Korean J. Math. **28** (2020), No. 2, pp. 295–309
http://dx.doi.org/10.11568/kjm.2020.28.2.295

# ON THE GROWTH OF SOLUTIONS OF SOME NON-LINEAR COMPLEX DIFFERENTIAL EQUATIONS

Dilip Chandra Pramanik, Jayanta Roy, and Kapil Roy†

Abstract. In this paper we study the growth of solutions of some non-linear complex differential equations in connection to Brück conjecture using the theory of complex differential equation.

## 1. Introduction and main results

In this paper, by meromorphic function we shall always mean a meromorphic function in the complex plane. We adopt the standard notations in the Nevanlinna Theory of meromorphic functions as explained in [4, 6, 10, 11]. It will be convenient to let $E$ denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence.

For any non-constant meromorphic function $f(z)$, we denote by $S(r, f)$ any quantity satisfying $S(r, f) = \circ(T(r, f))$ as $r \to \infty, r \notin E$, where $T(r, f)$ is the Nevanlinna characteristic function of $f$. A meromorphic function $a(z)$ is said to be small with respect to $f(z)$ if $T(r, a) = S(r, f)$. We denote by $S(f)$ the collection of all small functions with respect to $f$. Clearly $\mathbb{C} \cup \{\infty\} \in S(f)$ and $S(f)$ is a field over the set of complex

Received January 3, 2019. Revised May 22, 2020. Accepted May 25, 2020.
2010 Mathematics Subject Classification: 30D35, 30D30.
Key words and phrases: Entire function, Brück conjecture, Small function, Differential polynomial.
† This work was supported by the Council of Scientific and Industrial Research, ExtraMural Research Division, CSIR Complex, Library Avenue, Pusa, New Delhi-110012, India, Under the sanctioned file no. 09/285(0069)/2016-EMR-I..

numbers.

For any two non-constant meromorphic functions $f$ and $g$, and $a \in S(f) \cap S(g)$, we say that $f$ and $g$ share $a$ IM(CM) provided that $f - a$ and $g - a$ have the same zeros ignoring(counting) multiplicities.

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function. We define by $\mu(r, f) = max\{|a_n|r^n : n = 0, 1, 2, ...\}$ the maximum term of $f$ and by $\nu(r, f) = max\{m : \mu(r, f) = |a_m|r^m\}$ the central index of $f$. In this paper we also need the following definition:

DEFINITION 1.1. Let $f(z)$ be a non-constant entire function. Then the order $\sigma(f)$, the lower order $\mu(f)$ and the hyper-order $\sigma_2(f)$ of $f(z)$ are defined as follows:

$$\sigma(f) = \limsup_{r\to+\infty} \frac{\log T(r,f)}{\log r} = \limsup_{r\to+\infty} \frac{\log\log M(r,f)}{\log r}$$

$$\mu(f) = \liminf_{r\to+\infty} \frac{\log T(r,f)}{\log r} = \liminf_{r\to+\infty} \frac{\log\log M(r,f)}{\log r}$$

$$\sigma_2(f) = \limsup_{r\to+\infty} \frac{\log\log T(r,f)}{\log r} = \limsup_{r\to+\infty} \frac{\log\log\log M(r,f)}{\log r},$$

where and in the sequel

$$M(r,f) = \max_{|z|=r} |f(z)|.$$

In 1976, Rubel and Yang [9] proved that if a non-constant entire function $f$ and its derivative $f'$ share two distinct finite complex numbers CM, then $f \equiv f'$. What will be the relation between $f$ and $f'$, if an entire function $f$ and its derivative $f'$ share one finite complex number CM ?

In 1996 Brück [1] made the following conjecture:

CONJECTURE 1.1. *Let $f$ be a non-constant entire function satisfying $\sigma_2(f) < \infty$, where $\sigma_2(f)$is not a positive integer. If $f$ and $f'$ share one finite complex number $a$ CM, then*

$$\frac{f' - a}{f - a} = c,$$

*for some finite complex number $c \neq 0$.*

In the same paper, Brück showed that the conjecture is true when $a = 0$. He also proved that the conjecture is true for $a \neq 0$ provided that $f$ satisfies the additional assumption $N(r, 0; f') = S(r, f)$ and in this case the order restriction on $f$ can be omitted.

