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ALMOST ζ- CONTRACTION ON M- METRIC SPACE

M. Pitchaimani and K. Saravanan*

Abstract. In this paper, we initiate the concept of almost ζ- contractions via Simulation functions to find fixed points on M- metric spaces, and prove some related fixed points results for such mappings. Moreover an illustration is provided to show the applicability of our obtained results.

1. Introduction and Preliminaries

In 1994, Matthews [2] introduced partial metric spaces as a generalization of a standard metric space which has nonzero self distance. That is the distance between two identical point need not be zero. Then many authors formulated and proved fixed point results in partial metric spaces.
In 2014, Mehdi Asadi et al [1] formulated the concept of M-metric space as a generalization of partial metric space, as well as, a generalization of metric spaces. Vasile Berinde used weak contractions to find fixed points in various spaces. Isik [3] introduced ζ- contraction to prove existence of fixed points. In this paper, we proved the existence of fixed points in M-metric space using almost ζ -contraction. For more studies of fixed point results for different contraction refer [4-11] and reference therein.
In this section we will recall the basic notions of M-metric space and ζ-contraction. The following notion will be used in the presentation,

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* Corresponding author.

Definition 1.1. [1] Let X be a nonempty set. If the function $m : X \times X \to \mathbb{R}^+$ satisfies the following conditions for all $x, y, z \in X$,

1. $m(x,x) = m(y,y) = m(x,y) \iff x = y$,
2. $m_{xy} \leq m(x,y)$,
3. $m(x,y) = m(y,x)$,
4. $(m(x,y) - m_{xy}) \leq (m(x,z) - m_{xz}) + (m(z,y) - m_{zy})$,

Then the pair (X,m) is called $M-$ metric space, where

1. $m_{xy} := \min\{m(x,x), m(y,y)\}$,
2. $M_{xy} := \max\{m(x,x), m(y,y)\}$.

Remarks 1.2. [1] For every $x, y \in X$,

1. $0 \leq M_{xy} + m_{xy} = m(x,x) + m(y,y)$,
2. $0 \leq M_{xy} - m_{xy} = |m(x,x) - m(y,y)|$,
3. $M_{xy} - m_{xy} \leq (M_{xz} - m_{xz}) + (M_{zy} - m_{zy})$.

Example 1.3. Let $X = [0, \infty)$. Then $m(x,y) = \dfrac{x+y}{2}$ on X is $M-$ metric space.

Example 1.4. Let (X,m) be an $M-$ metric space. Put

1. $m^w(x,y) = m(x,y) - 2m_{xy} + M_{xy}$,
2. $m^s(x,y) = m(x,y) - m_{xy}$ when $x \neq y$ and $m^s(x,y) = 0$ if $x = y$.

Example 1.5. Let $X = \{1,2,3\}$ Define $m(1,1) = 1, m(2,2) = 9, m(3,3) = 5$,
$m(1,2) = m(2,1) = 10, m(1,3) = m(3,1) = 7, m(3,2) = m(2,3) = 7$.
Then (X,m) is M-metric space.

In [1], author describes each m metric on X generates a T_0 topology τ_m on X.
The set $\{B_m(x,\epsilon) : x \in X, \epsilon > 0\}$, where $B_m(x,\epsilon) = \{y \in X : m(x,y) < m_{xy} + \epsilon\}$, for all $x \in X$ and $\epsilon > 0$, forms a base of τ_m.

Definition 1.6. Let (X,m) be a M-metric space. Then,

1. A sequence $\{x_n\}$ in a M-metric space (X,m) converges to a point $x \in X$ if and only if
$$\lim_{n\to\infty} (m(x_n,x) - m_{x_n x}) = 0. \tag{1}$$
2. A sequence $\{x_n\}$ in a M-metric space (X,m) is called an m-cauchy sequence if
$$\lim_{n\to\infty} (m(x_n,x_m) - m_{x_n x_m}), \quad \lim_{n\to\infty} (M_{x_n x_m} - m_{x_n x_m})$$

exist.(and are finite)

3. M- metric space (X, m) is said to be complete if every m-Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_m, to a point $x \in X$ such that

$$(\lim_{n\to\infty}(m(x_n, x) - m_{x_n x}) = 0 \ \ \& \lim_{n\to\infty}(M_{x_n x} - m_{x_n x}) = 0)$$

