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2-COLOR RADO NUMBER FOR $x_1 + x_2 + \cdots + x_n = y_1 + y_2 = z$

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Abstract. An r-color Rado number $N = R(\mathcal{L}, r)$ for a system $\mathcal{L}$ of equations is the least integer, provided it exists, such that for every r-coloring of the set $\{1, 2, \ldots, N\}$, there is a monochromatic solution to $\mathcal{L}$. In this paper, we study the 2-color Rado number $R(\mathcal{E}, 2)$ for $\mathcal{E} : x_1 + x_2 + \cdots + x_n = y_1 + y_2 = z$ when $n \geq 4$.

1. Introduction

For $a, b \in \mathbb{N}$ with $a < b$, let $[a, b] = \{a, a+1, \ldots, b\}$. A function $c : [1, N] \to [1, r]$ is called an r-coloring of the set $[1, N]$. A solution $\{x_1, x_2, \ldots, x_n\}$ to an equation L is said to be monochromatic if $c(x_1) = c(x_2) = \cdots = c(x_n)$.

In 1916 Schur [17] proved the existence of the number $N = S(r)$ such that for a given integer $r \geq 2$ and every r-coloring of the set $[1, N]$, there exists a monochromatic solution to $x + y = z$. The least such integer is called the r-color Schur number $S(r)$. There are some known Schur numbers such as $S(2) = 5,\ S(3) = 14,\ S(4) = 45$ [18] and $S(5) = 161$ [5], but it is unknown yet for $r \geq 6$. Motivated by the Schur numbers, Rado considered the same problem for a system of linear equations instead

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of the single equation $x + y = z$. He found the necessary and sufficient conditions to determine if an arbitrary system of linear equations admits a monochromatic solution for every coloring of the natural numbers with a finite number of colors [3,10]. If such a system always has a monochromatic solution, then there is N such that for every r-coloring of $[1, N]$ this system has a monochromatic solution. The least number N satisfying this property is called the r-color Rado number for the system.

The results on Rado number has been conducted mainly in 2-color for a specific linear equation. As the most natural generalization of the 2-color Schur number $S(2)$, Beutelspacher and Brestovansky [2] found the 2-color Rado number for $x_1 + x_2 + \cdots + x_{m-1} = x_m$. Harborth and Maasberg [6,7] studied the 2-color Rado number for $a(x+y) = bz$ which is another generalization of it.

Hopkins and Schaal [8] found the 2-color Rado number for some special classes of $\sum_{i=1}^{m-1} a_i x_i = x_m$ and conjectured for the general case. Guo and Sun [4] proved this conjecture. Robertson and Myers [11] computed the 2-color Rado number for some special classes of $x+y+kz = \ell w$, and Saracino and Wynne [16] obtained this number when $\ell = 3$. In [14,15], Saracino studied the 2-color Rado number for $x_1+x_2+\cdots+x_{m-1} = ax_m$. There are some interesting results [1,9,12] in two important variants of Rado numbers, disjunctive Rado numbers and off-diagonal Rado numbers.

Most of the results on Rado number have been limited on 2-color or r-color Rado number for single equation. Consider a system of linear equation $\mathcal{E} : x_1+x_2+\cdots+x_n = y_1+y_2 = z$. It is known that the 2-color Rado number for $x_1 + x_2 + \cdots + x_n = z$ is $n^2 + n - 1$ [2] and that the 2-color Rado number for $x_1+x_2+\cdots+x_n = y_1+y_2$ is $\lceil \frac{n}{2} \lceil \frac{n}{2} \rceil \rceil$ [13]. In this paper we show that the 2-color Rado number for the system of equations $\mathcal{E}$ is $n^2 + n - 1$, which is the same with that for $x_1 + x_2 + \cdots + x_n = z$.

2. Main Result

Lemma 1. [2] *For $n \geq 4$, the 2-color Rado number for $x_1 + x_2 + \cdots + x_n = z$ is $n^2 + n - 1$.*

Consider the system of equation $\mathcal{E} : x_1+x_2+\cdots+x_n = y_1+y_2 = z$ for $n \geq 4$. By Lemma 1, the 2-color Rado number $R(\mathcal{E}, 2)$ for $\mathcal{E}$ is greater than or equals to $n^2 + n - 1$. Thus when $N \geq n^2 + n - 1$, if we find a monochromatic solution to $\mathcal{E}$, then we can prove that the 2-color Rado number for $\mathcal{E}$ is $n^2 + n - 1$.