Gundersen and Yang [3] proved that the conjecture is true for functions of finite order.

THEOREM 1.1. *Let $f$ be a non-constant entire function of finite order. If $f$ and $f'$ share one finite complex number $a$ CM, then*

$$\frac{f' - a}{f - a} = c,$$

*for some finite complex number $c \neq 0$.*

In 2009, Chang and Zhu [2] proved that Theorem 1.1 remains valid when the complex number $a$ is replaced by a function.

THEOREM 1.2. *Let $f$ be a non-constant entire function of finite order and $a = a(z)(\not\equiv 0)$ be an entire function such that $\sigma(a) < \sigma(f) < \infty$. If $f$ and $f'$ share $a$ CM, then*

$$\frac{f' - a}{f - a} = c,$$

*for some finite complex number $c \neq 0$.*

In 2016, Li and Yi [8] investigated the Brück conjecture and proved that Theorem 1.2 remains true when $f'$ is replaced by a linear differential polynomial of $f$, namely $L(f) = f^{(k)} + a_{k-1} f^{(k-1)} + ... + a_1 f' + a_0 f$, where $k$ is a positive integer and $a_{k-1}, ..., a_0$ are complex constants. They proved the following result:

THEOREM 1.3. *Let $f$ be a non-constant entire function such that $\sigma(f) < \infty$, and let $a(\not\equiv 0)$ be an entire function such that $\sigma(a) < \sigma(f)$. If $f - a$ and $L[f] - a$ share 0 CM, where $L[f]$ is defined as above, then $\sigma(f) = 1$ and one of the following two cases will occur:*

*(i) $L[f] - a = c(f - a)$, where $c$ is some non-zero constant.*

*(ii) $f$ is a solution of the equation $L[f] - a = (f - a)e^{p_1 z + p_0}$ such that $\sigma(f) = \mu(f) = 1$, where not all $a_0, a_1, ..., a_{k-1}$ are zeros, $p_1 \neq 0$ and $p_0$ are complex numbers.*

QUESTION 1.1. It is an interesting question to investigate that what will happen if we replace the linear differential polynomial by a non-linear differential polynomial in Theorem 1.3.

In this connection we need the following definition:

Let $n_{0j}, n_{1j}, n_{2j}, ..., n_{kj}$ are non-negative integers. The expression

$$M_j\left[f\right] = f^{n_{0j}} \left(f^{(1)}\right)^{n_{1j}} \left(f^{(2)}\right)^{n_{2j}} ... \left(f^{(k)}\right)^{n_{kj}},$$

is called a differential monomial generated by $f$ of degree $d(M_j) = \sum\limits_{i=0}^{k} n_{ij}$ and weight $\Gamma_{M_j} = \sum\limits_{i=0}^{k}(i+1)n_{ij}$. The sum

$$P\left[f\right] = \sum_{j=1}^{l} a_j M_j\left[f\right],$$

is called a differential polynomial generated by $f$ of degree $\overline{d}(P) = \max\left\{d(M_j) : 1 \leq j \leq l\right\}$ and weight $\Gamma_P = \max\left\{\Gamma_{M_j} : 1 \leq j \leq l\right\}$, where $a_j$ is complex constant for $j = 1, 2, ..., l$. The numbers $\underline{d}_P = \min\left\{d(M_j) : 1 \leq j \leq l\right\}$ and $k$ (the highest order of the derivative of $f$ in $P\left[f\right]$) are called respectively the lower degree and the order of $P\left[f\right]$. $P\left[f\right]$ is said to be homogeneous differential polynomial of degree $d$ if $\overline{d}_P = \underline{d}_P = d$. $P\left[f\right]$ is called a linear differential polynomial generated by $f$ if $\overline{d}_P = 1$. Otherwise, $P\left[f\right]$ is called non-linear differential polynomial. We denote by $Q_j = \Gamma_{M_j} - d(M_j) = \sum_{i=1}^{k} i.n_{ij}$ for $1 \leq j \leq l$.

In this paper we prove the following theorems which improve and generalizes Theorems 1.1, 1.2 and 1.3.

