KYUNGPOOK Math. J. 60(2020), 361-373
https://doi.org/10.5666/KMJ.2020.60.2.361
pISSN 1225-6951 eISSN 0454-8124

Canal Surfaces in Pseudo-Galilean 3-Spaces

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Abstract. In this paper, we define admissible canal surfaces with isotropic radius vectors in pseudo-Galilean 3-spaces and we obtaine their position vectors. We also attain some important results by considering their Gauss and mean curvatures.

1. Introduction

A canal surface is defined as an envelope of a one-parameter set of spheres, centered at a spine curve $\gamma(s)$ with radius $r(s)$. When $r(s)$ is a constant function, the canal surface is the envelope of a moving sphere and is called a pipe surface. Canal surfaces have wide applications in CAGD, such as construction of blending surfaces, shape reconstruction, transition surfaces between pipes, and robotic path planning. An envelope of a 1-parameter family of surfaces is constructed in the same way as we construct a 1-parameter family of curves. The family is described by a differentiable function $F(x, y, z, \lambda) = 0$, where λ is a parameter. When λ can be eliminated from the equations

$$F(x, y, z, \lambda) = 0$$

and

$$\frac{\partial F(x, y, z, \lambda)}{\partial \lambda} = 0$$

we get the envelope, which is a surface described implicitly as $G(x, y, z) = 0$. For example, for a 1-parameter family of planes, we get a developable surface [3, 5].

A general canal surface is an envelope of a 1-parameter family of surfaces. The envelope of a 1-parameter family $s \longrightarrow S^2(s)$ of spheres in $\mathbb{R}^3$ is called a general canal surface [3]. The curve formed by the centers of the spheres is called center curve of the canal surface. The radius of general canal surface is the function r such that $r(s)$ is the radius of the sphere $S^2(s)$.

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Received March 4, 2018; revised February 9, 2019; accepted June 27, 2019.
2010 Mathematics Subject Classification: 53A35, 53B25.
Key words and phrases: pseudo-Galilean space, canal surface, tubular surface.

Suppose that the center curve of a canal surface is a unit speed curve $\gamma : I \to \mathbb{R}^3$. The general canal surface can be parametrized by the formula

$$C(s,t) = \gamma(s) - R(s)\,T(s) - Q(s)\cos(t)\,N(s) + Q(s)\sin(t)\,B(s) \tag{1.1}$$

where

$$R(s) = r(s)r'(s)$$
$$Q(s) = \pm r(s)\sqrt{1 - r'(s)^2}$$

and $T(s)$, $N(s)$, $B(s)$ are the unit tangent, the principal normal, the binormal vectors of the center curve $\gamma(s)$. All the tubes and the surfaces of revolution are subclass of the general canal surface.

Theorem 1.1. *Let M be a canal surface. The center curve of M is a straight line if and only if M is a surface of revolution for which no normal line to the surface is parallel to the axis of revolution. The following conditions are equivalent for a canal surface M:*

(i) *M is a tube parametrized by (1.1);*

(ii) *the radius of M is constant;*

(iii) *the radius vector of each sphere in family that defines the canal surface M meets the center curve orthogonally [3].*

2. Canal Surfaces in Pseudo-Galilean Space

Pseudo-Galilean geometry is one of the real Cayley-Klein geometries of projective signature (0, 0, +, -) [6]. The absolute of Pseudo-Galilean geometry is an ordered triple $\{w, f, I\}$ where w is the ideal (absolute) plane, f is line in w and I is the fixed hyperbolic involution of points of f. The Pseudo-Galilean scalar product g can be written as

$$g(A,B) = \begin{cases} a_1 b_1, & \text{if} \quad a_1 \neq 0 \ \vee b_1 \neq 0 \\ a_2 b_2 - a_3 b_3, & \text{if} \quad a_1 = 0 \ \wedge b_1 = 0 \end{cases} \tag{2.1}$$

where $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$, and the Pseudo-Galilean norm of the vector $A = (a_1, a_2, a_3)$ is defined by

$$\|A\| = \begin{cases} a_1, & \text{if} \quad a_1 \neq 0 \\ \sqrt{(a_2)^2 - (a_3)^2}, & \text{if} \quad a_1 = 0. \end{cases}$$

The vector $A = (a_1, a_2, a_3)$ is said to be non-isotropic if $a_1 \neq 0$. The Pseudo-Galilean cross product is defined for $A = (a_1, a_2, a_3)$, $B = (b_1, b_2, b_3)$ by

$$A \wedge_{G_3^1} B = \begin{vmatrix} 0 & -e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

[1, 2, 4, 7]. All unit non-isotropic vectors are in the form $(1, a_2, a_3)$, for isotropic vectors $a_1 = 0$. There are four types of isotropic vectors: spacelike $((a_2)^2 - (a_3)^2 > 0)$, timelike $((a_2)^2 - (a_3)^2 < 0)$ and two types of lightlike $(a_2 = \pm a_3)$ vectors. A non-lightlike isotropic vector is a unit vector if $(a_2)^2 - (a_3)^2 = \pm 1$.