THEOREM 1. *If $n \geq 4$, then the 2-color Rado number for $\mathcal{E}$ is n^2+n-1.*

Since the 2-color Rado number for $x_1+x_2+\cdots+x_n = z$ is n^2+n-1, we have $R(\mathcal{E}, 2) \geq n^2+n-1$. Thus it suffices to prove $R(\mathcal{E}, 2) \leq n^2+n-1$. Let $c : [1, n^2+n-1] \to \{0, 1\}$ be a 2-coloring and let $S_c(\mathcal{E})$ be the set of all $[(x_1, x_2, \ldots, x_n), (y_1, y_2), z]$ such that $x_1 + x_2 + \cdots + x_n = y_1 + y_2 = z$, $c(x_i) = c(y_j) = c(z)$ and $x_i, y_j, z \in [1, n^2 + n - 1]$ for all $i = 1, 2, \ldots, n$ and $j = 1, 2$. The inequality $R(\mathcal{E}, 2) \leq n^2+n-1$ follows from $S_c(\mathcal{E}) \neq \emptyset$.

Suppose that $S_c(\mathcal{E}) = \emptyset$. We want to find a contradiction in each case. The proof consists of case by case considerations. We divide all the cases into following 18 cases.

- $c(1) = 0$
  - $c(n) = 0$
    - $c(2) = 0$
      - $c(n^2) = 0 \cdots (1)$
      - $c(n^2) = 1$
        - $c(n^2 - n + 1) = 0 \cdots (2)$
        - $c(n^2 - n + 1) = 1 \cdots (3)$
    - $c(2) = 1$
      - $c(2n) = 0$
        - $c(n^2) = 0$
          - $c(n^2 + n - 1) = 0 \cdots (4)$
          - $c(n^2 + n - 1) = 1 \cdots (5)$
        - $c(n^2) = 1$
          - $c(n^2 + 2) = 0 \cdots (6)$
          - $c(n^2 + 2) = 1 \cdots (7)$
      - $c(2n) = 1$
        - $c(n^2) = 0 \cdots (8)$
        - $c(n^2) = 1$
          - $c(n^2 + n - 1) = 0 \cdots (9)$
          - $c(n^2 + n - 1) = 1 \cdots (10)$
  - $c(n) = 1$
    - $c(n^2) = 0$
      - $c(n + 1) = 0 \cdots (11)$
      - $c(n + 1) = 1$
        - $c(n^2 + n - 1) = 0$
          - $c(n + 2) = 0 \cdots (12)$
          - $c(n + 2) = 1,\ c(2n) = 0 \cdots (13)$
          - $c(n + 2) = c(2n) = 1 \cdots (14)$
        - $c(n^2 + n - 1) = 1 \cdots (15)$
    - $c(n^2) = 1$
      - $c(n^2 - 1) = 0 \cdots (16)$
      - $c(n^2 - 1) = 1$
        - $c(n - 1) = 0 \cdots (17)$
        - $c(n - 1) = 1 \cdots (18)$

Case (1): $c(n) = c(2) = c(n^2) = 0$.

From the assumption, we have the following.

$c(n-1) = 1$, since otherwise $[(1, \ldots, 1), (n-1, 1), n] \in S_c(\mathcal{E})$,
$c(n-2) = 1$, since otherwise $[(1, \ldots, 1), (n-2, 2), n] \in S_c(\mathcal{E})$,
$c(2n) = 1$, since otherwise $[(2, \ldots, 2), (n, n), 2n] \in S_c(\mathcal{E})$,
$c(n^2 - n) = 1$, since otherwise $[(n, \ldots, n), (n^2 - n, n), n^2] \in S_c(\mathcal{E})$,
$c(n^2 - 1) = 1$, since otherwise $[(n, \ldots, n), (n^2 - 1, 1), n^2] \in S_c(\mathcal{E})$.

Thus, $[(n-1, \ldots, n-1, n-2, n-2, 2n), (n^2-n, n-1), n^2-1] \in S_c(\mathcal{E})$. This is a contradiction.

Case (2): $c(n) = c(2) = 0,\ c(n^2) = 1, c(n^2 - n + 1) = 0$.

We have $c(n-1) = c(n-2) = c(2n) = 1$ by the same method as in Case (1). Also we have $c(n^2-n+2) = 1$ since otherwise $[(n, \ldots, n, 2), (n^2-n+1, 1), n^2 - n + 2] \in S_c(\mathcal{E})$.

