AGGREGATION OPERATORS OF CUBIC PICTURE FUZZY QUANTITIES AND THEIR APPLICATION IN DECISION SUPPORT SYSTEMS

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Abstract. The paper aim is to resolve the issue of ranking to the fuzzy numbers in decision analysis, artificial intelligence and optimization. In the literature lot of ideologies have been established for ranking to the fuzzy numbers, that ideologies have some restrictions and limitations. In this paper, we proposed a method based on cubic picture fuzzy information's, for ranking to defeat the existing restrictions. Further introduced some cubic picture fuzzy algebraic and cubic picture fuzzy algebraic* aggregated operators for aggregated the information. Finally, a multi-attribute decision making problem is assumed as a practical application to establish the appropriateness and suitability of the proposed ranking approach.

1. Introduction

Structure of fuzzy set (FSs), try different approaches to handle uncertainty and imprecision. With the growing problems of the system in our daily lives, it is very difficult for every decision-maker to get the best selection from the set of reasonable ones. Thus, to make the best conclusion, one realizes the causes that can inspire decision-makers to control such an undeniable theory. Researchers are involved in describing behavior by developing undefined data to make a good product within

the desired amount of accuracy. As a matter of fact , it is necessary to achieve imprecision and disinclination in the data. Thus, dealing with such uncertain data in decision-making problems can not always be effective in classical or crisp methods. Thus, in 1965, Zadeh [31] presented the idea of a fuzzy set that includes the membership function. After the useful application of fuzzy sets Attanassov [10] proposed the idea of intuitionistic fuzzy set (IFS) to fulfill the lack of non-membership degree in fuzzy sets. An IFS includes the membership and non-membership degrees. IFSs [10, 12] and interval-valued intuitionistic fuzzy sets (IV-IFSs) [11] are the proper additions of FSs, that can be managed and defined the hesitations in the given records and be identify their membership and non-membership degrees.

From the above study, we can be noted that mostly proposed operators and decision making for the IFSs and IVIFSs have been positively applied in dissimilar parts, but an IFSs there are some situations in practicality which could not be symbolized. For example, the people's opinions during the voting occasion containing many answers which could not be perfectly described an IFS of such kinds: yes, abstain, no, refusal. Such type of difficulties is not point out by FSs or IFSs. Therefore, in such condition to handle computational types of difficulties, Cuong [14] presented a new concept which was named as picture fuzzy set (PFS). For notation we refer to [15–17]. PFSs deal with three functions, named as, membership, neutral and non-membership degrees. Now a day, the problems in the picture fuzzy sets have been studied by some authors such as a correlation coefficient for the PFS has been presented by Singh [27], and the measure of a generalized picture distance and their application to be solved the grouping difficulties in the picture fuzzy environment has been presented by Son [28]. For a picture fuzzy weighted cross-entropy and used to develop a concept for the different alternatives ranking Wei [29] introduced a best decision-making technique. Thus, holding the encouragement from the information that the picture fuzzy sets have been the unlimited potential for modeling the inexact and confusing data in the real-world applications. Jun at all. [18], presented a new concept which is named as cubic set. Based on the combination of IVFSs extension and FSs extension, Jun. work on cubic set was quite remarkable. He establishes cubic set with their operational properties and applied to BCK/BCI-algebras. For notation we refer to [19, 20, 23, 24].

In this paper, the extend the structure of cubic sets to the picture fuzzy sets. Ashraf et al. [3] presented the concepts of positive-internal (neutral-internal, negative-internal) cubic picture fuzzy sets and positive-external (neutral -external, negative-external) cubic picture fuzzy sets, and explore associated properties. Since in decision making aggregation operators play a vital role, therefore in this paper we introduced cubic picture aggregation operators, namely the Cubic picture fuzzy weighted average (CPFWA), weighted average* (CPFWA*), order weighted average (CPFOWA), order weighted average* (CPFOWA*), hybrid weighted average (CPFHWA), hybrid weighted average* (CPFHO WA*) operators, and established the comparison technique for ranking the alternatives.

The paper aims are: (a) to propose the idea of cubic picture fuzzy set (b) some debate on its basic operational characteristics, (c) For ranking the CPFNs, to propose score and accuracy functions, (d) established cubic picture fuzzy averaging aggregation operators (e) propose MADM method based on these aggregation operators under cubic picture fuzzy information. The establish method have verified with an application to a petroleum circulation center evaluation problem for viewing their effectiveness as well as reliability.

2. Preliminaries

DEFINITION. [18] Let $R \neq \phi$ be a universe set. Then the cubic set can be written as:

$$J = \left\{ \left\langle r, \widetilde{P}_j(r), P_j(r) \middle| \ r \in R \right\rangle \right\},$$

where $\widetilde{P}_j(r)$ is an IVFS in R and $P_j(r)$ is a FS in R. For simplicity we denoted the cubic set as, $J = \left\langle \widetilde{P}_j, P_j \right\rangle$.

DEFINITION. [14] Let $R \neq \phi$ be a universe set. Then the picture fuzzy set can be written as:

$$J = \left\{ \left\langle r, P_j(r), I_j(r), N_j(r) \middle| \ r \in R \right\rangle \right\}.$$

where $P_j : R \to [0,1]$, $I_j : R \to [0,1]$ and $N_j : R \to [0,1]$ are indicated the positive membership degree of the element r to the set R, the neutral membership degree of the element r to the set R and the negative membership degree of the element r to the set R respectively. Also following

condition are satisfy by P_j, I_j and N_j are; $0 \leq P_j(r) + I_j(r) + N_j(r) \leq 1$ for all $r \in R$.