LEMMA 1.7. *[1] Assume that $x_n \to x$ and $y_n \to y$ as $n \to \infty$ in an M- metric space (X, m). Then*

$$\lim_{n\to\infty}(m(x_n, y_n) - m_{x_n y_n}) = m(x, y) - m_{xy}.$$

Proof. We have
$|(m(x_n, y_n) - m_{x_n y_n}) - (m(x, y) - m_{xy})| \leq (m(x_n, x) - m_{x_n x}) + (m(y, y_n) - m_{y y_n})$ □

LEMMA 1.8. *[1] Assume that $x_n \to x$ as $n \to \infty$ in M- metric space (X, m). Then $\lim_{n\to\infty}(m(x_n, y) - m_{x_n y}) = m(x, y) - m_{xy}$.*

LEMMA 1.9. *[1] Assume that $x_n \to x$ and $x_n \to y$ as $n \to \infty$ in M-metric space (X, m). Then $x = y$.*

LEMMA 1.10. *[1] Let $\{x_n\}$ be a sequence in $M-$metric space (X, m), such that*
$\exists\, r \in [0, 1)$,

$$m(x_{n+1}, x_n) \leq r m(x_n, x_{n-1}), \quad \forall n \in \mathbb{N} \tag{1}$$

Then

1. $\lim_{n\to\infty} m(x_n, x_{n-1}) = 0$,
2. $\lim_{n\to\infty} m(x_n, x_n) = 0$,
3. $\lim_{n\to\infty} m_{x_m x_n} = 0$,
4. *$\{x_n\}$ is an m-Cauchy sequence.*

DEFINITION 1.11. [3] Let $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be a mapping, then ζ is called a **simulation function** if it satisfies the following conditions:

1. $\zeta(0, 0) = 0$,
2. $\zeta(t, s) < s - t$ for all $t, s > 0$,
3. If $\{t_n\}, \{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$. then

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0. \tag{2}$$

4. If $\{t_n\}, \{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ and
$t_n < s_n$ for all $n \in \mathbb{N}$, then Equation (2) is satisfied.

We denote the set of all simulation functions by $\mathcal{Z}$

EXAMPLE 1.12. [3] Let $\zeta_i : [0, \infty) \times [0, \infty) \to \mathbb{R}, i = 1, 2, 3$ be defined by

1. $\zeta_1(t, s) = \psi(s) - \phi(t)$ for all $t, s \in [0, \infty)$, where $\phi, \psi : [0, \infty) \to [0, \infty)$ are two continuous functions such that $\psi(t) = \phi(t) = 0$ if and only if $t = 0$ and
$\psi(t) < t \leq \phi(t)$ for all $t > 0$.
2. $\zeta_2 = s - \dfrac{f(t, s)}{g(t, s)} t$ for all $t, s \in [0, \infty)$, where $f, g : [0, \infty)^2 \to (0, \infty)$ are two continuous functions with respect to each variable such that $f(t, s) > g(t, s)$ for all $t, s > 0$.
3. $\zeta_3(t, s) = s - \phi(s) - t$ for all $t, s \in [0, \infty)$, where $\phi : [0, \infty) \to [0, \infty)$ is a continuous function such that $\phi(t) = 0$ if and only if $t = 0$.

DEFINITION 1.13. Let (X, d) be a metric space and $\zeta \in \mathcal{Z}$. We say that $T : X \to X$ is an almost ζ-contraction with respect to $\zeta \in \mathcal{Z}$, if there is a constant $\theta \geq 0$ such that

$$\zeta(d(Tx, Ty), d(x, y) + \theta N(x, y)) \geq 0.$$

for all $x, y \in X$, where

$$N(x, y) = \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

If T is an almost ζ-contraction with respect to $\zeta \in \mathcal{Z}$, then

$$d(Tx, Ty) < d(x, y) + \theta N(x, y)$$

for all $x, y \in X$.

DEFINITION 1.14. Let (X, d) be a metric space and $\zeta \in \mathcal{Z}$. We say that $T : X \to X$ is an $\lambda - L$ almost $\zeta-$ Contraction if there is a constant $L \geq 0$ and $\lambda \in (0, 1)$ such that

$$\zeta(d(Tx, Ty), \lambda(x, y) + LN(x, y)) \geq 0.$$

for all $x, y \in X$, where

$$N(x, y) = \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

DEFINITION 1.15. Let (X, d) be a metric space and $T : X \to X$ be a self mapping, then
$Fix_T(X) = \{x \in X : \ Tx = x\}$, $Fix_T(X)$ denotes the collection of all fixed points of T in X.