An admissible curve $\gamma : I \subseteq R \to G_3^1$ is defined by

$$\gamma(s) = (s, y(s), z(s)) \,. \tag{2.2}$$

where s is arc length parameter. The curvature $\kappa(s)$ and the torsion $\tau(s)$ are defined by

$$\kappa(s) = \sqrt{\left|(y''(s))^2 - (z''(s))^2\right|}\,, \quad \tau(x) = \frac{y''(s)z'''(s) - y'''(s)z''(s)}{\kappa^2(s)} \,. \tag{2.3}$$

An admissible curve has no inflection points, no isotropic tangents or normals whose projections on the absolute plane would be light-like vectors. The associated trihedron is given by

$$\begin{aligned} T(s) &= \gamma'(s) = (1, y'(s), z'(s)) \\ N(s) &= \frac{1}{\kappa(s)}\,(0, y''(s), z''(s)) \\ B(s) &= \frac{1}{\kappa(s)}\,(0, \epsilon z''(s), \epsilon y''(s)) \end{aligned} \tag{2.4}$$

where $\epsilon = \mp 1$, chosen by criterion $\det(T(s), N(s), B(s)) = 1$ means that

$$\left|(y''(s))^2 - (z''(s))^2\right| = \epsilon\left((y''(s))^2 - (z''(s))^2\right).$$

The curve $\gamma(s)$ given in (2.2) is timelike (resp. spacelike) if $N(s)$ is a spacelike(resp. timelike) vector. The principal normal vector or simply normal is spacelike if $\varepsilon = 1$ and timelike if $\varepsilon = -1$. For derivatives of the tangent (vector) $T(s)$, the normal $N(s)$ and the binormal $B(s)$, respectively, the following Serret-Frenet formulas hold

$$T'(s) = \kappa(s)N(s), \;\; N'(s) = \tau(s)B(s), \;\; B'(s) = \tau(s)N(s). \tag{2.5}$$

On the other hand, a C^r-surface, $r \geq 2$, is a subset $\Phi \subset G_3^1$ for which there exists an open subset D of $\mathbb{R}^2$ and C^r-mapping $X : D \to G_3^1$ satisfying $\Phi = X(D)$. A C^r surface $\Phi \subset G_3^1$ is called regular if X is an immersion, and Φ is called simple if X is an embedding. It is admissible if it does not have pseudo-Euclidean tangent planes. If we denote

$$\begin{gathered} X = X(x(u_1, u_2), y(u_1, u_2), z(u_1, u_2)) \\ x_{,i} = \frac{\partial x}{\partial u_i}, \;\; y_{,i} = \frac{\partial y}{\partial u_i}, \;\; z_{,i} = \frac{\partial z}{\partial u_i}, i = 1, 2 \end{gathered}$$

then, a surface is admissible if and only if $x_{,i} \neq 0$, for some $i = 1, 2$.

Let $\Phi \subset G_3^1$ be a regular admissible surface. Then, the unit normal vector field of a surface $X(u, v)$ is equal to

$$\eta(u,v) = \frac{(0, x_{,1}z_{,2} - x_{,2}z_{,1}, x_{,1}y_{,2} - x_{,2}y_{,1})}{W(u,v)}, \tag{2.6}$$
$$W(u,v) = \sqrt{\left|(x_{,1}y_{,2} - x_{,2}y_{,1})^2 - (x_{,1}z_{,2} - x_{,2}z_{,1})^2\right|}.$$

The function $W(u, v)$ is equal to the pseudo-Galilean norm of the isotropic vector $x_{,1}X_{,2} - x_{,2}X_{,1}$. Vector defined by

$$\sigma = \frac{(x_{,1}X_{,2} - x_{,2}X_{,1})}{W}$$

is called a side tangential vector. We will not consider surfaces with $W(u, v) = 0$, i.e. surfaces having lightlike side tangential vector (lightlike surfaces).