THEOREM 1.4. *Let $f$ be a non-constant entire function with $\sigma(f) < \infty$ and let $a(\not\equiv 0)$ be entire function such that $\sigma(a) < \sigma(f)$. If $f^d(z) - a(z)$ and $P[f] - a(z)$ share 0 CM, where $P[f] = M[f] + \sum\limits_{j=1}^{l} a_j M_j\left[f\right]$ is a differential polynomial of $f$ of degree $d$, and $M[f]$ is a differential monomial of $f$ of degree $d$. Then $\sigma(f) = 1$ and one of the following two cases will occur:*

*(i) $f$ is a solution of the equation $P[f] - a(z) = c(f^d - a(z))$, where $c$ is some non-zero constant.*

*(ii) $f$ is a solution of the equation $P[f] - a(z) = (f^d - a(z))e^{p_1 z + p_0}$ such that $\sigma(f) = \mu(f) = 1$, where not all $a_1$, $a_2$,..., $a_l$ are zeros, $p_1 \neq 0$ and $p_0$ are complex numbers.*

Proceeding as in the proof of Theorem 1.4 of this paper, we can prove the following theorem.

THEOREM 1.5. *Let $f$ be a non-constant entire function such that $\sigma(f) < \infty$ and let $a(\not\equiv 0)$ and $\beta$ be entire functions such that $\max\{\sigma(a), \sigma(\beta)\} < \sigma(f)$. If $f^d(z) - a(z)$ and $P[f] + \beta(z) - a(z)$ share 0 $CM$, where $P[f] = M[f] + \sum_{j=1}^{l} a_j M_j[f]$ is a differential polynomial of $f$ of degree $d$, and $M[f]$ is a differential monomial of $f$ of degree $d$. Then $\sigma(f) = 1$ and one of the following two cases will occur:*

*(i) $f$ is a solution of the equation $P[f]+\beta(z)-a(z) = c(f^d(z)-a(z))$, where $c$ is some non-zero constant.*

*(ii) $f$ is a solution of the equation $P[f] + \beta(z) - a(z) = (f^d(z) - a(z))e^{p_1 z + p_0}$ such that $\sigma(f) = \mu(f) = 1$, where not all $a_1$, $a_2$,..., $a_l$ are zeros, $p_1 \neq 0$ and $p_0$ are complex numbers.*

From Theorem 1.4 we get the following corollary:

COROLLARY 1.1. *Let $f$ be a non-constant entire function such that $\sigma(f) < \infty$ and let $a(\not\equiv 0)$ be entire function such that $\sigma(a) < \sigma(f)$. If $f^d(z) - a(z)$ and $M[f] - a(z)$ share 0 $CM$, where $M[f]$ is a differential monomial of $f$ of degree $d$. Then $\sigma(f) = 1$ and $f$ is a solution of the equation $M[f] - a(z) = c(f^d - a(z))$, where $c$ is some non-zero constant.*

From Theorem 1.5 we get the following corollary:

COROLLARY 1.2. *Let $f$ be a non-constant entire function such that $\sigma(f) < \infty$ and let $a(\not\equiv 0)$ and $\beta$ be entire functions such that $\max\{\sigma(a), \sigma(\beta)\} < \sigma(f)$. If $f^d(z) - a(z)$ and $M[f] + \beta(z) - a(z)$ share 0 $CM$, where $M[f]$ is a differential monomial of $f$ of degree $d$. Then $\sigma(f) = 1$ and $f$ is a solution of the equation $M[f] + \beta(z) - a(z) = c(f^d(z) - a(z))$, where $c$*

*is some non-zero constant.*

The following is the supportive example of (i) of Theorem 1.4.

EXAMPLE 1.1. Let $f(z) = 1 - e^z$ and $P[f] = f'f + f$. Then $\sigma(f) = 1$ and $P[f] - a(z) = c(f^2(z) - a(z))$, where $c = 1$ and $a(z) = z + 1$.

The following is the supportive example of (ii) of Theorem 1.4.

EXAMPLE 1.2. Let $f(z) = 1+e^z$ and $P[f] = f^2-(f'')^2-f'+2$. Then $P[f] - 1$ and $f^2 - 1$ share 0 CM, $\sigma(f) = 1$ and $P[f] - 1 = (f^2 - 1)e^{-z}$.