DEFINITION. [14] Let $R \neq \phi$ be a universe set. Then the interval valued picture fuzzy set (IVPFSs) can be written as:

$$J = \left\{ \left\langle r, \widetilde{P}_j(r), \widetilde{I}_j(r), \widetilde{N}_j(r) | \; r \in R \right\rangle \right\}.$$

An IVPFS in a set R is indicated by $\widetilde{P}_j : R \to \Omega$, $\widetilde{I}_j : R \to \Omega$ and $\widetilde{N}_j : R \to \Omega$. The functions $\widetilde{P}_j(r), \widetilde{I}_j(r)$ and $\widetilde{N}_j(r)$ denoted the positive membership degree of the element r to the set R, the neutral membership degree of the element r to the set R and the negative membership degree of the element r to the set R respectively, where Ω be the collection of all closed subintervals of $[0, 1]$. Also $\widetilde{P}_j(r), \widetilde{I}_j(r)$ and $\widetilde{N}_j(r)$ satisfy the following condition; $\forall r \in R, 0 \leq Sup\left(\widetilde{P}_j(r)\right) + Sup\left(\widetilde{I}_j(r)\right) + Sup\left(\widetilde{N}_j(r)\right) \leq 1$.

3. Cubic picture fuzzy sets and their Operations

DEFINITION. [1, 3, 4] Let $R \neq \phi$ be a universe set. Then the cubic picture fuzzy set (CPFS) can be written as:

$$C = \left\{ \left\langle r, \widetilde{J}_c(r), J_c(r) | \; r \in R \right\rangle \right\},$$

where $\widetilde{J}_c(r)$ is an IVPFS in R and $J_c(r)$ is a PFS in R. For simplicity we denoted the CPFS as, $C = \left\langle \widetilde{J}_c, J_c \right\rangle$.

DEFINITION. Let $R \neq \phi$ be a universe set. Then the cubic picture fuzzy set $B = \left\langle \widetilde{J}_b, J_b \right\rangle$ in R is said to be:

(1) Positive-internal if $P_b^-(r) \leq P_b(r) \leq P_b^+(r), \; \forall \; r \in R$.
(2) Neutral-internal if $I_b^-(r) \leq I_b(r) \leq I_b^+(r), \; \forall \; r \in R$.
(3) Nagative-internal if $N_b^-(r) \leq N_b(r) \leq N_b^+(r), \; \forall \; r \in R$.

If the cubic picture fuzzy set $B = \left\langle \widetilde{J}_b, J_b \right\rangle$ in R is satisfies all the above properties then the cubic picture fuzzy set is said to be an internal cubic picture fuzzy set in R.

DEFINITION. Let $R \neq \phi$ be a universe set. Then the cubic picture fuzzy set $B = \left\langle \widetilde{J}_b, J_b \right\rangle$ in R is said to be:

(1) Positive-external if $P_b(r) \notin \left(P_b^-(r), P_b^+(r)\right)$, $\forall\, r \in R$.

(2) Neutral-external if $I_b(r) \notin \left(I_b^-(r), I_b^+(r)\right)$, $\forall\, r \in R$.

(3) Nagative-external if $N_b(r) \notin \left(N_b^-(r), N_b^+(r)\right)$, $\forall\, r \in R$.

If the cubic picture fuzzy set $B = \left\langle \widetilde{J_b}, J_b \right\rangle$ in R is satisfies all the above properties then the cubic picture fuzzy set is said to be an external cubic picture fuzzy set in R.

PROPOSITION 3.1. *Let $R \neq \phi$ be a universe set and the pair $B = \left\langle \widetilde{J_b}, J_b \right\rangle$ be a CPFS in R which is not external. Then there exists $r \in R$ such that, $P_b(r) \in \left(P_b^-(r), P_b^+(r)\right)$, $I_b(r) \in \left(I_b^-(r), I_b^+(r)\right)$ and $N_b(r) \in \left(N_b^-(r), N_b^+(r)\right)$.*

Proof. Straightforward. □

DEFINITION. Let A and B are two CPFNs, some operations on CPFNs are defined as follows:

(1) $A \subseteq B$ iff $\forall r \in R$, $\widetilde{P}_a(r) \leq \widetilde{P}_b(r)$, $\widetilde{I}_a(r) \geq \widetilde{I}_b(r)$, $\widetilde{N}_a(r) \geq \widetilde{N}_b(r)$, $P_a(r) \leq P_b(r)$, $I_a(r) \geq I_b(r)$ and $N_a(r) \geq N_b(r)$. Where $\widetilde{P}_a(r) \leq \widetilde{P}_b(r)$ means that $\left[P_a^-(r) \leq P_b^-(r),\; P_a^+(r) \leq P_b^+(r)\right]$ and similar as for the degree of neutral and non-membership.

(2) Union of two CPFNs $A \cup B$

$$= \left\{ \begin{array}{c} \left\{\max\left\{\widetilde{P}_a, \widetilde{P}_b\right\}, \min\left\{\widetilde{I}_a, \widetilde{I}_b\right\}, \min\left\{\widetilde{N}_a, \widetilde{N}_b\right\}\right\}, \\ \left\{\max\left\{P_a, P_b\right\}, \min\left\{I_a, I_b\right\}, \min\left\{N_a, N_b\right\}\right\} \end{array} \right\}, \text{ where}$$

$\max\left\{\widetilde{P}_a, \widetilde{P}_b\right\} = \left[\max\left\{P_a^-, P_b^-\right\}, \max\left\{P_a^+, P_b^+\right\}\right]$ and similar as for the degree of neutral and non-membership.

(3) Intersection of two CPFNs $A \cup B$

$$= \left\{ \begin{array}{c} \left\{\min\left\{\widetilde{P}_a, \widetilde{P}_b\right\}, \min\left\{\widetilde{I}_a, \widetilde{I}_b\right\}, \max\left\{\widetilde{N}_a, \widetilde{N}_b\right\}\right\}, \\ \left\{\min\left\{P_a, P_b\right\}, \min\left\{I_a, I_b\right\}, \max\left\{N_a, N_b\right\}\right\} \end{array} \right\}, \text{ where}$$

$\min\left\{\widetilde{P}_a, \widetilde{P}_b\right\} = \left[\min\left\{P_a^-, P_b^-\right\}, \min\left\{P_a^+, P_b^+\right\}\right]$ and similar as for the degree of neutral and non-membership.