LEMMA 1.16. *If an almost ζ-contraction has a fixed point in a metric space, then it is unique.*

Proof. Let (X, m) be a M-metric space and $T : X \to X$ be an almost ζ-contraction. Suppose that there are two distinct fixed points $u, v \in X$ of the mapping T. Then, $d(u, v) > 0$. Therefore, we have

$$\begin{array}{rcl} 0 & \leq & \zeta(m(Tu,Tv), m(u,v) + \theta N(u,v)) \\ & = & \zeta(m(Tu,Tv), m(u,v) + \theta \min\{m(u,Tu), m(v,Tv), m(u,Tv), m(v,Tu)\}) \\ & = & \zeta(m(u,v), m(u,v)) \\ & < & m(u,v) - m(u,v) \\ & = & 0. \end{array}$$

which is a contradiction. Thus, the fixed point of T in X is unique. Our main result is as follows. □

2. Main Results

THEOREM 2.1. *Let (X, m) be a complete M- metric space, $T : X \to X$ be an almost ζ-contraction with respect to $\zeta \in \mathcal{Z}$, then T has a unique fixed point, and for every initial point $x_0 \in X$, the Picard sequence $\{T^n x_0\}$ converges to x in X, such that $x \in Fix_T(X)$.*

Proof. Take $x_0 \in X$ and consider the Picard sequence $\{x_n = T^n x_0 = Tx_{n-1}\}_{n \geq 0}$. If $x_{n_0} = x_{n_0+1}$ for some n_0, then x_{n_0} is a fixed point of T. Hence, for the rest of the proof, we assume that $m(x_n, x_{n+1}) > 0$ for all $n \geq 0$.
This proof is divided into three parts. In the first step, we prove that,

$$\lim_{n \to \infty} m(x_n, x_{n+1}) = 0$$

Consider,

$$m(x_{n_k}, x_{n_{k+1}}) = 0 \quad for\ some \quad n_k \in \mathbb{N}. \tag{3}$$

Then we have $m(x_{n_{k+1}}, x_{n_{k+2}}) = 0$.
If not $m(x_{n_{k+1}}, x_{n_{k+2}}) > 0$

$$0 < \zeta(m(x_{n_{k+1}}, x_{n_{k+2}}), m(x_{n_k}, x_{n_{k+1}}) + \theta N(x_{n_k}, x_{n_{k+1}})) \tag{4}$$

where $N(x_{n_k}, x_{n_{k+1}})$
$= \min\{m(x_{n_k}, x_{n_{k+1}}), m(x_{n_{k+1}}, x_{n_{k+2}}), m(x_{n_k}, x_{n_{k+2}}), m(x_{n_{k+1}}, x_{n_{k+1}})\}$ From (3) we have $N(x_{n_k}, x_{n_{k+1}}) = 0$.
By the definition of ζ function we've

$$\begin{aligned} 0 &< \zeta(m(x_{n_{k+1}}, x_{n_{k+2}}), m(x_{n_k}, x_{n_{k+1}}) + \theta N(x_{n_k}, x_{n_{k+1}})) \\ &= m(x_{n_k}, x_{n_{k+1}}) - m(x_{n_{k+1}}, x_{n_{k+2}}) \\ &= -m(x_{n_{k+1}}, x_{n_{k+2}}) \\ &\leq 0. \end{aligned}$$

which implies that

$$\lim_{n\to\infty} m(x_{n_{k+1}}, x_{n_{k+2}}) = 0$$

Now, we will prove $m(x_n, x_{n+1})$ is a non decreasing sequence and bounded below by zero.
we know that

$$m(x_n, x_{n+1}) > 0$$

$$\begin{aligned} m(x_n, x_{n+1}) &= m(Tx_{n-1}, Tx_n) \\ &\leq m(x_{n-1}, x_n) + \theta N(x_{n-1}, x_n) \\ m(x_n, x_{n+1}) - m(x_{n-1}, x_n) &\leq \theta N(x_{n-1}, x_n) \end{aligned}$$