EXAMPLE 1.3. Let $f(z) = a(z) = e^z$ and $P[f] = f'^2 - f^2 + 2f - 1$. Then $f^2-a$ and $P[f]-a$ share 0 CM and $\sigma(f) = \sigma(a) = 1$ but $P[f]-a = e^{-z}(f^2 - a)$. This example shows that the condition "$\sigma(a) < \sigma(f)$" in (i) of Theorem 1.4 is the best possible.

THEOREM 1.6. *In Theorem 1.4 if we replace the condition "$\sigma(a) < \sigma(f)$" by "$\sigma(a) < \mu(f)$" and all other conditions remains the same, then also the conclusion of the theorem is true.*

THEOREM 1.7. *In Theorem 1.5 if we replace the condition "$\max\{\sigma(a), \sigma(\beta)\} < \sigma(f)$" by "$\max\{\sigma(a), \sigma(\beta)\} < \mu(f)$" and all other conditions remains the same, then also the conclusion of the theorem is true.*

From Theorem 1.6 we get the following corollary:

COROLLARY 1.3. *Let $f$ be a non-constant entire function such that $\sigma(f) < \infty$ and let $a \not\equiv 0$ be entire function such that $\sigma(a) < \mu(f)$. If $f^d(z) - a(z)$ and $M[f] - a(z)$ share 0 $CM$, where $M[f]$ is a differential monomial of $f$ of degree $d$. Then $\sigma(f) = 1$ and $f$ is a solution of the equation $M[f] - a(z) = c(f^d - a(z))$, where $c$ is some non-zero constant.*

From Theorem 1.7 we get the following corollary:

COROLLARY 1.4. *Let $f$ be a non-constant entire function such that $\sigma(f) < \infty$ and let $a \not\equiv 0$ and $\beta$ be entire functions such that $\max\{\sigma(a), \sigma(\beta)\} < \mu(f)$. If $f^d(z) - a(z)$ and $M[f] + \beta(z) - a(z)$ share 0 $CM$, where $M[f]$ is a differential monomial of $f$ of degree $d$. Then $\sigma(f) = 1$ and $f$ is a solution of the equation $M[f] + \beta(z) - a(z) = c(f^d(z) - a(z))$, where $c$ is some non-zero constant.*

## 2. Preparatory Lemmas

In this section we state some lemmas needed in the sequel.

LEMMA 2.1. *[6] Let $f(z)$ be a transcendental entire function, $\nu(r,f)$ be the central index of $f(z)$. Then there exists a set $E \subset (1,+\infty)$ with finite logarithmic measure such that for some point $z$ satisfying $|z| = r \notin [0,1] \cup E$ and $|f(z)| = M(r,f)$, we get*

$$\frac{f^{(j)}(z)}{f(z)} = \left\{ \frac{\nu(r,f)}{z} \right\}^j (1 + o(1)), \text{ for } j \in N.$$

LEMMA 2.2. *[5] Let $f(z)$ be an entire function of finite order $\sigma(f) = \sigma < +\infty$ and let $\nu(r,f)$ be the central index of $f$. Then*

$$\sigma(f) = \limsup_{r\to+\infty} \frac{\log \nu(r,f)}{\log r}$$

*and*

$$\mu(f) = \liminf_{r\to+\infty} \frac{\log \nu(r,f)}{\log r}.$$

*And if $f$ is a transcendental entire function of hyper order $\sigma_2(f)$, then*

$$\limsup_{r\to+\infty} \frac{\log\log \nu(r,f)}{\log r} = \sigma_2(f)$$

LEMMA 2.3. *[7] Let $f(z)$ be a transcendental entire function and let $E \subset [1,+\infty)$ be a set having finite logarithmic measure. Then there exists $\left\{ z_n = r_n e^{i\theta_n} \right\}$ such that $|f(z_n)| = M(r_n,f), \theta_n \in [0,2\pi)$, $\lim_{n\to+\infty} \theta_n = \theta_0 \in [0,2\pi)$, $r_n \notin E$ and if $0 < \sigma(f) < +\infty$, then for any given $\varepsilon > 0$ and sufficiently large $r_n$,*

$$r_n^{\sigma(f)-\varepsilon} < \nu(r_n,f) < r_n^{\sigma(f)+\varepsilon}.$$

LEMMA 2.4. *( [6], Corollary 2.3.4) Let $f$ be a transcendental meromorphic function and $k$ be a positive integer. Then $m(r, f^{(k)}/f) = S(r,f)$, outside of a possible exceptional set $E$ of finite linear measure, and if $f$ is of finite order of growth, then $m(r, f^{(k)}/f) = O(\log r)$.*