DEFINITION. Let $R \neq \phi$ be a universe set and the pair $A = \left\langle \widetilde{J_a}, J_a \right\rangle$ be cubic picture fuzzy set in R. The compliment is denoted and defined as

$$A^c = \left\langle \widetilde{J^c}_a, J_a^c \right\rangle = \left\{r, \left\langle \widetilde{N}_a(r), \widetilde{I}_a(r), \widetilde{P}_a(r) \right\rangle, \left\langle N_a(r), I_a(r), P_a(r) \right\rangle \,|\, r \in R \right\}.$$

DEFINITION. Let $R \neq \phi$ be a universe set and $A = \left\langle \widetilde{J_a}, J_a \right\rangle$ and $B = \left\langle \widetilde{J_b}, J_b \right\rangle$ be any two CPFNs in R and $\tau \geq 0$. Then the operations of CPFNs can be defined and denotes as

(1) $A \oplus B = \left\langle \widetilde{J_a} \oplus \widetilde{J_b}, J_a \oplus J_b \right\rangle$

$$= \left(\left(\begin{array}{c} \left[P_a^- + P_b^- - P_a^- \cdot P_b^- , P_a^+ + P_b^+ - P_a^+ \cdot P_b^+ \right], \\ \left[I_a^- \cdot I_b^-, \ I_a^+ \cdot I_b^+ \right], \left[\ N_a^- \cdot N_b^-, \ N_a^+ \cdot N_b^+ \right] \\ \{ P_a + P_b - P_a \cdot P_b, \ I_a \cdot I_b, \ N_a \cdot N_b \} \end{array} \right), \right).$$

(2) $A \oplus^* B = \left\langle \widetilde{J_a} \oplus \widetilde{J_b}, J_a \otimes J_b \right\rangle$

$$= \left(\left(\begin{array}{c} \left[P_a^- + P_b^- - P_a^- \cdot P_b^- , P_a^+ + P_b^+ - P_a^+ \cdot P_b^+ \right], \\ \left[I_a^- \cdot I_b^-, \ I_a^+ \cdot I_b^+ \right], \left[\ N_a^- \cdot N_b^-, \ N_a^+ \cdot N_b^+ \right] \\ \{ P_a \cdot P_b, \ I_a \cdot I_b, \ N_a + N_b - N_a \cdot N_b \} \end{array} \right), \right).$$

(3) $A \otimes B = \left\langle \widetilde{J_a} \otimes \widetilde{J_b}, J_a \otimes J_b \right\rangle$

$$= \left(\left(\begin{array}{c} \left[\ P_a^- \cdot P_b^-, \ P_a^+ \cdot P_b^+ \right], \left[I_a^- \cdot I_b^-, \ I_a^+ \cdot I_b^+ \right], \\ \left[N_a^- + N_b^- - N_a^- \cdot N_b^- , N_a^+ + N_b^+ - N_a^+ \cdot N_b^+ \right] \\ \{ P_a \cdot P_b, \ I_a \cdot I_b, \ N_a + N_b - N_a \cdot N_b \} \end{array} \right), \right).$$

(4) $A \otimes^* B = \left\langle \widetilde{J_a} \otimes \widetilde{J_b}, J_a \oplus J_b \right\rangle$

$$= \left(\left(\begin{array}{c} \left[\ P_a^- \cdot P_b^-, \ P_a^+ \cdot P_b^+ \right], \left[I_a^- \cdot I_b^-, \ I_a^+ \cdot I_b^+ \right], \\ \left[N_a^- + N_b^- - N_a^- \cdot N_b^- , N_a^+ + N_b^+ - N_a^+ \cdot N_b^+ \right] \\ \{ P_a + P_b - P_a \cdot P_b, \ I_a \cdot I_b, \ N_a \cdot N_b \} \end{array} \right), \right).$$

(5) $\tau \cdot A = \left\langle \tau \otimes \widetilde{J_a}, \tau \otimes J_a \right\rangle$

$$= \left(\left(\begin{array}{c} \left[1 - (1 - P_a^-)^\tau, \ 1 - (1 - P_a^+)^\tau \right], \\ \left[(I_a^-)^\tau, \ (I_a^+)^\tau \right], \left[\ (N_a^-)^\tau, \ (N_a^+)^\tau \right] \end{array} \right), \{ 1 - (1 - P_a)^\tau, \ (I_a)^\tau, (N_a)^\tau \} \right).$$

(6) $\tau \cdot^* A = \left\langle \tau \otimes \widetilde{J_a}, (J_a)^\tau \right\rangle$

$$= \left(\left(\begin{array}{c} \left[1 - (1 - P_a^-)^\tau, \ 1 - (1 - P_a^+)^\tau \right], \\ \left[(I_a^-)^\tau, \ (I_a^+)^\tau \right], \left[\ (N_a^-)^\tau, \ (N_a^+)^\tau \right] \end{array} \right), \{ (P_a)^\tau, \ (I_a)^\tau, 1 - (1 - N_a)^\tau \} \right).$$

(7) $A^\tau = \left\langle \left(\widetilde{J_a} \right)^\tau, (J_a)^\tau \right\rangle$

$$= \left(\left(\begin{array}{c} \left[(P_a^-)^\tau, \ (P_a^+)^\tau \right], \left[(I_a^-)^\tau, \ (I_a^+)^\tau \right], \\ \left[1 - (1 - N_a^-)^\tau, \ 1 - (1 - N_a^+)^\tau \right] \end{array} \right), \{ (P_a)^\tau, \ (I_a)^\tau, 1 - (1 - N_a)^\tau \} \right).$$

(8) $A^{*\tau} = \left\langle \left(\widetilde{J_a} \right)^\tau, \tau \otimes J_a \right\rangle$

$$= \left(\left(\begin{array}{c} \left[(P_a^-)^\tau, \ (P_a^+)^\tau \right], \left[(I_a^-)^\tau, \ (I_a^+)^\tau \right], \\ \left[1 - (1 - N_a^-)^\tau, \ 1 - (1 - N_a^+)^\tau \right] \end{array} \right), \{ 1 - (1 - P_a)^\tau, \ (I_a)^\tau, (N_a)^\tau \} \right).$$

3.1. Comparison Rules for CPFNs. This section proposed ranking approaches to evaluating the rank of the alternatives.

DEFINITION. [4] Let $R \neq \phi$ be a universe set and $A = \left\langle \widetilde{J_a}, J_a \right\rangle$ be any CPFN in R. Then

(1) $sco(A) = \frac{\left(P_a^- + P_a^+ + 1 - I_a^- - I_a^+ - N_a^- - N_a^+\right) + (P_a + 1 - I_a + 1 - N_a)}{6}$ which denoted as score function.