$$m(x_n, x_{n+1}) \leq m(x_{n-1}, x_n) \tag{5}$$

Because

$$N(x_{n-1}, x_n) = \min\{m(x_{n-1}, x_n), m(x_n, x_n), m(x_n, x_{n+1}), m(x_{n+1}, x_{n+1})\}$$

when $n \to \infty$ then $N(x_{n-1}, x_n) = 0$.
From (5), $m(x_n, x_{n-1})$ is non-decreasing sequence bounded below by zero.
Now, we will show that $\{x_n\}$ is bounded.
Suppose $\{x_n\}$ is unbounded.
Then there exists $n_k > n$ such that

$$m(x_{n_k}, x_{n_{k+1}}) > h(say)$$

which means that $m(x_{n_{k-1}}, x_{n_k}) < h$.
Consider two sequences $\{s_n\}$ and $\{t_n\}$ where $s_n = m(x_n, x_{n+1})$. $t_n = m(x_{n-1}, x_n)$

$$\begin{aligned} 0 &\leq \lim_{n\to\infty} \sup \zeta(m(x_{n-1}, x_n), m(x_n, x_{n+1}) \\ 0 &\leq \lim_{n\to\infty} (m(x_n, x_{n+1}) - m(x_{n-1}, x_n)) \\ 0 &\leq -\lim_{n\to\infty} m(x_{n-1}, x_n) \end{aligned}$$

This shows that $m(x_{n-1}, x_n) < 0$, which is not possible.
So we must have $h = 0$.

$$\lim_{n\to\infty} m(x_{n-1}, x_n) = 0.$$

Now we will prove that $\{x_n\}$ is m-Cauchy sequence in (X, m).
We have

$$\lim_{n\to\infty} m(x_n, x_{n+1}) = 0,$$

$$\text{(6)} \quad i.e., \; m(x_n, x_{n+1}) \leq M_{x_n x_{n+1}} \leq 0 \implies \lim_{n\to\infty} M_{x_n x_{n+1}} = 0.$$

From the definition of M-metric and (6)

$$m_{x_n x_{n+1}} = \min(m(x_n, x_n), m(x_{n+1}, x_{n+1})) = 0$$

Let us take

$$a_{n,m} = m(x_n, x_m) - m_{x_n x_m}$$

$\lim_{n,m\to\infty} a_{n,m} \neq 0$, there exists $l > 0$ and $\{l_k\} \subset \mathbb{N}$, such that $a_{n,m}(x_{l_k}, x_{n_k}) \geq l$ which means that $a_{n,m}(x_{n_k}, x_{l_{k-1}}) < l$

$$\begin{aligned} l &\leq a_{n,m}(x_{l_k}, x_{n_k}) \\ &\leq a_{n,m}(x_{l_k}, x_{l_{k-1}}) + a_{n,m}(x_{l_{k-1}}, x_{n_k}) \\ &< a_{n,m}(x_{l_k}, x_{l_{k-1}}) + l \end{aligned}$$

This implies that

$$\lim_{k\to\infty} a_{n,m}(x_{l_k}, x_{n_k}) = l$$

which means that

$$\lim_{k\to\infty} m(x_{l_k}, x_{l_{k-1}}) = l + m_{l_k l_{k-1}}.$$

On the other hand,

$$\lim_{k\to\infty} m(x_{l_k}, x_{l_{k-1}}) = l$$

$$\begin{aligned} l &\leq a_{n,m}(x_{l_k}, x_{n_k}) \\ &\leq a_{n,m}(x_{l_k}, x_{l_{k+1}}) + a_{n,m}(x_{l_{k+1}}, x_{n_{k+1}}) + a_{n,m}(x_{n_{k+1}}, x_{n_k}) \\ l &\leq a_{n,m}(x_{l_{k+1}}, x_{n_{k+1}}) \\ &\leq a_{n,m}(x_{l_{k+1}}, x_{l_k}) + a_{n,m}(x_{l_k}, x_{n_k}) + a_{n,m}(x_{n_k}, x_{n_{k+1}}) \end{aligned}$$