LEMMA 2.5. *[8] Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function, let $\mu(r,f)$ be the maximum term of $f$, and let $\nu(r,f)$ be the central index.*

*Then for* $0 < r < R$ *we have*

$$M(r,f) < \mu(r,f)\left\{\nu(R,f) + \frac{R}{R-r}\right\}.$$

LEMMA 2.6. *( [6], Lemma 1.1.2) Let* $g : (0,+\infty) \to R,\ h : (0,+\infty) \to R$ *be monotone increasing functions such that* $g(r) \le h(r)$ *outside of an exceptional set* $F$ *of finite logarithmic measure. Then for any* $\alpha > 1$, *there exists* $r_0 > 0$ *such that* $g(r) \le h(r^\alpha)$ *for all* $r > r_0$.

## 3. Proof of Main Theorems

In this section we present the proof of the main result of the paper.

Proof of Theorem 1.4:

Since $f^d - a$ and $P[f] - a$ share 0 $CM$, we get

$$\frac{P[f]-a}{f^d-a} = e^\phi, \tag{3.1}$$

where $\phi$ is an entire function. Again from $\sigma(a) < \sigma(f)$, we have $\sigma(f) > 0$, which implies that $f$ is a transcendental entire function.

Now, we consider the following two cases:

**Case I:**

$$\liminf_{r\to\infty} \frac{\log \nu(r,f)}{\log r} > 1. \tag{3.2}$$

Then from (3.2) and Lemma 2.2, we get

$$\mu(f) = \liminf_{r\to\infty} \frac{\log \nu(r,f)}{\log r} > 1. \tag{3.3}$$

Since $f$ is a transcendental entire function, we have

$$M(r,f) \to \infty \ \ as \ \ r \to \infty. \tag{3.4}$$

Again since $f$ is a transcendental entire function, by Lemma 2.1 there exist subset $F_j \subset (1,\infty)$ $(1 \le j \le n)$ with finite logarithmic measure such that for some point $z_r = re^{i\theta(r)}$, $(\theta(r) \in [0, 2\pi))$ satisfying $|z_r| = r \notin F_j$ and $M(r, f) = |f(z_r)|$, we have

(3.5)
$$\frac{f^{(j)}(z_r)}{f(z_r)} = \left(\frac{\nu(r,f)}{z_r}\right)^j \{1+o(1)\}\ (1 \le j \le n),\ \text{as } r \notin \cup_{j=1}^n F_j \text{ and } r \to \infty.$$

By Definition 1.1, Lemma 2.6, Definition 1.1.1 and Theorem 1.1.3 from [12] and the assumption $\sigma(a) < \sigma(f)$, there exists an infinite sequence of points $z_{r_n} = r_n e^{i\theta(r_n)}$ satisfying $M(r_n, f) = |f(z_{r_n})|$, where $r_n \in I \setminus \cup_{j=1}^n F_j, I \subseteq R^+$ is a subset with logarithmic measure $\int_I \frac{dt}{t} = \infty$ such that

$$\lim_{r_n \to \infty} \frac{\log \log M(r_n, f)}{\log r_n} = \sigma(f) \tag{3.6}$$

and

$$\lim_{r_n \to \infty} \frac{M(r_n, a)}{M(r_n, f)} = 0. \tag{3.7}$$

Since

$$\frac{P[f] - a}{f^d - a} = \frac{\frac{P[f]}{f^d} - \frac{a}{f^d}}{1 - \frac{a}{f^d}}, \tag{3.8}$$

using (3.2),(3.4)-(3.7) in (3.8) we get

$$\frac{P[f](z_{r_n}) - a(z_{r_n})}{f^d(z_{r_n}) - a(z_{r_n})} = R\left(\frac{\nu(r_n, f)}{z_{r_n}}\right)^Q \{1 + o(1)\},\ \text{as } r_n \to \infty, \tag{3.9}$$

where $Q = max\{\Gamma_M - d(M) : M \text{ is a monomial in } P[f]\}$ and $R$ is a complex number.