Thus, $[(n-1, \ldots, n-1, n-2, 2n), (n^2 - n + 2, n - 2), n^2]$ satisfies $\mathcal{E}$. This is a contradiction.

Case (3): $c(n) = c(2) = 0,\ c(n^2) = c(n^2 - n + 1) = 1$.

We have $c(n-1) = c(n-2) = c(2n) = 1$ by the same method as in Case (1). Thus, $[(n-1, \ldots, n-1, n-2, 2n), (n^2 - n + 1, n - 1), n^2] \in S_c(\mathcal{E})$, This is a contradiction.

Case (4): $c(n) = 0,\ c(2) = 1,\ c(2n) = c(n^2) = c(n^2 + n - 1) = 0$.

From the assumption, we have the following.

$c(n^2 - n) = 1$, since otherwise $[(n, \ldots, n), (n^2 - n, n), n^2] \in S_c(\mathcal{E})$,
$c(n^2 - 2n) = 1$, since otherwise $[(n, \ldots, n), (n^2 - 2n, 2n), n^2] \in S_c(\mathcal{E})$,
$c(n-1) = 1$, since otherwise $[(1, \ldots, 1, n^2), (n^2, n-1), n^2 + n - 1] \in S_c(\mathcal{E})$,
$c(n+1) = 1$, since otherwise $[(1, \ldots, 1, n+1), (n, n), 2n] \in S_c(\mathcal{E})$,
$c(n^2 - n + 2) = 0$, since otherwise $[(n+1, \ldots, n+1, 2, 2), (n^2 - n, 2), n^2 - n + 2] \in S_c(\mathcal{E})$.
$c(n-2) = 1$, since otherwise $[(n, \ldots, n), (n^2 - n + 2, n - 2), n^2] \in S_c(\mathcal{E})$,
$c(n^2 - n - 1) = 0$, since otherwise $[(n-1, \ldots, n-1, n-2), (n^2 - 2n, n - 1), n^2 - n - 1] \in S_c(\mathcal{E})$,

Thus, $[(1, \ldots, 1, n^2), (n^2 - n - 1, 2n), n^2 + n - 1] \in S_c(\mathcal{E})$, This is a contradiction.

Case (5): $c(n) = 0,\ c(2) = 1,\ c(2n) = c(n^2) = 0,\ c(n^2 + n - 1) = 1$.

From the assumption, we have the following.

$c(n+1) = 1$, since otherwise $[(1, \ldots, 1, n+1), (n, n), 2n] \in S_c(\mathcal{E})$,
$c(n^2 - n) = 1$, since otherwise $[(n, \ldots, n), (n^2 - n, n), n^2] \in S_c(\mathcal{E})$,
$c(n^2 + 1) = 1$, since otherwise $[(n, \ldots, n, 2n, 1), (n^2, 1), n^2 + 1] \in S_c(\mathcal{E})$,

Thus, $[(n+1, \ldots, n+1, 2), (n^2 - n, n + 1), n^2 + 1] \in S_c(\mathcal{E})$, This is a contradiction.

Case (6): $c(n) = 0,\ c(2) = 1,\ c(2n) = 0,\ c(n^2) = 1,\ c(n^2 + 2) = 0$.

From the assumption, we have the following.

$c(n+1) = 1$, since otherwise $[(1, \ldots, 1, n+1), (n, n), 2n] \in S_c(\mathcal{E})$,

$c(n^2-n+2) = 1$, since otherwise $[(n, \ldots, n, 2n, 2n, 1, 1), (n^2-n+2, n), n^2+2] \in S_c(\mathcal{E})$,
$c(n^2-2n+2) = 1$, since otherwise $[(n, \ldots, n, 2n, 2n, 1, 1), (n^2-2n+2, 2n), n^2+2] \in S_c(\mathcal{E})$,
$c(n^2+1) = 1$, since otherwise $[(n, \ldots, n, 2n, 2n, 1, 1), (n^2+1, 1), n^2+2] \in S_c(\mathcal{E})$,
$c(n-1) = 0$, since otherwise $[(n+1, \ldots, n+1, 2), (n^2-n+2, n-1), n^2+1] \in S_c(\mathcal{E})$,
$c(2n-1) = 1$, since otherwise $[(1, \ldots, 1, n), (n, n-1), 2n-1] \in S_c(\mathcal{E})$,

Thus, $[(n+1, \ldots, n+1, 2), (n^2-2n+2, 2n-1), n^2+1] \in S_c(\mathcal{E})$, This is a contradiction.