Since the normal vector field satisfies $g(\eta, \eta) = \epsilon = \pm 1$, we distinguish two basic types of admissible surfaces: spacelike surfaces having timelike surface normals ($\epsilon = -1$) and timelike surfaces having spacelike normals ($\epsilon = 1$).

The first fundamental form of a surface is induced from the metric of the ambient space G_3^1

$$ds^2 = (g_1 du_1 + g_2 du_2)^2 + \delta(h_{11} du_1^2 + 2h_{12} du_1 du_2 + h_{22} du_2^2), \tag{2.7}$$

where $g_i = x_{,i}$, $h_{ij} = g\left(\widetilde{X}_{,i}, \widetilde{X}_{,j}\right)$ and

$$\delta = \begin{cases} 0 & ; \text{ if direction } du_1 : du_2 \text{ is non-isotropic} \\ 1 & ; \text{ if direction } du_1 : du_2 \text{ is isotropic.} \end{cases}$$

By $(\widetilde{\ })$ above of a vector is denoted the projection of a vector on the pseudo-Euclidean $yz-$plane. The Gaussian curvature of a surface is defined by means of the coefficients of the second fundamental form

$$K = -\epsilon \frac{L_{11}L_{22} - L_{12}^2}{W^2}. \tag{2.8}$$

The second fundamental form II is given by

$$II = L_{11}\ du_1^2 + 2L_{12}\ du_1 du_2 + L_{22} du_2^2$$

where L_{ij} are the normal components of $X_{,i,j}$, $i, j = 1, 2$. It holds

$$L_{ij} = \epsilon g\left(\left(\frac{x_{,1}\widetilde{X}_{,i,j} - x_{,i,j}\widetilde{X}_{,1}}{x_{,1}}\right), \eta\right) = \epsilon g\left(\left(\frac{x_{,2}\widetilde{X}_{,i,j} - x_{,i,j}\widetilde{X}_{,1}}{x_{,2}}\right), \eta\right). \tag{2.9}$$

The mean curvature of a surface is defined by [4, 7]

$$H = -\epsilon \frac{(g_2)^2 L_{11} - 2g_1 g_2 L_{12} + (g_1)^2 L_{22}}{2W^2}. \tag{2.10}$$

In pseudo-Galilean geometry, there are two types of sphere depending radius vector whether it is an isotropic vector or it is a non-isotropic vector. Spheres with non-isotropic radius vector are pseudo-Euclidean circles in $yz-$plane and spheres with isotropic radius vector are parallel planes such as $x = \pm r$. Pseudo-Euclidean circles intersect the absolute line f. There are three kinds of pseudo-Euclidean circles; circles with timelike radius vector ($H^1_\pm(r)$), spacelike radius vector ($S^1_\pm(r)$) and lightlike radius vector, where

$$S^1_\pm(r) = \left\{X \ \in yz - plane | g\left(X, X\right) = r^2\right\}$$

and

$$H^1_\pm(r) = \left\{X \in yz - plane | g\left(X, X\right) = -r^2\right\}.$$

Definition 2.1. The envelope of a 1-parameter family $r \to S^1_\pm(r)$ (or $r \to H^1_\pm(r)$) of pseudo-Euclidean circles in G^1_3 is called a *canal surface* in pseudo-Galilean 3-space. The curve formed by the centers of the pseudo-Euclidean circles is called *center curve* of the canal surface. The *radius* of the canal surface is the function r such that $r(s)$ is the radius of the pseudo-Euclidean circles $S^1_\pm(s)$ (or $H^1_\pm(s)$).

Let us consider $C\left(s,t\right) - \gamma\left(s\right)$ is a isotropic vector of $H^1_\pm(r)$ then, the envelope of a 1-parameter family $r \to H^1_\pm(r)$ in G^1_3 is spacelike canal surface and since $C\left(s,t\right) - \gamma\left(s\right) \in Sp\left\{T\left(s\right), N\left(s\right), B\left(s\right)\right\}$ and $C\left(s,t\right)$ is non-isotropic then, we have

$$C\left(s,t\right) = \gamma\left(s\right) + \psi\left(s,t\right) T\left(s\right) + \varphi\left(s,t\right) N\left(s\right) + \omega\left(s,t\right) B\left(s\right) \tag{2.11}$$

and $\psi\left(s,t\right) = 0$. In the case that the centered curve is a spacelike curve, we can write

$$g\left(C\left(s,t\right) - \gamma\left(s\right), C\left(s,t\right) - \gamma\left(s\right)\right) = -\varphi^2\left(s,t\right) + \omega^2\left(s,t\right) = -r\left(s\right)^2. \tag{2.12}$$