From the above, we get

$$\lim_{n\to\infty} m(x_{l_{k+1}}, x_{n_{k+1}}) = l$$

Let $u_n = m(x_{l_k}, x_{l_{k-1}})$ and $v_n = m(x_{l_{k+1}}, x_{n_{k+1}})$

$$\begin{aligned} 0 &\leq \lim_{n\to\infty} \sup \zeta(u_n, v_n) \\ \implies & \quad v_n - u_n < 0 \end{aligned}$$

This contradicts that

$$\lim_{n\to\infty} m(x_{n_k}, x_{l_k}) = 0.$$

$$\implies \lim_{n,m\to\infty} m(x_n, x_m) - m_{x_n x_m} = 0.$$

$$\implies \lim_{n,m\to\infty} m(x_n, x_m) = m_{x_n x_m}$$

$\therefore$ $\{x_n\}$ is $m - Cauchy$ in X. Since X is a complete $M-$metric space, the Picard sequence $x_n \to x \in X$ in τ_m topology that means

$$\lim_{n\to\infty} m(x_n, x) - m_{x_n x} = 0$$

and also

$$\lim_{n\to\infty} m(x_n, x_n) - m_{x_n x_n} = 0$$

Since T is continuous

$$\lim_{n\to\infty} m(Tx_n, Tx) - m_{Tx_n Tx} = 0$$

$$\lim_{n\to\infty} (m(x_n, x) + \theta N(x_n, x) - m_{Tx_n Tx}) = 0$$

$$\lim_{n\to\infty} m(x_n, x) - m_{x_{n+1} Tx} = 0$$

$$\lim_{n\to\infty} m_{x_{n+1} Tx} = 0$$

So we have

$$\lim_{n\to\infty} (m(x_n, x)) = 0$$

By lemma 1.10

$$\lim_{n\to\infty} m(x_n, Tx) - m_{x_n Tx} = \lim_{n\to\infty} m(x, Tx) - m_{xTx}$$

$$m(x, Tx) = 0 \quad as \quad n \to \infty.$$

Hence

$$m(Tx, x) = m(x, x) = 0 \implies Tx = x$$

$$x \in Fix_T(X).$$

□

Theorem 2.2. *Let (X, m) be a complete $M-$metric space, $T : X \to X$ be an $\lambda - L$ almost ζ-contraction with respect to $\zeta \in \mathcal{Z}$, then T has a fixed point, and for every initial point $x_0 \in X$, the picard sequence $\{T^n x_0\}$ converges to x in X, such that $x \in Fix_T(X)$.*

Proof. Take $x_0 \in X$ and consider the Picard sequence $\{x_n = T^n x_0 = Tx_{n-1}\}_{n\geq 0}$.
If $x_{n_0} = x_{n0+1}$ for some n_0, then x_{n_0} is a fixed point of T. Hence, for the rest of the proof, we assume that $m(x_n, x_{n+1}) > 0$ for all $n \geq 0$.
We will prove

$$\lim_{n\to\infty} m(x_n, x_{n+1}) = 0.$$

Consider,

$$m(x_{n_k}, x_{n_{k+1}}) = 0 \; for \; some \; n_k \in \mathbb{N} \tag{7}$$

Then we have $m(x_{n_{k+1}}, x_{n_{k+2}}) = 0$.
Suppose that $m(x_{n_{k+1}}, x_{n_{k+2}}) > 0$

$$0 < \zeta(m(x_{n_{k+1}}, x_{n_{k+2}}), \lambda m(x_{n_k}, x_{n_{k+1}}) + LN(x_{n_k}, x_{n_{k+1}})) \tag{8}$$

where $N(x_{n_k}, x_{n_{k+1}})$
$= \min\{m(x_{n_k}, x_{n_{k+1}}), m(x_{n_{k+1}}, x_{n_{k+2}}), m(x_{n_k}, x_{n_{k+2}}), m(x_{n_{k+1}}, x_{n_{k+1}})\}$ From (7) we have $N(x_{n_k}, x_{n_{k+1}}) = 0$ By the definition of ζ function we've

$$\begin{aligned} 0 &< \zeta(m(x_{n_{k+1}}, x_{n_{k+2}}), \lambda m(x_{n_k}, x_{n_{k+1}}) + LN(x_{n_k}, x_{n_{k+1}})) \\ &= m(x_{n_k}, x_{n_{k+1}}) - \lambda m(x_{n_{k+1}}, x_{n_{k+2}}) \\ &= -\lambda m(x_{n_{k+1}}, x_{n_{k+2}}) \end{aligned}$$