(2) $acu(A) = \frac{\left(P_a^- + P_a^+ - N_a^- - N_a^+\right) + (P_a - N_a)}{6}$ which denoted as accuracy function.

DEFINITION. [4] Let $A = \left\langle \widetilde{J_a}, J_a \right\rangle$ and $B = \left\langle \widetilde{J_b}, J_b \right\rangle$ be any two CPFNs in R. Then by using the Definition 3.1, equating technique can be described as,

(a) If $sco(A) > sco(B)$, then $A > B$.

(b) If $sco(A) = sco(B)$, and $acu(A) > acu(B)$, then $A > B$.

(c) If $sco(A) = sco(B)$, $acu(A) = acu(B)$ then $A = B$.

3.2. Cubic Picture Fuzzy Weighted Averaging Aggregation Operators.

DEFINITION. Let any collection $A_l = \left\langle \widetilde{J_{a_l}}, J_{a_l} \right\rangle$, $l \in N$ of CPFNs and $CPFWA : CPFN^n \to CPFN$, then CPFWA describe as,

$$CPFWA\left(A_1, A_2, ..., A_n\right) = \sum_{l=1}^{n} \tau_l \cdot A_l,$$

In which weight vector $\tau = \{\tau_1, \tau_2, ..., \tau_n\}$ of $A_l = \left\langle \widetilde{J_{a_l}}, J_{a_l} \right\rangle$, $l \in N$, with $\tau_l \geq 0$ and $\sum_{l=1}^{n} \tau_l = 1$.

THEOREM 3.2. *Let any collection* $A_l = \left\langle \widetilde{J_{a_l}}, J_{a_l} \right\rangle$, $l \in N$ *of CPFNs. Then using the Idea propose in Definition 3.2 and operational laws of CPFNs, we attain following outcome which are,*

$CPFWA\left(A_1, A_2, ..., A_n\right)$
$$= \left(\left(\begin{array}{c} \left[1 - \Pi_{l=1}^{n}(1 - P_{a_l}^-)^{\tau_l}, \ 1 - \Pi_{l=1}^{n}(1 - P_{a_l}^+)^{\tau_l}\right], \\ \left[\Pi_{l=1}^{n}(I_{a_l}^-)^{\tau_l}, \ \Pi_{l=1}^{n}(I_{a_l}^+)^{\tau_l}\right], \ \left[\Pi_{l=1}^{n}(N_{a_l}^-)^{\tau_l}, \ \Pi_{l=1}^{n}(N_{a_l}^+)^{\tau l}\right] \\ \{1 - \Pi_{l=1}^{n}(1 - P_{a_l})^{\tau_l}, \Pi_{l=1}^{n}(I_{a_l})^{\tau_l}, \Pi_{l=1}^{n}(N_{a_l})^{\tau_l}\} \end{array} \right), \right).$$

Where the weight vector $\tau = \{\tau_1, \tau_2, ..., \tau_n\}$ *of* $A_l = \left\langle \widetilde{J_{a_l}}, J_{a_l} \right\rangle$, $l \in N$, *with* $\tau_l \geq 0$ *and* $\sum_{l=1}^{n} \tau_l = 1$.

Proof. We done the prove by utilizing the technique of mathematical induction. Therefore we follows as

(a) For $n = 2$, we have

$$CPFWA\,(A_1, A_2)$$

$$= \tau_1 \cdot A_1 + \tau_2 \cdot A_2$$

$$= \left(\left(\begin{array}{c} \left[1 - (1 - P_{a_1}^-)^{\tau_1}, 1 - (1 - P_{a_1}^+)^{\tau_1}\right], \\ \left[(I_{a_1}^-)^{\tau_1}, \ (I_{a_1}^+)^{\tau_1}\right], \ \left[(N_{a_1}^-)^{\tau_1}, \ (N_{a_1}^+)^{\tau_1}\right] \\ \{1 - (1 - P_{a_1})^{\tau_1}, (I_{a_1})^{\tau_1}, (N_{a_1})^{\tau_1}\} \end{array} \right), \right) +$$

$$\left(\left(\begin{array}{c} \left[1 - (1 - P_{a_2}^-)^{\tau_2}, 1 - (1 - P_{a_2}^+)^{\tau_2}\right], \\ \left[(I_{a_2}^-)^{\tau_2}, \ (I_{a_2}^+)^{\tau_2}\right], \ \left[(N_{a_2}^-)^{\tau_2}, \ (N_{a_2}^+)^{\tau_2}\right] \\ \{1 - (1 - P_{a_2})^{\tau_2}, (I_{a_2})^{\tau_2}, (N_{a_2})^{\tau_2}\} \end{array} \right), \right)$$

$$= \left(\left(\begin{array}{c} \left[1 - \Pi_{l=1}^2(1 - P_{a_l}^-)^{\tau_l}, 1 - \Pi_{l=1}^2(1 - P_{a_l}^+)^{\tau_l}\right], \\ \left[\Pi_{l=1}^2(I_{a_l}^-)^{\tau_l}, \ \Pi_{l=1}^2(I_{a_l}^+)^{\tau_l}\right], \ \left[\Pi_{l=1}^2(N_{a_l}^-)^{\tau_l}, \ \Pi_{l=1}^2(N_{a_l}^+)^{\tau_l}\right] \\ \{1 - \Pi_{l=1}^2(1 - P_{a_l})^{\tau_l}, \Pi_{l=1}^2(I_{a_l})^{\tau_l}, \Pi_{l=1}^2(N_{a_l})^{\tau_l}\} \end{array} \right), \right)$$