From (3.9), we have

(3.10)
$$\log \left| \frac{P[f](z_{r_n}) - a(z_{r_n})}{f^d(z_{r_n}) - a(z_{r_n})} \right| = Q\{\log \nu(r_n, f) - \log r_n\} + o(1),\ \text{as } r_n \to \infty.$$

From (3.1), Lemma 2.4 and the condition $\sigma(a) < \sigma(f) < \infty$, we get

$$\begin{aligned} T(r, e^{\phi}) &\leq 2T(r, f) + O(\log r) \\ \Rightarrow \log T(r, e^{\phi}) &\leq \log T(r, f) + O(\log\log r) \\ \Rightarrow \frac{\log T(r, e^{\phi})}{\log r} &\leq \frac{\log T(r, f)}{\log r} + O(1) \\ \Rightarrow \sigma(e^{\phi}) &\leq \sigma(f) < \infty \text{ as } r \to \infty, \end{aligned} \tag{3.11}$$

which implies that $\phi$ is a polynomial.

Let

$$\phi = p_m z^m + p_{m-1} z^{m-1} + ... + p_1 z + p_0, \tag{3.12}$$

where $p_0, p_1, ..., p_{m-1}, p_m$ are complex constants with $p_m \neq 0$.

It follows from (3.12) that $\lim_{|z|\to\infty} |\phi(z)/p_m z^m| = 1$ and $|\phi(z)/p_m z^m| > \frac{1}{e}$ as $|z| > r_0$, when $r_0$ is a sufficiently large positive number. From this and (3.1), we get

$$m \log|z| + \log|p_m| - 1 \leq \log|\phi(z)| \leq |\log\log e^{\phi}| = \left|\log\log \frac{P[f] - a}{f^d - a}\right| \text{ as } |z| \to \infty. \tag{3.13}$$

From (3.9), (3.13), Lemma 2.2 and the condition $\sigma(f) < \infty$, we get

$$\begin{aligned} & m \log|z_{r_n}| + \log|p_m| - 1 \\ \leq & \left|\log\log \frac{P[f](z_{r_n}) - a(z_{r_n})}{f^d(z_{r_n}) - a(z_{r_n})}\right| \\ = & \left|\log\left|\log \frac{P[f](z_{r_n}) - a(z_{r_n})}{f^d(z_{r_n}) - a(z_{r_n})}\right|\right| + i\, arg\left(\log \frac{P[f](z_{r_n}) - a(z_{r_n})}{f^d(z_{r_n}) - a(z_{r_n})}\right) \\ \leq & \left|\log\left|\log \frac{P[f](z_{r_n}) - a(z_{r_n})}{f^d(z_{r_n}) - a(z_{r_n})}\right|\right| + 2\pi \\ \leq & \log\log \nu(r_n, f) + \log\log r_n + O(1) \\ \leq & 2\log\log r_n + O(1), \text{ as } r_n \to \infty \end{aligned}$$

$$\Rightarrow m \log|z_{r_n}| + \log|p_m| - 1 \leq 2\log\log r_n + O(1), \text{ as } r_n \to \infty \tag{3.14}$$

which is impossible. Thus $\phi$ is a constant and so (3.9) becomes

$$\left(\frac{\nu(r_n, f)}{z_{r_n}}\right)^Q \{1 + o(1)\} = c \text{ as } r_n \to \infty, \tag{3.15}$$

where $c$ is some non-zero constant.

From (3.15), we get

$$\lim_{r_n \to \infty} \frac{\log \nu(r_n, f)}{\log r_n} = 1. \tag{3.16}$$

By Lemma 2.5, we know that

$$M(r_n, f) < \mu(r_n)\{\nu(2r_n, f) + 2\} = |a_{\nu(r_n,f)}| r_n^{\nu(r_n,f)}\{\nu(2r_n, f) + 2\}. \tag{3.17}$$

Since $|a_j| < M_1$, for all non-negative integer $j$ and some constant $M_1 > 0$, we get from (3.17) that

$$\log\log M(r_n, f) \leq \log \nu(r_n, f) + \log\log \nu(2r_n, f) + \log\log r_n + C_1, \tag{3.18}$$

where $C_1 > 0$ is a suitable constant.