Case (7): $c(n) = 0,\ c(2) = 1,\ c(2n) = 0,\ c(n^2) = c(n^2+2) = 1$.

From the assumption, we have the following.

$c(n+1) = 1$, since otherwise $[(1, \ldots, 1, n+1), (n, n), 2n] \in S_c(\mathcal{E})$,
$c(n+2) = 0$, since otherwise $[(n+1, \ldots, n+1, n+2, 2), (n^2, 2), n^2+2] \in S_c(\mathcal{E})$,
$c(3) = 0$, since otherwise $[(n+1, \ldots, n+1, 3), (n^2, 2), n^2+2] \in S_c(\mathcal{E})$,
$c(n-1) = 1$, since otherwise $[(1, \ldots, 1, 3), (n-1, 3), n+2] \in S_c(\mathcal{E})$,
$c(n^2-2n+4) = 0$, since otherwise $[(2, \ldots, 2, n^2-2n+4), (n^2, 2), n^2+2] \in S_c(\mathcal{E})$,
$c(n^2-2n+1) = 1$, since otherwise $[(n, \ldots, n, 1, 3), (n^2-2n+1, 3), n^2-2n+4] \in S_c(\mathcal{E})$,
$c(n^2-2n+3) = 1$, since otherwise $[(n, \ldots, n, 1, 3), (n^2-2n+3, 1), n^2-2n+4] \in S_c(\mathcal{E})$,

Thus, $[(n-1, \ldots, n-1, 2), (n^2-2n+1, 2), n^2-2n+3] \in S_c(\mathcal{E})$, This is a contradiction.

Case (8): $c(n) = 0,\ c(2) = c(2n) = 1,\ c(n^2) = 0$.

From the assumption, we have the following.

$c(2n-2) = 0$, since otherwise $[(2, \ldots, 2), (2n-2, 2), 2n] \in S_c(\mathcal{E})$,
$c(n-1) = 1$, since otherwise $[(1, \ldots, 1, n-1), (n-1, n-1), 2n-2] \in S_c(\mathcal{E})$,
$c(n^2-1) = 1$, since otherwise $[(n, \ldots, n), (n^2-1, 1), n^2] \in S_c(\mathcal{E})$,
$c(n^2+1) = 0$, since otherwise $[(n, \ldots, n-1, 2n), (n^2-1, 2), n^2+1] \in S_c(\mathcal{E})$,
$c(n+1) = 1$, since otherwise $[(n, \ldots, n, n+1), (n^2, 1), n^2+1] \in S_c(\mathcal{E})$,

Thus, $[(2, \ldots, 2), (n-1, n+1), 2n] \in S_c(\mathcal{E})$, This is a contradiction.

Case (9): $c(n) = 0,\ c(2) = c(2n) = c(n^2) = 1,\ c(n^2 + n - 1) = 0$.

We have $c(2n - 2) = 0$ and $c(n - 1) = 1$ by the same method as in Case (8). Also we have

$c(n+1) = 0$, since otherwise $[(2, \dots, 2), (n+1, n-1), 2n] \in S_c(\mathcal{E})$,
$c(n^2 - 2) = 1$, since otherwise $[(n + 1, \dots, n + 1, n), (n^2 - 2, n + 1), n^2 + n - 1] \in S_c(\mathcal{E})$,
$c(2n-1) = 0$, since otherwise $[(n-1, \dots, n-1, 2n-1), (n^2-2, 2), n^2] \in S_c(\mathcal{E})$,
$c(n - 2) = 0$, since otherwise $[(n - 1, \dots, n - 1, n - 2, 2n), (n^2 - 2, 2), n^2] \in S_c(\mathcal{E})$,

Thus, $[(1, \dots, 1, n), (n-2, n+1), 2n-1] \in S_c(\mathcal{E})$, This is a contradiction.

Case (10): $c(n) = 0,\ c(2) = c(2n) = c(n^2) = c(n^2 + n - 1) = 1$.