(b) Suppose that outcome is true for $n = z$ that is,

$$CPFWA\,(A_1, A_2, ..., A_z)$$

$$= \left(\left(\begin{array}{c} \left[1 - \Pi_{l=1}^z(1 - P_{a_l}^-)^{\tau_l}, 1 - \Pi_{l=1}^z(1 - P_{a_l}^+)^{\tau_l}\right], \\ \left[\Pi_{l=1}^z(I_{a_l}^-)^{\tau_l}, \ \Pi_{l=1}^z(I_{a_l}^+)^{\tau_l}\right], \ \left[\Pi_{l=1}^z(N_{a_l}^-)^{\tau_l}, \ \Pi_{l=1}^z(N_{a_l}^+)^{\tau_l}\right] \\ \{1 - \Pi_{l=1}^z(1 - P_{a_l})^{\tau_l}, \Pi_{l=1}^z(I_{a_l})^{\tau_l}, \Pi_{l=1}^z(N_{a_l})^{\tau_l}\} \end{array} \right), \right),$$

(c) Now we have to prove that outcome is true for $n = z + 1$, by utilizing the (a) & (b) we have

$$CPFWA\,(A_1, A_2, ..., A_z, A_{z+1})$$

$$= \sum_{l=1}^{z} \tau_l \cdot A_l + \tau_{z+1} \cdot A_{z+1}$$

$$= \left(\left(\begin{array}{c} \left[1 - \Pi_{l=1}^z(1 - P_{a_l}^-)^{\tau_l}, 1 - \Pi_{l=1}^z(1 - P_{a_l}^+)^{\tau_l}\right], \\ \left[\Pi_{l=1}^z(I_{a_l}^-)^{\tau_l}, \ \Pi_{l=1}^z(I_{a_l}^+)^{\tau_l}\right], \ \left[\Pi_{l=1}^z(N_{a_l}^-)^{\tau_l}, \ \Pi_{l=1}^z(N_{a_l}^+)^{\tau_l}\right] \\ \{1 - \Pi_{l=1}^z(1 - P_{a_l})^{\tau_l}, \Pi_{l=1}^z(I_{a_l})^{\tau_l}, \Pi_{l=1}^z(N_{a_l})^{\tau_l}\} \end{array} \right), \right)$$

$$+ \left(\left(\begin{array}{c} \left[1 - (1 - P_{a_{z+1}}^-)^{\tau_{z+1}}, 1 - (1 - P_{a_{z+1}}^+)^{\tau_{z+1}}\right], \\ \left[(I_{a_{z+1}}^-)^{\tau_{z+1}}, \ (I_{a_{z+1}}^+)^{\tau_{z+1}}\right], \ \left[(N_{a_{z+1}}^-)^{\tau_{z+1}}, \ (N_{a_{z+1}}^+)^{\tau_{z+1}}\right] \\ \{1 - (1 - P_{a_{z+1}})^{\tau_{z+1}}, (I_{a_{z+1}})^{\tau_{z+1}}, (N_{a_{z+1}})^{\tau_{z+1}}\} \end{array} \right), \right)$$

$$= \left(\left(\begin{array}{c} \left[1 - \Pi_{l=1}^{z+1}(1 - P_{a_l}^-)^{\tau_l}, 1 - \Pi_{l=1}^{z+1}(1 - P_{a_l}^+)^{\tau_l}\right], \\ \left[\Pi_{l=1}^{z+1}(I_{a_l}^-)^{\tau_l}, \ \Pi_{l=1}^{z+1}(I_{a_l}^+)^{\tau_l}\right], \ \left[\Pi_{l=1}^{z+1}(N_{a_l}^-)^{\tau_l}, \ \Pi_{l=1}^{z+1}(N_{a_l}^+)^{\tau_l}\right] \\ \{1 - \Pi_{l=1}^{z+1}(1 - P_{a_l})^{\tau_l}, \Pi_{l=1}^{z+1}(I_{a_l})^{\tau_l}, \Pi_{l=1}^{z+1}(N_{a_l})^{\tau_l}\} \end{array} \right), \right)$$

i.e., Outcome is satisfy for $n = z + 1$.
which done the proof. $\qquad\square$

DEFINITION. Let any collection $A_l = \left\langle \widetilde{J_{a_l}}, J_{a_l} \right\rangle$, $l \in N$ of CPFNs and $CPFWA^* : CPFN^n \to CPFN$, then $\widetilde{CPFWA^*}$ describe as,

$$CPFWA^* (A_1, A_2, ..., A_n) = \sum_{l=1}^{n} \tau_l \cdot^* A_l,$$

In which weight vector $\tau = \{\tau_1, \tau_2, ..., \tau_n\}$ of $A_l = \left\langle \widetilde{J_{a_l}}, J_{a_l} \right\rangle$, $l \in N$, with $\tau_l \geq 0$ and $\sum_{l=1}^{n} \tau_l = 1$.

THEOREM 3.3. *Let any collection* $A_l = \left\langle \widetilde{J_{a_l}}, J_{a_l} \right\rangle$, $l \in N$ *of CPFNs. Then using the Idea propose in Definition 3.2 and operational laws of CPFNs, we attain following outcome which are,*

$CPFWA^* (A_1, A_2, ..., A_n)$
$$= \left(\left(\begin{array}{c} \left[1 - \Pi_{l=1}^{n}(1 - P_{a_l}^{-})^{\tau_l}, 1 - \Pi_{l=1}^{n}(1 - P_{a_l}^{+})^{\tau_l} \right], \\ \left[\Pi_{l=1}^{n}(I_{a_l}^{-})^{\tau_l}, \Pi_{l=1}^{n}(I_{a_l}^{+})^{\tau_l} \right], \left[\Pi_{l=1}^{n}(N_{a_l}^{-})^{\tau_l}, \Pi_{l=1}^{n}(N_{a_l}^{+})^{\tau_l} \right] \\ \{\Pi_{l=1}^{n}(P_{a_l})^{\tau_l}, \Pi_{l=1}^{n}(I_{a_l})^{\tau_l}, 1 - \Pi_{l=1}^{n}(1 - N_{a_l})^{\tau_l} \} \end{array} \right), \right) .$$

Where the weight vector $\tau = \{\tau_1, \tau_2, ..., \tau_n\}$ *of* $A_l = \left\langle \widetilde{J_{a_l}}, J_{a_l} \right\rangle$, $l \in N$, *with* $\tau_l \geq 0$ *and* $\sum_{l=1}^{n} \tau_l = 1$.