From Lemma 2.2 and the condition $\sigma(f) < \infty$, we get

$$\log \nu(2r_n, f) < \{1 + o(1)\}(\log r_n + \log 2) \text{ as } r \to \infty. \tag{3.19}$$

From (3.16), (3.18) and (3.19) we get

$$\begin{aligned} \log\log M(r_n, f) &\leq \log \nu(r_n, f) + 2\log\log r_n + o(1) \\ &\leq \log \nu(r_n, f)\{1 + o(1)\}, \ as\ r_n \to \infty \end{aligned}$$

$$\Rightarrow \frac{\log\log M(r_n, f)}{\log r_n} \leq \frac{\log \nu(r_n, f)}{\log r_n}. \tag{3.20}$$

By (3.6), (3.16) and (3.20), we get

$$\sigma(f) \leq 1. \tag{3.21}$$

which is a contradiction by the fact $\mu(f) \leq \sigma(f)$ and (3.3).

**Case II:** Suppose that

$$\liminf_{r \to \infty} \frac{\log \nu(r, f)}{\log r} \leq 1. \tag{3.22}$$

Then from (3.21) and Lemma 2.2, we get

$$\mu(f) \leq 1. \tag{3.23}$$

We consider the following two subcases:

**Subcase I:** Suppose that

$$\sigma(f) > 1. \tag{3.24}$$

By (3.24), Definition 1.1, Lemma 2.6, Definition 1.1.1, Theorem 1.1.3 from [12] and the assumption $\sigma(a) < \sigma(f)$, there exists an infinite sequence of points $z_{r_n} = r_n e^{i\theta(r_n)}$ satisfying $M(r_n, f) = |f(z_{r_n})|$, where $r_n \in I \setminus \cup_{j=1}^{n} F_j,\ I \subseteq R^+$ is a subset with logarithmic measure $\int_I \frac{dt}{t} = \infty$, such that (3.6) and (3.7) hold. Next proceeding in the same manner an in Case I we get (3.21), which contradicts (3.24).

**Subcase II:** Suppose that

$$\sigma(f) \leq 1. \tag{3.25}$$

We will show that

$$\sigma(f) = 1. \tag{3.26}$$

Suppose that

$$\sigma(f) < 1. \tag{3.27}$$

Then from (3.27) and (3.11), we get $\sigma(e^{\phi}) \leq \sigma(f) < 1$, which implies that $\phi$ is a constant and so is $e^{\phi}$. Thus (3.1) becomes

$$\frac{P[f] - a}{f^d - a} = c, \tag{3.28}$$

where $c$ is some non-zero constant.

Re-writing (3.28), we get

$$\frac{M[f]}{f^d} + \sum_{j=1}^{l} a_j \frac{M_j[f]}{f^d} - \frac{a}{f}\frac{1}{f^{d-1}} \;\; = \;\; c\left(1 - \frac{a}{f}\frac{1}{f^{d-1}}\right). \tag{3.29}$$

By Lemma 2.6, Definition 1.1.1, Theorem 1.1.3 from [12] and the assumption $\sigma(a) < \sigma(f)$, there exists an infinite sequence of points $z_{r_n} = r_n e^{i\theta(r_n)}$ satisfying $M(r_n, f) = |f(z_{r_n})|$, where $r_n \in I \setminus \cup_{j=1}^{n} F_j,\ I \subseteq R^+$ is a subset with logarithmic measure $\int_I \frac{dt}{t} = \infty$, such that (3.6) and (3.7) hold and from 3.29 we have

(3.30)

$$\left(\frac{\nu(r_n, f)}{z_{r_n}}\right)^{\Gamma_M - d(M)} \{1+o(1)\} + \sum_{j=1}^{l} a_j \left(\frac{\nu(r_n, f)}{z_{r_n}}\right)^{Q_j} . \frac{1}{f(z_{r_n})^{d-d(M_j)}} \{1+o(1)\} = c$$

as $r_n \to \infty$.

From Lemma 2.3, we get

$$\nu(r_n, f) \leq r_n^{\sigma(f)+\varepsilon_0}, \tag{3.31}$$

as $r_n \geq R_0$, where $\varepsilon_0 = (1-\sigma(f))/2$ and $R_0$ is sufficiently large positive number.