By differentiating (2.12) with respect to s and t, we get

$$\varphi\left(s,t\right)\varphi_s\left(s,t\right) - \omega\left(s,t\right)\omega_s\left(s,t\right) = r'\left(s\right) r\left(s\right) \tag{2.13}$$

$$\varphi\left(s,t\right)\varphi_t\left(s,t\right) - \omega\left(s,t\right)\omega_t\left(s,t\right) = 0 \tag{2.14}$$

from the equations (2.12), (2.13) and (2.14), we obtain

$$\omega\left(s,t\right) = r\left(s\right)\sinh(t)\ ,\ \varphi\left(s,t\right) = r\left(s\right)\cosh(t).$$

Thus, we have the following corollary.

Corollary 2.2. *Let $\gamma(s)$ be an admissible spacelike curve with arclenght parameter in pseudo-Galilean 3-space. Then, position vector of spacelike canal surface with spacelike centered curve is*

$$C(s,t) = \gamma(s) + r(s)\cosh(t)N(s) + r(s)\sinh(t)B(s). \tag{2.15}$$

By using (2.5) and (2.15), natural bases $\{C_s, C_t\}$ are

$$\begin{aligned} C_s &= T(s) + \{r'\cosh(t) + r\tau\sinh(t)\}N(s) + \{r'\sinh(t) + r\tau\cosh(t)\}B(s) \\ C_t &= r\sinh(t)N(s) + r\cosh(t)B(s) \end{aligned}$$

and from (2.7) the coefficients h_{ij} and g_i are

$$h_{11} = r^2(s)\tau^2(s) - (r'(s))^2, \;\; h_{12} = h_{21} = r^2(s)\tau(s), \;\; h_{22} = r^2(s)$$
$$g_1 = 1, \;\; g_2 = 0.$$

Thus, the first fundamental form of spacelike canal surface is

$$I_C = \left(1 + r^2(s)\tau^2(s) - (r'(s))^2\right)ds^2 + 2r^2(s)\tau(s)\,dsdt + r^2(s)\,dt^2.$$

By using (2.5), the second derivations (2.15) are

$$\begin{aligned} C_{ss} &= \left\{\kappa + (2r'\tau + r\tau')\sinh(t) + \left(r\tau^2 + r''\right)\cosh(t)\right\}N(s) \\ &\quad + \left\{(2r'\tau + r\tau')\cosh(t) + \left(r\tau^2 + r''\right)\sinh(t)\right\}B(s) \\ C_{tt} &= r\cosh(t)N(s) + r\sinh(t)B(s) \\ C_{ts} &= \{r'\sinh(t) + r\tau\cosh(t)\}N(s) + \{r\tau\sinh(t) + r'\cosh(t)\}B(s) \end{aligned}$$

and the unit normal vector is

$$\eta(s,t) = \cosh(t)N(s) + \sinh(t)B(s).$$

From (2.9) coefficients L_{ij} are

$$L_{11} = r(s)\tau^2(s) + r''(s) + \kappa(s)\cosh(t), \;\; L_{12} = L_{21} = r(s)\tau(s), \;\; L_{22} = r(s)$$

and the second fundamental form is

$$II_C = \left(r(s)\tau^2(s) + r''(s) + \kappa(s)\cosh(t)\right)ds^2 + 2r(s)\tau(s)\,dsdt + r(s)\,dt^2.$$

Thus, from (2.8) and (2.10), Gauss and mean curvatures are

$$K(s,t) = \frac{r''(s) + \kappa(s)\cosh(t)}{r(s)}, \;\; H(s,t) = \frac{1}{2r(s)}. \tag{2.16}$$

In the case that $K(s,t) = 0$, the centered curve has to be planar and there are two K-flat canal surfaces for $r(s) = c_1 s + c_2$ and $r(s) = c$.

Hence, from (2.2), (2.3), (2.4), (2.15) and (2.16), we have the following theorem.