Proof. Proof is similar to 3.2. $\qquad\square$

DEFINITION. Let any collection $A_l = \left\langle \widetilde{J_{a_l}}, J_{a_l} \right\rangle$, $l \in N$ of CPFNs, and $CPFOWA : CPFN^n \to CPFN$, then the CPFOWA describe as,

$$CPFOWA(A_1, A_2, ..., A_n) = \sum_{l=1}^{n} \tau_l \cdot A_{\eta(l)},$$

with dimensions n, where *lth* biggest weighted value is $A_{\eta(l)}$ consequently by total order $A_{\eta(1)} \geq A_{\eta(2)} \geq ... \geq A_{\eta(n)}$. $\tau = \{\tau_1, \tau_2, ..., \tau_n\}$ is the wëight vectőrs such that $\tau_l \geq 0$, $l \in N$ and $\Sigma_{l=1}^{n} \tau_l = 1$.

THEOREM 3.4. *Let any collection* $A_l = \left\langle \widetilde{J_{a_l}}, J_{a_l} \right\rangle$, $l \in N$ *of CPFNs. Then using the Idea propose in Definition 3.2 and operational laws of CPFNs, we attain following outcome which are,*

$CPFOWA\left(A_1, A_2, ..., A_n\right)$

$$= \left(\left(\begin{array}{c} \left[1 - \Pi_{l=1}^n(1 - P_{a_{\eta(l)}}^-)^{\tau_l}, \ 1 - \Pi_{l=1}^n(1 - P_{a_{\eta(l)}}^+)^{\tau_l}\right], \\ \left[\Pi_{l=1}^n(I_{a_{\eta(l)}}^-)^{\tau_l}, \ \Pi_{l=1}^n(I_{a_{\eta(l)}}^+)^{\tau_l}\right], \ \left[\Pi_{l=1}^n(N_{a_{\eta(l)}}^-)^{\tau_l}, \ \Pi_{l=1}^n(N_{a_{\eta(l)}}^+)^{\tau_l}\right] \\ \left\{1 - \Pi_{l=1}^n(1 - P_{a_{\eta(l)}})^{\tau_l}, \Pi_{l=1}^n(I_{a_{\eta(l)}})^{\tau_l}, \Pi_{l=1}^n(N_{a_{\eta(l)}})^{\tau_l}\right\} \end{array} \right), \right).$$

Where $\tau = \{\tau_1, \tau_2, ..., \tau_n\}$ be the weight vector of $A_l = \left\langle \widetilde{J_{a_l}}, J_{a_l} \right\rangle$, $l \in N$, with $\tau_l \geq 0$ and $\sum_{l=1}^n \tau_l = 1$ and $A_{\eta(l)}$ is lth largest value consequently by total order $A_{\eta(1)} \geq A_{\eta(2)} \geq ... \geq A_{\eta(n)}$.

Proof. Proof is similar to 3.2. □

DEFINITION. Let any collection $A_l = \left\langle \widetilde{J_{a_l}}, J_{a_l} \right\rangle$, $l \in N$ of CPFNs and $CPFOWA^* : CPFN^n \to CPFN$, then the CPFOWA* describe as,

$$CPFOWA^*(A_1, A_2, ..., A_n) = \sum_{l=1}^n \tau_l \cdot^* A_{\eta(l)},$$

with dimensions n, where ;th biggest weighted value is $A_{\eta(l)}$ consequently by total order $A_{\eta(1)} \geq A_{\eta(2)} \geq ... \geq A_{\eta(n)}$. $\tau = \{\tau_1, \tau_2, ..., \tau_n\}$ is the wëight vectőrs such that $\tau_l \geq 0$, $l \in N$ and sum will be exactly 1.

THEOREM 3.5. *Let any collection $A_l = \left\langle \widetilde{J_{a_l}}, J_{a_l} \right\rangle$, $l \in N$ of CPFNs. Then using the Idea propose in Definition 3.2 and operational laws of CPFNs, we attain following outcome which are,*

$CPFOWA^*\left(A_1, A_2, ..., A_n\right)$

$$= \left(\left(\begin{array}{c} \left[1 - \Pi_{l=1}^n(1 - P_{a_{\eta(l)}}^-)^{\tau_l}, 1 - \Pi_{l=1}^n(1 - P_{a_{\eta(l)}}^+)^{\tau_l}\right], \\ \left[\Pi_{l=1}^n(I_{a_{\eta(l)}}^-)^{\tau_l}, \ \Pi_{l=1}^n(I_{a_{\eta(l)}}^+)^{\tau_l}\right], \ \left[\Pi_{l=1}^n(N_{a_{\eta(l)}}^-)^{\tau_l}, \ \Pi_{l=1}^n(N_{a_{\eta(l)}}^+)^{\tau_l}\right] \\ \left\{\Pi_{l=1}^n(P_{a_{\eta(l)}})^{\tau_l}, \Pi_{l=1}^n(I_{a_{\eta(l)}})^{\tau_l}, 1 - \Pi_{l=1}^n(1 - N_{a_{\eta(l)}})^{\tau_l}\right\} \end{array} \right), \right).$$

Where $\tau = \{\tau_1, \tau_2, ..., \tau_n\}$ be the weight vector of $A_l = \left\langle \widetilde{J_{a_l}}, J_{a_l} \right\rangle$, $l \in N$, with $\tau_l \geq 0$ and $\sum_{l=1}^n \tau_l = 1$ and $A_{\eta(l)}$ is lth largest value consequently by total order $A_{\eta(1)} \geq A_{\eta(2)} \geq ... \geq A_{\eta(n)}$.