We have $c(n - 1) = 1$ and $c(2n - 2) = 0$ by the same method as in Case (9). Also we have $c(n^2 - n + 1) = 0$, since otherwise $[(2, \dots, 2, n^2 - n + 1), (n^2, n - 1), n^2 + n - 1] \in S_c(\mathcal{E})$. We have $c(n^2 - n) = 1$, since otherwise $[(n, \dots, n, 1), (n^2 - n, 1), n^2 - n + 1] \in S_c(\mathcal{E})$.

Also we have $c(n^2 - 2n + 1) = 0$, since otherwise $[(n - 1, \dots, n - 1), (n^2 - 2n + 1, n - 1), n^2 - n] \in S_c(\mathcal{E})$. And we have $c(n^2 - n + 1) = 1$, since otherwise $[(n, \dots, n, 1), (n^2 - 2n + 1, n), n^2 - n + 1] \in S_c(\mathcal{E})$,

Thus, $[(2, \dots, 2, n^2 - n + 1), (n^2, n - 1), n^2 + n - 1] \in S_c(\mathcal{E})$. This is a contradiction.

Case (11): $c(n) = 1,\ c(n^2) = c(n + 1) = 0$.

From the assumption, we have the following.

$c(n^2 - 1) = 1$, since otherwise $[(n + 1, \dots, n + 1, 1), (n^2 - 1, 1), n^2] \in S_c(\mathcal{E})$,
$c(n^2 - n - 1) = 1$, since otherwise $[(n + 1, \dots, n + 1, 1), (n^2 - n - 1, n + 1), n^2] \in S_c(\mathcal{E})$,
$c(n-1) = 0$, since otherwise $[(n, \dots, n, n-1), (n^2-n-1, n), n^2-1] \in S_c(\mathcal{E})$,
$c(n^2 - n + 1) = 1$, since otherwise $[(n + 1, \dots, n + 1, 1), (n^2 - n + 1, n - 1), n^2] \in S_c(\mathcal{E})$,
$c(2) = 1$, since otherwise $[(1, \dots, 1, 2), (n - 1, 2), n + 1] \in S_c(\mathcal{E})$,
$c(2n) = 1$, since otherwise $[(1, \dots, 1, n + 1), (n - 1, n + 1), 2n] \in S_c(\mathcal{E})$,

Thus, $[(2, \dots, 2), (n, n), 2n] \in S_c(\mathcal{E})$, This is a contradiction.

Case (12): $c(n) = 1,\ c(n^2) = 0,\ c(n + 1) = 1,\ c(n^2 + n - 1) = c(n + 2) = 0$.

From the assumption, we have the following.

$c(n-1) = 1$, since otherwise $[(n+2, \ldots, n+2, 1), (n^2, n-1), n^2 + n - 1] \in S_c(\mathcal{E})$,
$c(n^2 + n - 2) = 1$, since otherwise $[(n+2, \ldots, n+2, 1), (n^2 + n - 2, 1), n^2 + n - 1] \in S_c(\mathcal{E})$,
$c(n^2-2) = 0$, since otherwise $[(n+1, \ldots, n+1, n, n), (n^2-2, n), n^2 + n - 2] \in S_c(\mathcal{E})$,
$c(n^2 - 3) = 0$, since otherwise $[(n+1, \ldots, n+1, n, n), (n^2 - 3, n + 1), n^2 + n - 2] \in S_c(\mathcal{E})$,

Thus, $[(n+2, \ldots, n+2, 1, 1), (n^2 - 3, 1), n^2 - 2] \in S_c(\mathcal{E})$, This is a contradiction.

Case (13): $c(n) = 1,\ c(n^2) = 0,\ c(n+1) = 1,\ c(n^2 + n - 1) = 0,\ c(n+2) = 1,\ c(2n) = 0$.

From the assumption, we have $c(n-1) = c(n^2-n-1) = c(n^2+n-2) = 1$, since otherwise $[(1, \ldots, 1, n^2), (n^2, n-1), n^2 + n - 1] \in S_c(\mathcal{E})$, and $n^2 + (n-1) = (n^2 - n - 1) + 2n = (n^2 + n - 2) + 1$. We also have the following

$c(2n-1) = 0$, since otherwise $[(n+1, \ldots, n+1, n, n), (n^2 - n - 1, 2n - 1), n^2 + n - 2] \in S_c(\mathcal{E})$,
$c(2) = 1$, since otherwise $[(2, \ldots, 2), (2n-1, 1), 2n] \in S_c(\mathcal{E})$,
$c(n^2 - n + 1) = 0$, since otherwise $[(n-1, \ldots, n-1, n), (n^2 - n - 1, 2), n^2 - n + 1] \in S_c(\mathcal{E})$,
$c(n^2 - 1) = 0$, since otherwise $[(n+1, \ldots, n+1, n-1), (n^2 - 1, n - 1), n^2 + n - 2] \in S_c(\mathcal{E})$,

Thus, $[(1, \ldots, 1, n^2-n+1), (n^2-1, 1), n^2] \in S_c(\mathcal{E})$, This is a contradiction.