which gives the contradiction $m(x_{n_{k+1}}, x_{n_{k+2}}) < 0$. This implies that

$$\lim_{n\to\infty} m(x_{n_{k+1}}, x_{n_{k+2}}) = 0.$$

Now, we will prove $m(x_n, x_{n+1})$ is a non decreasing sequence which is bounded below by zero.
we know that

$$m(x_n, x_{n+1}) > 0$$

$$\begin{aligned} m(x_n, x_{n+1}) &= m(Tx_{n-1}, Tx_n) \\ &\leq \lambda m(x_{n-1}, x_n) + LN(x_{n-1}, x_n) \\ m(x_n, x_{n+1}) - \lambda m(x_{n-1}, x_n) &\leq LN(x_{n-1}, x_n) \\ m(x_n, x_{n+1}) &\leq \lambda m(x_{n-1}, x_n) \end{aligned}$$

Because

$$N(x_{n-1}, x_n) = \min\{m(x_{n-1}, x_n), m(x_n, x_n), m(x_n, x_{n+1}), m(x_{n+1}, x_{n+1})\}$$

when $n \to \infty$ then $N(x_{n-1}, x_n) = 0$.
From (3), $m(x_n, x_{n-1})$ is non-decreasing sequence bounded below by zero.
Now, we will show that $\{x_n\}$ is bounded.
Suppose $\{x_n\}$ is unbounded.
Then there exists $n_k > n$ such that

$$m(x_{n_k}, x_{n_{k+1}}) > h(say)$$

which means that $m(x_{n_{k-1}}, x_{n_k}) < h$.
Consider two sequences $\{s_n\}$ and $\{t_n\}$ where $s_n = m(x_n, x_{n+1})$. $t_n = m(x_{n-1}, x_n)$

$$\begin{aligned} 0 &\leq \lim_{n\to\infty} \sup \zeta(m(x_{n-1}, x_n), m(x_n, x_{n+1}) \\ 0 &\leq \lim_{n\to\infty} (m(x_n, x_{n+1}) - m(x_{n-1}, x_n)) \\ 0 &\leq -\lim_{n\to\infty} m(x_{n-1}, x_n) \end{aligned}$$

This shows that $m(x_{n-1}, x_n) < 0$, which is not possible.
So we must have $h = 0$.

$$\lim_{n\to\infty} m(x_{n-1}, x_n) = 0$$

. Using the same argument in the theorem 2.1, we can easily show that $\{x_n\}$ is m-Cauchy. Since X is a complete $M-$ metric space, the Picard sequence $x_n \to x \in X$ in τ_m topology that means,

$$\lim_{n\to\infty} m(x_n, x) - m_{x_n x} = 0$$

and also

$$\lim_{n\to\infty} m(x_n, x_n) - m_{x_n x_n} = 0$$

Since T is continuous

$$\lim_{n\to\infty} m(Tx_n, Tx) - m_{Tx_n Tx} = 0$$

$$\lim_{n\to\infty} (\lambda^n m(x_n, x) + LN(x_n, x) - m_{Tx_n Tx}) = 0$$

As $n \to \infty$, $\lambda^n \to \infty$.

$$\lim_{n\to\infty} m_{x_{n+1} Tx} = 0.$$

So we have

$$\lim_{n\to\infty} (m(x_n, x)) = 0.$$

By lemma 1.10,

$$\lim_{n\to\infty} m(x_n, Tx) - m_{x_nTx} = \lim_{n\to\infty} m(x, Tx) - m_{xTx}$$

$$m(x, Tx) = 0 \;\; as \;\; n \to \infty.$$

Hence

$$m(Tx, x) = m(x, x) = 0 \implies Tx = x$$

$$x \in Fix_T(X).$$

This completes the proof. □

Corollary 2.3. *Suppose that (X, m) is a complete $M-$metric space and $T : X \to X$ is a mapping satisfying*

$$m(Tx, Ty) \leq \delta m(x, y) + LN(y, Ty), \;\; for\ all\ x, y \in X,$$

for $\delta \in (0, 1)$ and some $L \geq 0$. Then

1. $Fix_T(X) = \{x \in X : Tx = x\} \neq \emptyset$
2. *For arbitrary $x_0 \in X$, the Picard iteration $x_{n+1} = Tx_n$ for $n \geq 0$ converges to some $x \in Fix_T(X)$.*