Proof. Proof is similar to 3.2. □

The proposed aggregation operators ($AOps$) fulfilled following properties obviously.

(1) Idempotency: Let any collection $A_l = \left\langle \widetilde{J_{a_l}}, J_{a_l} \right\rangle$, $l \in N$ of CPFNs.
If all of $A_l = \left\langle \widetilde{J_{a_l}}, J_{a_l} \right\rangle$, $l \in N$ are identical. Then

$$AOps\left(A_1, A_2, ..., A_n\right) = A.$$

(2) Boundedness:Let any collection $A_l = \left\langle \widetilde{J_{a_l}}, J_{a_l} \right\rangle$, $l \in N$ of CPFNs.
Let $A_l^- = \left\{ \begin{array}{c} \left\{\min_l P_{a_l}^-, \min_l I_{a_l}^-, \max_l N_{a_l}^-\right\}, \\ \left\{\min_l P_{a_l}, \min_l I_{a_l}, \max_l N_{a_l}\right\} \end{array} \right\}$ and
$A_l^+ = \left\{ \begin{array}{c} \left\{\max_l P_{a_l}^+, \min_l I_{a_l}^+, \min_l N_{a_l}^+\right\}, \\ \left\{\max_l P_{a_l}, \min_l I_{a_l}, \min_l N_{a_l}\right\} \end{array} \right\}$ for whole $l \in N$, therefore

$$A_l^- \subseteq AOps\left(A_1, A_2, ..., A_n\right) = A_l^+.$$

(3) Monotonically: Let any collection $e_l = \left\langle \widetilde{J_{e_l}}, J_{e_l} \right\rangle$, $l \in N$, of CPFNs.
If it satisfy that $A_l \subseteq e_l$ for whole $l \in N$, then

$$AOps\left(A_1, A_2, ..., A_n\right) = AOps\left(e_1, e_2, ..., e_n\right).$$

4. Decision Support Methodology

In this section, method for solving the MADM problems by utilizing the Cubic picture fuzzy aggregation operators is established. Suppose that any finite collection $C = \{c_1, c_2, ..., c_m\}$ of m alternatives and any finite collection $G = \{g_1, g_2, ..., g_n\}$ of n attributes. Let $A = [A_{jk}] = \left[\left\langle \widetilde{J_{a_{jk}}}, J_{a_{jk}} \right\rangle\right]_{m \times n}$

be the decision matrics. (DMs), where $\left\langle \widetilde{J_{a_{jk}}}, J_{a_{jk}} \right\rangle$ are the collection of CPFNs. If weight vector $\tau = \{\tau_1, \tau_2, ..., \tau_n\}$ of attribute, with $\tau_l \geq 0$ and $\sum_{l=1}^{n} \tau_l = 1$. Then, technique of handling the MADM problems are listed below:

Step-1: Normalized the given Decision Matrix. In extensively, there are attributes which have two types; (1) benefit criteria (2) worse criteria. If given criteria is worse type then we use given below equation to modify the worse criteria into benefit criteria,

$$A^c = \left\langle \widetilde{J^c}_a, J^c_a \right\rangle = \left\{r, \left\langle \widetilde{N_a}(r), \widetilde{I_a}(r), \widetilde{P_a}(r) \right\rangle, \left\langle N_a(r), I_a(r), P_a(r) \right\rangle | r \in R \right\}.$$

Where A^c is the complement of A. If given criteria is benefit type then no need to be normalized.

Step-2: We use weighted averaging operator to aggregate the CPFNs and calculated separately for each alternative $a_i(i = 1, 2, ..., n)$.

Step-3: In this step we use Definition 3.1 to calculate the score and accuracy functions.

Step-4: On the basis of score and accuracy functions, we gives the ranks to alternatives $a_i(i = 1, 2, ..., m)$ according to Definition 3.1, to choice the best one option.

4.1. A Descriptive Example. In this section, the proposed ranking method is applied to deal with circulation center evaluation problem. Consider a committee of decision-maker to perform the evaluation and to select the most suitable circulation center, among the three circulation centers G_1, G_2 and G_3. The decision maker evaluates the circulation center according to four attributes, which are given as follows: (1) Cost of transportation (C_1), (2) the load of capacity (C_2), (3) the satisfy demand with minimum delay (C_3), and (4) the security (C_4). The importance and the average weights of the attributes from the decision making committee that are $\tau = (0.224, 0.236, 0.304, 0.236)^T$. Now we enlist the suitability ratings of alternatives versus the four attribute are presented. According to the suitability ratings of three alternatives, G_1, G_2 and G_3 under four attributes C_1, C_2, C_3 and C_4 can be obtained as shown in below table.

petroleum circulation center information		
(C_1)		
G_1	$\{\langle [0.32, 0.43], [0.09, 0.15], [0.22, 0.31] \rangle, \langle 0.44, 0.13, 0.36 \rangle\}$	
G_2	$\{\langle [0.10, 0.18], [0.13, 0.23], [0.29, 0.32] \rangle, \langle 0.50, 0.08, 0.36 \rangle\}$	
G_3	$\{\langle [0.22, 0.40], [0.12, 0.10], [0.28, 0.39] \rangle, \langle 0.40, 0.23, 0.30 \rangle\}$	
(C_2)		
G_1	$\{\langle [0.20, 0.30], [0.05, 0.16], [0.39, 0.41] \rangle, \langle 0.35, 0.08, 0.40 \rangle\}$	
G_2	$\{\langle [0.11, 0.18], [0.29, 0.33], [0.15, 0.20] \rangle, \langle 0.13, 0.29, 0.24 \rangle\}$	
G_3	$\{\langle [0.33, 0.39], [0.09, 0.18], [0.27, 0.29] \rangle, \langle 0.26, 0.27, 0.28 \rangle\}$	

petroleum circulation center information

	(C_3)
G_1	$\{\langle[0.20,0.30],[0.15,0.26],[0.39,0.41]\rangle,\langle0.41,0.08,0.40\rangle\}$
G_2	$\{\langle[0.14,0.28],[0.29,0.39],[0.18,0.28]\rangle,\langle0.13,0.43,0.24\rangle\}$
G_3	$\{\langle[0.41,0.43],[0.09,0.18],[0.27,0.33]\rangle,\langle0.26,0.27,0.28\rangle\}$