From (3.27) and (3.31), we get

$$\lim_{r_n\to\infty} \left|\frac{\nu(r_n, f)}{z_{r_n}}\right|^{Q_j} \leq \lim_{r_n\to\infty} r_n^{(\frac{\sigma(f)-1}{2})Q_j} = 0 \text{ for } 1 \leq j \leq l \tag{3.32}$$

and

$$\lim_{r_n\to\infty} \left|\frac{\nu(r_n, f)}{z_{r_n}}\right|^{\Gamma_M - d(M)} \leq \lim_{r_n\to\infty} r_n^{(\frac{\sigma(f)-1}{2})(\Gamma_M - d(M))} = 0. \tag{3.33}$$

From (3.30), (3.32) and (3.33) we get $c = 0$, which is a contradiction. Therefore we get

$$\sigma(f) = 1. \tag{3.34}$$

From (3.11) and (3.34) we get $\sigma(e^{\phi}) \leq 1$ and it follows that $\phi$ is a polynomial of degree $\deg(\phi) \leq 1$. If $\phi$ is a constant, then from (3.1) we get the conclusion $(i)$ of Theorem 1.2.

Next suppose that $\phi$ is a polynomial degree $\deg(\phi) = 1$. Thus

$$\phi(z) = p_1 z + p_0,$$

where $p_1 \neq 0$ and $p_0$ are complex number.

First of all we prove that $\mu(f) = 1$.

From (3.34) it follows that $\mu(f) \leq 1$.

Let us suppose that $\mu(f) < 1$.

By Definition 1.1 there exists an infinite sequence of positive numbers $r_n$ such that

$$\lim_{r_n\to\infty} \frac{\log T(r_n, f)}{\log r_n} = \mu(f).$$

Again from (3.11), we get

$$\mu(e^{\phi}) \leq \lim_{r_n\to\infty} \frac{\log T(r_n, e^{\phi})}{\log r_n} \leq \lim_{r_n\to\infty} \frac{\log T(r_n, f)}{\log r_n} = \mu(f) < 1.$$

$$\Rightarrow \mu(e^{\phi}) < 1,$$

which is a contradiction. Therefore $\mu(f) = 1$.

Secondly, we will prove that not all $a_1, a_2, ..., a_l$ are zero. Suppose that $a_j = 0$ for $1 \leq j \leq l$, then we have

$$M[f] - a(z) = (f^d - a(z))e^{p_1 + p_0}. \tag{3.35}$$

From Definition 1.1, Lemma 2.6, Definition 1.1.1, Theorem 1.1.3 from [12] and the assumption $\sigma(a) < \sigma(f)$, there exists an infinite sequence of points $z_{r_n} = r_n e^{i\theta(r_n)}$ satisfying $M(r_n, f) = |f(z_{r_n})|$, where $r_n \in I \backslash \cup_{j=1}^{n} F_j$, $I \subseteq R^+$ is a subset with logarithmic measure $\int_I \frac{dt}{t} = \infty$, such that (3.6) and (3.7) holds.

From (3.6), (3.7) and (3.35), we get

$$\left(\frac{\nu(r_n, f)}{z_{r_n}}\right)^{\Gamma_M - d(M)} \{1 + o(1)\} = e^{p_1 z + p_0} \text{ as } r_n \to \infty. \tag{3.36}$$

From (3.36), we get

$$\begin{aligned} |p_1| r_n - |p_0| &= |p_1||z_{r_n}| - |p_0| \\ &\leq |p_1 z_{r_n} + p_0| \\ &\leq \left|\log e^{p_1 z_{r_n} + p_0}\right| + O(1) \\ &\leq (\Gamma_M - d(M))|\log \nu(r_n, f) - \log r_n| + O(1) \\ &\leq (\Gamma_M - d(M))\{\sigma(f) + 2\} \log r_n + O(1) \text{ as } r_n \to \infty, \end{aligned}$$

which is a contradiction, since $p_1 \not\equiv 0$. This completes the proof of (ii) of Theorem 1.4.

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**Dilip Chandra Pramanik**
Department of Mathematics, University of North Bengal
Darjeeling-734013, West Bengal, India
*E-mail*: dcpramanik.nbu2012@gmail.com

**Jayanta Roy**
Department of Mathematics, University of North Bengal
Darjeeling-734013, West Bengal, India
*E-mail*: jayantaroy983269@yahoo.com

**Kapil Roy**
Department of Mathematics, University of North Bengal
Darjeeling-734013, West Bengal, India
*E-mail*: roykapil692@gmail.com