proves the claim.

Now, substituting yz instead of y in (12), we get

$$d^2(x)\tau(yz) + \sigma(d(x))d(yz) + d(\sigma(x))f(yz) = 0.$$

and from this we obtain

$$\begin{aligned} d^2(x)\tau(yz) &+ \sigma(d(x))d(y)\tau(z) + \sigma(d(x))\sigma(y)d(z) \\ &+ d(\sigma(x))f(y)\tau(z) + d(\sigma(x))\sigma(y)d(y) = 0. \end{aligned}$$

Rewrite the equation above and use Lemma 2.10, to find

$$(d^2(x)\tau(y) + d(\sigma(x))f(y))\tau(z) + \sigma(d(x))d(y)\tau(z) + 2\sigma(d(x))\sigma(y)d(z) = 0. \tag{3.15}$$

By using our claim and the equations (13),(14), and (15) we get

$$2\sigma(d(x))\sigma(y)d(z) = 0 \ \forall x, y, z \in N.$$

Since N is 2-torsion free, we have $d(N)Nd(N) = 0$. It follows that $d(N) = 0$ which is a contradiction. This completes the proof. $\square$

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