	(C_4)
G_1	$\{\langle[0.39,0.41],[0.15,0.20],[0.20,0.30]\rangle,\langle0.40,0.08,0.41\rangle\}$
G_2	$\{\langle[0.18,0.28],[0.29,0.39],[0.14,0.28]\rangle,\langle0.24,0.43,0.13\rangle\}$
G_3	$\{\langle[0.27,0.33],[0.09,0.18],[0.41,0.43]\rangle,\langle0.28,0.27,0.26\rangle\}$

Step-1: The above information contain the uniformity so we do no need to normalized.

Step-2: Operate the aggregation operator to evaluate DMs of cubic picture fuzzy information.

Aggregated Information (Utilizing CPFWA)

G_1	$\{\langle[0.276,0.357],[0.103,0.0.192],[0.293,0.357]\rangle,\langle0.401,0.089,0.392\rangle\}$
G_2	$\{\langle[0.133,0.235],[0.242,0.333],[0.180,0.266]\rangle,\langle0.255,0.268,0.227\rangle\}$
G_3	$\{\langle[0.319,0.391],[0.095,0.157],[0.300,0.353]\rangle,\langle0.298,0.260,0.279\rangle\}$

Aggregated Information (Utilizing CPFWA*)

G_1	$\{\langle[0.276,0.357],[0.103,0.192],[0.293,0.357]\rangle,\langle0.398,0.089,0.393\rangle\}$
G_2	$\{\langle[0.133,0.235],[0.242,0.333],[0.180,0.266]\rangle,\langle0.203,0.268,0.244\rangle\}$
G_3	$\{\langle[0.319,0.391],[0.095,0.157],[0.300,0.353]\rangle,\langle0.291,0.260,0.279\rangle\}$

Aggregated Information (Utilizing CPFOWA)

G_1	$\{\langle[0.367,0.416],[0.101,0.177],[0.240,0.308]\rangle,\langle0.338,0.172,0.324\rangle\}$
G_2	$\{\langle[0.221,0.330],[0.097,0.171],[0.366,0.410]\rangle,\langle0.364,0.135,0.338\rangle\}$
G_3	$\{\langle[0.133,0.235],[0.242,0.333],[0.180,0.266]\rangle,\langle0.255,0.268,0.227\rangle\}$

Aggregated Information (Utilizing CPFOWA*)

G_1	$\{\langle[0.367,0.416],[0.101,0.177],[0.240,0.308]\rangle,\langle0.323,0.172,0.330\rangle\}$
G_2	$\{\langle[0.221,0.330],[0.097,0.171],[0.366,0.410]\rangle,\langle0.358,0.135,0.347\rangle\}$
G_3	$\{\langle[0.133,0.235],[0.242,0.333],[0.180,0.266]\rangle,\langle0.203,0.268,0.244\rangle\}$

Step-3: By using Definition 3.1 to calculate score of the give attributes as,

Operators	$sc(G_1)$	$sc(G_2)$	$sc(G_3)$
CPFWA	0.268	0.184	0.260
CPFWA*	0.267	0.173	0.259
CPFOWA	0.299	0.233	0.184
CPFOWA*	0.296	0.232	0.173

Step-4: By using comparison approach define in Definition 3.1 for ranking the attributes, tabular representation is given below,

Calculation result of each petroleum circulation center		
	Score	Ranking
CPFWA	$sco(G_1) > sco(G_3) > sco(G_2)$	$G_1 > G_3 > G_2$
CPFWA*	$sco(G_1) > sco(G_3) > sco(G_2)$	$G_1 > G_3 > G_2$
CPFOWA	$sco(G_1) > sco(G_2) > sco(G_3)$	$G_1 > G_2 > G_3$
CPFOWA*	$sco(G_1) > sco(G_2) > sco(G_3)$	$G_1 > G_2 > G_3$
CPFHWA	$sco(G_1) > sco(G_2) > sco(G_3)$	$G_1 > G_2 > G_3$
CPFHWA*	$sco(G_1) > sco(G_2) > sco(G_3)$	$G_1 > G_2 > G_3$

Hence G_1 is our best choice.

5. Conclusion

Aggregation of information is an important technique to tackles an MADM problems. Many aggregation operators for different sets have been introduced. B. C. Cuong's construction of picture fuzzy sets is of prodigious reputation but decision makers are somehow restricting in assigning values due to the condition on $P(x)$ $I(x)$ and $N(x)$. Dealing with such kind of circumstances, we propose a new structure by defining cubic picture fuzzy sets, a generalization of fuzzy sets whose combination of IVPFSs and PFSs. We have investigated some basic operations and properties, and proposed an extension principle of CPFS. Since in decision making problems aggregation operators play a vital role, therefore in this paper we introduced some aggregation operators, namely the Cubic picture fuzzy weighted average (CPFWA) operator, Cubic picture

fuzzy weighted average* (CPFWA*) operator, Cubic picture fuzzy order weighted average (CPFOWA) operator, Cubic picture fuzzy order weighted average* (CPFOWA*) operator, Cubic picture fuzzy hybrid weighted average (CPFHWA) operator and Cubic picture fuzzy hybrid weighted average* (CPFHWA*) operator. According to the importance of evaluating options and ranking fuzzy quantities in the design of algorithms that are used for solving fuzzy linear optimization problems such as simplex-based algorithms, an approach for ranking is proposed in this paper; to illustrate the usage, applicability, and advantages of the proposed ranking approach, it has been applied for evaluating the circulation center of petroleum as an applicable problem. We have outlined a practical example about of an arrangement of drive frameworks to check the created approach and to show its common uses and adaptability as compare to conventional methods.

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