Case (14): $c(n) = 1,\ c(n^2) = 0,\ c(n+1) = 1,\ c(n^2 + n - 1) = 0,\ c(n+2) = c(2n) = 1$.

From the assumption, we have the following.

$c(n-1) = 1$, since otherwise $[(1, \ldots, 1, n^2), (n^2, n-1), n^2 + n - 1] \in S_c(\mathcal{E})$,
$c(2) = 0$, since otherwise $[(2, \ldots, 2), (n+1, n-1), 2n] \in S_c(\mathcal{E})$,
$c(n^2+n-2) = 1$, since otherwise $[(1, \ldots, 1, n^2), (n^2+n-2, 1), n^2 + n - 1] \in S_c(\mathcal{E})$,
$c(n^2-2) = 0$, since otherwise $[(n+1, \ldots, n+1, n, n), (n^2-2, n), n^2 + n - 2] \in S_c(\mathcal{E})$,
$c(n^2-n-2) = 0$, since otherwise $[(n+1, \ldots, n+1, n, n), (n^2-n-2, 2n), n^2 + n - 2] \in S_c(\mathcal{E})$,

Thus, $[(1, \ldots, 1, 2, 2, 2, n^2 - n - 2), (n^2 - 2, 2), n^2] \in S_c(\mathcal{E})$, This is a contradiction.

Case (15): $c(n) = 1,\ c(n^2) = 0,\ c(n+1) = c(n^2+n-1) = 1$.

From the assumption, we have the following.

$c(n^2-1) = 0$, since otherwise $[(n+1,\dots,n+1,n),(n^2-1,n),n^2+n-1] \in S_c(\mathcal{E})$,
$c(n^2-2) = 0$, since otherwise $[(n+1,\dots,n+1,n),(n^2-2,n+1),n^2+n-1] \in S_c(\mathcal{E})$,
$c(n^2-n+1) = 1$, since otherwise $[(1,\dots,1,n^2-n+1),(n^2-1,1),n^2] \in S_c(\mathcal{E})$,
$c(2n-2) = 0$, since otherwise $[(n+1,\dots,n+1,n),(n^2-n+1,2n-2),n^2+n-1] \in S_c(\mathcal{E})$,
$c(n-1) = 1$, since otherwise $[(1,\dots,1,n-1),(n-1,n-1),2n-2] \in S_c(\mathcal{E})$,
$c(n^2-2n) = 0$, since otherwise $[(n-1,\dots,n-1,n),(n^2-2n,n+1),n^2-n+1] \in S_c(\mathcal{E})$,
$c(n^2-n-1) = 1$, since otherwise $[(1,\dots,1,n^2-n-1),(n^2-2n,2n-2),n^2-2] \in S_c(\mathcal{E})$,
$c(2) = 1$, since otherwise $[(2,\dots,2,n^2-2n),(n^2-2n,2n-2),n^2-2] \in S_c(\mathcal{E})$,

Thus, $[(n-1,\dots,n-1,n),(n^2-n-1,2),n^2-n+1] \in S_c(\mathcal{E})$, This is a contradiction.

Case (16): $c(n) = c(n^2) = 1,\ c(n^2-1) = 0$.

From the assumption, we have the following.