Theorem 2.3. *Let M be a spacelike canal surface with spacelike centered curve in pseudo-Galilean 3-space. Then, the followings are true.*

(i) *There is no minimal spacelike canal surface with spacelike centered curve,*

(ii) *Gauss and mean curvatures of M satisfy the relation*

$$K\left(s,t\right)-2H\left(s,t\right)\left(r''\left(s\right)+\kappa\left(s\right)\cosh(t)\right)=0,$$

(iii) *M is a $K-$flat if and only if M is a parabolic cone and its position vector is*

$$\begin{aligned} C(s,t) &= (s,(c_1 s+c_2)(c_3\cosh(t)\mp\sqrt{(c_3)^2+1}\sinh(t)) \\ &\quad ,(c_1 s+c_2)(\mp\sqrt{(c_3)^2+1}\cosh(t)+c_3\sinh(t))) \end{aligned}$$

where $c_1\neq 0$, $c_2,c_3\in\mathbb{R}$,(see figure 1.a),

(iv) *M is a $K-$flat spacelike tubular surface if and only if M is a parabolic cylinder and its position vector is*

$$C(s,t)=(s,c_1c_2\cosh(t)\mp c_1\sqrt{(c_2)^2+1}\sinh(t),\mp c_1\sqrt{(c_2)^2+1}\cosh(t)+c_1c_2\sinh(t))$$

where $c_1\in\mathbb{R}^+$, $c_2\in\mathbb{R}$, (see figure 1.b),

(v) *All the spacelike tubes with spacelike centered curve are positive-constant mean curvature surfaces.*

In the case that $C(s,t)$ is spacelike canal surface and centered curve is a timelike curve, we can write:

$$g\left(C\left(s,t\right)-\gamma\left(s\right),C\left(s,t\right)-\gamma\left(s\right)\right)=\varphi^2\left(s,t\right)-\omega^2\left(s,t\right)=-r^2\left(s\right). \tag{2.17}$$

By differentiating (2.17) with respect to s and t, we get

$$\omega\left(s,t\right)\omega_s\left(s,t\right)-\varphi\left(s,t\right)\varphi_s\left(s,t\right)=r'\left(s\right)r\left(s\right) \tag{2.18}$$

$$\omega\left(s,t\right)\omega_t\left(s,t\right)-\varphi\left(s,t\right)\varphi_t\left(s,t\right)=0 \tag{2.19}$$

then, we obtain

$$\omega\left(s,t\right)=r\left(s\right)\cosh(t)\ ,\ \varphi\left(s,t\right)=r\left(s\right)\sinh(t)$$

by using (2.17), (2.18) and (2.19).

Thus, we have the following corollary.

Corollary 2.4. *Let $\gamma(s)$ be an admissible timelike curve with arclenght parameter in pseudo-Galilean 3-space. Then, position vector of spacelike canal surface with timelike centered curve is*

$$C\left(s,t\right)=\gamma\left(s\right)+r\left(s\right)\sinh(t)N\left(s\right)+r\left(s\right)\cosh(t)B\left(s\right). \tag{2.20}$$

From (2.5) and (2.20), natural bases $\{C_s, C_t\}$ are

$$\begin{aligned} C_s &= T(s) + \{r' \sinh(t) + r\tau \cosh(t)\} N(s) + \{r' \cosh(t) + r\tau \sinh(t)\} B(s) \\ C_t &= r\cosh(t)N(s) + r\sinh(t)B(s) \end{aligned}$$

and from (2.7) the coefficients h_{ij} and g_i are

$$h_{11} = r^2(s)\,\tau^2(s) - (r'(s))^2, \;\; h_{12} = h_{21} = r^2(s)\,\tau(s), \;\; h_{22} = r^2(s)$$
$$g_1 = 1, \;\; g_2 = 0.$$

Thus, the first fundamental form is

$$I_C = \left(1 + r^2(s)\,\tau^2(s) - (r'(s))^2\right) ds^2 + 2r^2(s)\,\tau(s)\,dsdt + r^2(s)\,dt^2.$$

By using (2.5), the second derivations (2.20) are

$$\begin{aligned} C_{ss} &= \left\{\kappa + \left(r'' + r\tau^2\right)\sinh(t) + (2r'\tau + r\tau')\cosh(t)\right\} N(s) \\ &\quad + \left\{\left(r'' + r\tau^2\right)\cosh(t) + (2r'\tau + r\tau')\sinh(t)\right\} B(s) \\ C_{tt} &= r\sinh(t)N(s) + r\cosh(t)B(s) \\ C_{ts} &= \{r'\cosh(t) + r\tau\sinh(t)\} N(s) + \{r'\sinh(t) + r\tau\cosh(t)\} B(s) \end{aligned}$$

the unit normal vector is

$$\eta(s,t) = \sinh(t)N(s) + \cosh(t)B(s).$$

From (2.9), the coefficients L_{ij} are

$$L_{11} = \kappa(s)\sinh(t) - r(s)\,\tau^2(s) - r''(s), \; L_{12} = L_{21} = -r(s)\,\tau(s), \; L_{22} = -r(s)$$

and so the second fundamental form is

$$II_C = \left(\kappa(s)\sinh(t) - r(s)\,\tau^2(s) - r''(s)\right) ds^2 - 2r(s)\,\tau(s)\,dsdt - r(s)\,dt^2.$$

From (2.8) and (2.10), Gauss and mean curvatures are

$$K(s,t) = \frac{\kappa(s)\sinh(t) - r''(s)}{r(s)}, \;\; H(s,t) = \frac{1}{2r(s)} \tag{2.21}$$

respectively. In the case that $K(s,t) = 0$, the centered curve has to be planar and there are two K-flat canal surfaces for $r(s) = c_1 s + c_2$ and $r(s) = c$.