Corollary 2.4. *Suppose that (X, m) is a complete $M-$metric space and $T : X \to X$ is a mapping satisfying*

$$m(Tx, Ty) \leq \delta m(x, y) + LN(x, Tx), \;\; for\ all\ x, y \in X,$$

for $\delta \in (0, 1)$ and some $L \geq 0$. Then

1. $Fix_T(X) = \{x \in X : Tx = x\} \neq \emptyset$
2. *For arbitrary $x_0 \in X$, the Picard iteration $x_{n+1} = Tx_n$ for $n \geq 0$ converges to some $x \in Fix_T(X)$.*

Corollary 2.5. *Suppose that (X, m) is a complete $M-$metric space and $T : X \to X$ is a mapping satisfying*

$$m(Tx, Ty) \leq \delta m(x, y) + LN(x, Ty), \;\; for\ all\ x, y \in X,$$

for $\delta \in (0, 1)$ and some $L \geq 0$. Then

1. $Fix_T(X) = \{x \in X : Tx = x\} \neq \emptyset$
2. *For arbitrary $x_0 \in X$, the Picard iteration $x_{n+1} = Tx_n$ for $n \geq 0$ converges to some $x \in Fix_T(X)$.*

3. Consequences

In this section, we endeavor to learn existence and uniqueness of fixed points in $M-$metric space through example which supports our result.

EXAMPLE 3.1. Consider $X = [0,1]$ and $m : X \to [0,\infty)$ with $m(x,y) = \frac{x+y}{2}$. Then the mapping $T : X \to X$ is defined as $T(x) = \frac{x^2}{3}$. Here

$$\begin{aligned} N(x,Ty) &= \min\{m(x,y), m(Tx,y), m(x,Ty), m(Tx,Ty)\} \\ &= \min\left\{\frac{x+y}{2}, \frac{x^2+3y}{6}, \frac{3x+y^2}{6}, \frac{x^2+y^2}{6}\right\} \end{aligned}$$

By theorem

$$\zeta(m(Tx,Ty), m(x,y) + \theta N(x,y)) \geq 0.$$

That is

$$m(x,y) + \theta N(x,y) - m(Tx,Ty) \geq 0.$$

In this example we will discuss three cases.

Case(1) If $x = 0$ and $y = 0$ then the result is obvious.

Case(2) If $x = 1$ and $y = 1$. Then $m(T(1),T(1)) = m\left(\frac{1}{3},\frac{1}{3}\right) = \frac{1}{3}$ and $m(1,1) + \theta N(1,1) = \frac{1}{2} + \theta\frac{1}{3}$. Obviously $m(1,1) + \theta N(1,1) - m(T(1),T(1)) \geq 0$.

Case(3) If $x \in [0,a)$ and $y \in (b,1]$ where $0 < a < b < 1$.
In this case

$$\begin{aligned} N(a,b) &= \min\{m(a,b), m(Ta,b), m(a,Tb), m(Ta,Tb)\} \\ &= \min\left\{\frac{a+b}{2}, \frac{a^2+3b}{6}, \frac{3a+b^2}{6}, \frac{a^2+b^2}{6}\right\} \\ &= \frac{a^2+b^2}{6}. \end{aligned}$$

$$\zeta(m(Ta,Tb), m(a,b) + \theta N(a,b)) \geq 0.$$

$$\begin{aligned} m(a,b) + \theta N(a,b) - m(Ta,Tb) &= \frac{a+b}{2} + \theta\frac{a^2+b^2}{6} - \frac{a^2+b^2}{6} \\ &\geq 0. \end{aligned}$$

In all the cases T satisfies the hypothesis of theorem 2.1, and so T has a fixed point. That is $Fix_T(X) \neq \emptyset$.

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M. Pitchaimani
Ramanujan Institute for Advanced Study in Mathematics
University of Madras, Chennai-600 005 India
E-mail: mpitchaimani@yahoo.com

K. Saravanan
Ramanujan Institute for Advanced Study in Mathematics
University of Madras, Chennai-600 005 India
E-mail: saranrkm5@gmail.com