$c(n^2-n) = 0$, since otherwise $[(n,\dots,n),(n^2-n,n),n^2] \in S_c(\mathcal{E})$,
$c(n-1) = 1$, since otherwise $[(1,\dots,1,n^2-n),(n^2-n,n-1),n^2-1] \in S_c(\mathcal{E})$,
$c(n^2-2) = 1$, since otherwise $[(1,\dots,1,n^2-n),(n^2-2,1),n^2-1] \in S_c(\mathcal{E})$,
$c(2) = 0$, since otherwise $[(n,\dots,n),(n^2-2,2),n^2] \in S_c(\mathcal{E})$,
$c(n^2-n-1) = 0$, since otherwise $[(n,\dots,n,n-1,n-1),(n^2-n-1,n-1),n^2-2] \in S_c(\mathcal{E})$,
$c(n+1) = 1$, since otherwise $[(n+1,\dots,n+1,1,1),(n^2-n-1,1),n^2-n] \in S_c(\mathcal{E})$,
$c(n^2+n-1) = 0$, since otherwise $[(n+1,\dots,n+1,n),(n^2,n-1),n^2+n-1] \in S_c(\mathcal{E})$,
$c(n^2+n-2) = 1$, since otherwise $[(1,\dots,1,2,n^2-1),(n^2+n-2,1),n^2+n-1] \in S_c(\mathcal{E})$,

Thus, $[(n+1,\dots,n+1,n,n),(n^2-2,n),n^2+n-2] \in S_c(\mathcal{E})$, This is a contradiction.

Case (17): $c(n) = c(n^2) = c(n^2 - 1) = 1,\ c(n-1) = 0$.

We have $c(n^2 - n) = 0$ by the same method as in Case (16). Also we have

$c(n^2 - n - 1) = 1$, since otherwise $[(n-1, \dots, n-1), (n^2 - n - 1, 1), n^2 - n] \in S_c(\mathcal{E})$,
$c(n+1) = 0$, since otherwise $[(n, \dots, n), (n^2 - n - 1, n+1), n^2] \in S_c(\mathcal{E})$,
$c(2) = 1$, since otherwise $[(1, \dots, 1, 2), (n-1, 2), n+1] \in S_c(\mathcal{E})$,
$c(2n) = 0$, since otherwise $[(2, \dots, 2), (n, n), 2n] \in S_c(\mathcal{E})$,

Thus, $[(1, \dots, 1, n+1), (n-1, n+1), 2n] \in S_c(\mathcal{E})$, This is a contradiction.

Case (18): $c(n) = c(n^2) = c(n^2 - 1) = c(n-1) = 1$.

We have $c(n^2 - n) = 0$ by the same method as in Case (16). Also we have

$c(n^2 - n - 1) = 0$, since otherwise $[(n, \dots, n, n-1), (n^2 - n - 1, n), n^2 - 1] \in S_c(\mathcal{E})$,
$c(n+1) = 1$, since otherwise $[(n+1, \dots, n+1, 1, 1), (n^2 - n - 1, 1), n^2 - n] \in S_c(\mathcal{E})$,
$c(n^2 - n + 1) = 0$, since otherwise $[(n, \dots, n), (n^2 - n + 1, n-1), n^2] \in S_c(\mathcal{E})$,
$c(n^2 - 2n + 1) = 1$, since otherwise $[(1, \dots, 1, n^2 - 2n + 1), (n^2 - n - 1, 1), n^2 - n] \in S_c(\mathcal{E})$,
$c(2n-2) = 0$, since otherwise $[(n-1, \dots, n-1, 2n-2), (n^2 - 2n + 1, 2n-2), n^2 - 1] \in S_c(\mathcal{E})$,
$c(n-2) = 1$, since otherwise $[(n-2, \dots, n-2, 2n-2), (n^2 - n - 1, 1), n^2 - n] \in S_c(\mathcal{E})$,
$c(n^2 + n - 1) = 0$, since otherwise $[(n+1, \dots, n+1, n), (n^2, n-1), n^2 + n - 1] \in S_c(\mathcal{E})$,
$c(2) = 1$, since otherwise $[(2, \dots, 2, n^2 - n + 1), (n^2 - n + 1, 2n - 2), n^2 + n - 1] \in S_c(\mathcal{E})$,
$c(n^2 - n + 2) = 0$, since otherwise $[(n, \dots, n, n), (n^2 - n + 2, n-2), n^2] \in S_c(\mathcal{E})$,
$c(n^2 + 1) = 0$, since otherwise $[(n, \dots, n, n+1), (n^2 - 1, 2), n^2 + 1] \in S_c(\mathcal{E})$,
$c(n^2 - 2n + 3) = 1$, since otherwise $[(1, \dots, 1, n^2 - n + 2), (n^2 - 2n + 3, 2n - 2), n^2 + 1] \in S_c(\mathcal{E})$,

Thus, $[(n-1, \dots, n-1, 2), (n^2 - 2n + 1, 2), n^2 - 2n + 3] \in S_c(\mathcal{E})$, This is a contradiction.

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