Hence, from (2.2), (2.3), (2.4), (2.20) and (2.21), we have the following cases.

Theorem 2.5. *Let M be a spacelike canal surface with timelike centered curve in pseudo-Galilean 3-space. Then, the followings are true.*

(i) *There is no minimal spacelike canal surface with timelike centered curve,*

(ii) *Gauss and mean curvatures of M satisfy the relation*

$$K\left(s,t\right)+2H\left(s,t\right)\left(\kappa\sinh(t)-r''\right)=0,$$

(iii) *M is a $K-$flat if and only if M is a parabolic cone and its position vector is*

$$\begin{aligned}C(s,t) &= (s,(c_1 s+c_2)(c_3\sinh(t)\mp\sqrt{(c_3)^2-1}\cosh(t))\\ &\quad ,(c_1 s+c_2)(\mp\sqrt{(c_3)^2-1}\sinh(t)+c_3\cosh(t)))\end{aligned}$$

where $c_1\neq 0,\ c_2\in\mathbb{R}, c_3\in\mathbb{R}-[0,1),$ *(see figure 2.a),*

(iv) *M is a $K-$flat spacelike tubular surface if and only if M is a parabolic cylinder and its position vector is*

$$C(s,t)=(s,c_1c_2\sinh(t)\mp c_1\sqrt{(c_2)^2-1}\cosh(t),\mp c_1\sqrt{(c_2)^2-1}\sinh(t)+c_1c_2\cosh(t))$$

where $c_1\in\mathbb{R}^+,\ c_2\in\mathbb{R}-[0,1),$ *(see figure 2.b),*

(v) *All the spacelike tubes with timelike centered curve are positive-constant mean curvature surfaces.*

Accordingly, in the case that $C\left(s,t\right)-\gamma\left(s\right)$ is an isotropic radius vector of $S^1_{\pm}(r)$ then, the envelope of a 1-parameter family $s\to S^1_{\pm}(r)$ in G^1_3 is timelike canal surface and since $C\left(s,t\right)-\gamma\left(s\right)\in Sp\left\{T\left(s\right),N\left(s\right),B\left(s\right)\right\}$ and $C\left(s,t\right)$ is non-isotropic then, we have (2.11) and $\psi\left(s,t\right)=0$. If the centered curve is a timelike curve then, the position vector $C\left(s,t\right)$ is obtained in the same form of (2.15). From (2.7) and (2.9), coefficients of the first and the second fundamental forms are obtained as

$$\begin{aligned}&h_{11}=\left(r'\left(s\right)\right)^2-r^2\left(s\right)\tau^2\left(s\right),\ \ h_{12}=h_{21}=-r^2\left(s\right)\tau\left(s\right),\ \ h_{22}=-r^2\left(s\right)\\ &L_{11}=\kappa\left(s\right)\cosh(t)+r\left(s\right)\tau\left(s\right)^2+r''\left(s\right),\ L_{12}=L_{21}=r\left(s\right)\tau\left(s\right),\ L_{22}=r\left(s\right)\end{aligned}$$

and also from (2.8) and (2.10), the Gauss and the mean curvatures are

$$K\left(s,t\right)=-\frac{\kappa\left(s\right)\cosh(t)+r''\left(s\right)}{r\left(s\right)},\ \ H\left(s,t\right)=-\frac{1}{2r\left(s\right)}. \tag{2.22}$$

Thus, from (2.2), (2.3), (2.4), (2.15) and (2.22), we can give the following corollary.

Corollary 2.6. *Let M be a timelike canal surface with timelike centered curve in pseudo-Galilean 3-space. Then, the followings are true.*

(i) *There is no minimal timelike canal surface with timelike centered curve,*

(ii) *Gauss and mean curvatures of M satisfy the relation*

$$K\left(s,t\right)-2H\left(s,t\right)\left(\kappa\left(s\right)\cosh(t)+r''\left(s\right)\right)=0,$$

(iii) *M is a $K-$flat if and only if M is a parabolic cone and its position vector is*

$$\begin{aligned} C(s,t) &= (s,(c_1s+c_2)(c_3\cosh(t)\mp\sqrt{(c_3)^2-1}\sinh(t)) \\ &\quad ,(c_1s+c_2)(\mp\sqrt{(c_3)^2-1}\cosh(t)+c_3\sinh(t))) \end{aligned}$$

where $c_1 \neq 0$, $c_2, c_3 \in \mathbb{R}-[0,1)$, (see figure 3.a),

(iv) *M is a $K-$flat timelike tubular surface if and only if M is a parabolic cylinder and its position vector is*

$$C(s,t) = (s, c_1c_2\cosh(t)\mp c_1\sqrt{(c_2)^2-1}\sinh(t), \mp c_1\sqrt{(c_2)^2-1}\cosh(t)+c_1c_2\sinh(t))$$

where $c_1 \in \mathbb{R}^+$, $c_2 \in \mathbb{R}-[0,1)$, (see figure 3.b),

(v) *All the timelike tubes with timelike centered curve are negative-constant mean curvature surfaces.*

If the centered curve is a spacelike curve then, the position vector $C(s,t)$ is obtained in the same form of (2.20) and from (2.7) and (2.9), coefficients of the first and the second fundamental forms are

$$h_{11} = (r'(s))^2 - r^2(s)\tau^2(s), \quad h_{12} = h_{21} = -r^2(s)\tau(s), \quad h_{22} = -r^2(s)$$
$$L_{11} = \kappa(s)\sinh(t) - r(s)\tau^2(s) - r''(s), \; L_{12} = L_{21} = -r(s)\tau(s), \; L_{22} = -r(s)$$

and also from (2.8) and (2.10), the Gauss and the mean curvatures are

$$K(s,t) = \frac{r''(s) - \kappa(s)\sinh(t)}{r(s)}, \quad H(s,t) = \frac{-1}{2r(s)}. \tag{2.23}$$

We have the following cases, by using the equations (2.2), (2.3), (2.4), (2.20) and (2.23).

Corollary 2.7. *Let M be a timelike canal surface with spacelike centered curve in pseudo-Galilean 3-space. Then, the followings are true.*

(i) *There is no minimal timelike canal surface with spacelike centered curve,*

(ii) *Gauss and mean curvatures of M satisfy the relation*

$$K(s,t) + 2H(s,t)(r''(s) - \kappa(s)\sinh(t)) = 0,$$

(iii) *M is a $K-$flat if and only if M is a parabolic cone and its position vector is*

$$\begin{aligned} C(s,t) &= (s,(c_1s+c_2)(c_3\sinh(t)\mp\sqrt{(c_3)^2+1}\cosh(t)) \\ &\quad ,(c_1s+c_2)(\mp\sqrt{(c_3)^2+1}\sinh(t)+c_3\cosh(t))) \end{aligned}$$

where $c_1 \neq 0$, $c_2 \in \mathbb{R}$, $c_3 \in \mathbb{R}$, (see figure 4.a),

(iv) *M is a $K-$flat timelike tubular surface if and only if M is a parabolic cylinder and its position vector is*

$$C(s,t) = (s, c_1c_2\sinh(t) \mp c_1\sqrt{(c_2)^2+1}\cosh(t), \mp c_1\sqrt{(c_2)^2+1}\sinh(t) + c_1c_2\cosh(t))$$

where $c_1 \in \mathbb{R}^+$, $c_2 \in I$, (see figure 4.b),

(v) *All the timelike tubes with spacelike centered curve are negative-constant mean curvature surfaces.*

Now, we can summarise our study as in following theorem.

Theorem 2.8. *Let $\gamma : (a,b) \to G_3^1$ be an admissible curve in G_3^1 and M be a canal surface with the centered curve $\gamma(s)$ then, there are two types canal surfaces in G_3^1 such that,*
type-1: M is spacelike (timelike) canal surface and $\gamma(s)$ is spacelike (timelike) curve then, M is parametrized by

$$C_\mu(s,t) = \gamma(s) + r(s)\cosh(t)N(s) + r(s)\sinh(t)B(s),$$

type-2: M is spacelike (timelike) canal surface and $\gamma(s)$ is timelike (spacelike) curve then, M is parametrized by

$$C_\sigma(s,t) = \gamma(s) + r(s)\sinh(t)N(s) + r(s)\cosh(t)B(s).$$

In consideration of above theorem, we can give coefficients of the first fundamental forms, Gauss and mean curvatures as follow by taking $g_1 = 1,\ g_2 = 0$.

For the type-1 canal surfaces,

$$h_{11} = \mu r(s)^2 \tau(s)^2,\ \ h_{21} = h_{12} = \mu r(s)^2 \tau(s),\ \ h_{22} = \mu r(s)^2,$$

Gauss and mean curvatures are

$$K(s,t) = \frac{\mu(r''(s) + \kappa(s)\cosh(t))}{r(s)},\ \ H(s,t) = \frac{\mu}{2r(s)}.$$

For the type-2 canal surfaces,

$$h_{11} = \sigma r(s)^2 \tau(s)^2,\ \ h_{12} = h_{21} = \sigma r(s)^2 \tau(s),\ \ h_{22} = \sigma r(s)^2,$$

Gauss and mean curvatures are

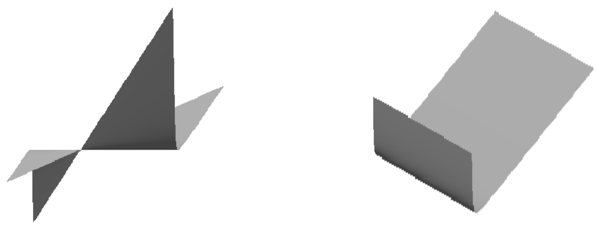
$$K(s,t) = \frac{\sigma(r''(s) + \kappa(s)\cosh(t))}{r(s)},\ \ H(s,t) = \frac{\sigma}{2r(s)}$$

where

$$\mu = \begin{cases} 1, & \text{if } M \text{ is a spacelike canal surface with spacelike centered curve} \\ -1, & \text{if } M \text{ is a timelike canal surface with timelike centered curve} \end{cases}$$

and

$$\sigma = \begin{cases} 1, & \text{if } M \text{ is a spacelike canal surface with timelike centered curve} \\ -1, & \text{if } M \text{ is a timelike canal surface with spacelike centered curve.} \end{cases}$$

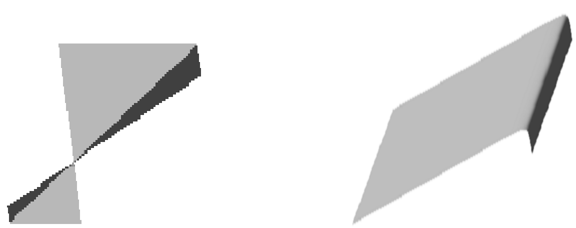


(a) (b)

Figure 1: For (a); $c_1 = 2, c_2 = 1, c_3 = 0, sign : (-)$, for (b); $c_1 = 2, c_2 = 1$.

(a) (b)

Figure 2: For (a); $c_1 = c_2 = 1, c_3 = 0$, for (b); $c_1 = 1, c_2 = 2$.



(a) (b)

Figure 3: For (a); $c_1 = c_2 = 1, c_3 = 2$, for (b); $c_1 = 1, c_2 = 2$.

(a) (b)

Figure 4: For (a); $c_1 = 2, c_2 = c_3 = 1$, for (b); $c_1 = 2, c_2 = 0$.

Acknowledgements. The authors are indebted to the referees for helpful suggestions and insights concerning the presentation of this paper.

References

[1] Z. Erjavec, *On generalization of Helices in the Galilean and the pseudo-Galilean space*,

[2] Z. Erjavec and B. Divjak, *The equiform differential geometry of curves in the pseudo-Galilean space*, Math. Commun., **13**(2008), 321–332.

[3] A. Gray, *Modern differential geometry Of curves and surfaces*, CRC Press, BocaRaton Ann Arbor London Tokyo, 1993.

[4] A. O. Öğrenmiş and M. Ergüt, *On the Gauss map of ruled surfaces of type II in 3-dimensional pseudo-Galilean space*, Bol. Soc. Paran. Mat., **31(1)**(2013), 145–152.

[5] G. Öztürk, B. Bulca, B. K. Bayram and K. Arslan, *On canal surfaces in E^3*, Selçuk J. Appl. Math., **11(2)**(2010), 103–108.

[6] O. Röschel, *Die Geometrie des Galileischen Raumes*, Forschungszentrum Graz, Mathematisch-Statistische Sektion, Graz, 1985. J. Math. Res., **6(3)**(2014), 39–50.

[7] Z. M. Sipus and B. Divjak, *Surfaces of constant curvature in the pseudo-Galilean space*, Int. J. Math. Math. Sci., (2012), Art. ID 375264, 28 pp.