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POWERS OF INTEGERS WITH ARITHMETIC TABLES

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Abstract. Any powers of 11 are easily obtained from the Pascal triangle. In this work we study powering and negative powering of any k digit integers by means of certain arithmetic tables.

1. Introduction

Pascal triangle is an arithmetic table(AT) of $(x+1)^m$ which gives connections among various areas in mathematics. One of the interesting features is a relationship with powers of 11 ([5], [6], [7] and [8]), indeed $11^9 = 2357947691$ is obtained immediately from the 9th row $(1, 9, 36, 84, 126, 126, 84, 36, 9, 1)$ of Pascal triangle. However it seems no one has asked about powering of any integer by certain AT.

A purpose of the work is to investigate powers and even negative exponent powers of any integers by utilizing AT of certain polynomials. We denote by $T^{(a_k,\cdots,a_0)}$ the AT of $(a_kx^k + \cdots + a_1x + a_0)^m$ expanded in descending order. We may refer [1], [3] and [4] for the AT $T^{(1,\cdots,1)}$. As an analog of powering of 11 through Pascal triangle $T^{(1,1)}$, we ask about powers of $k+1$ digit integer $a_k \cdots a_0$ via $T^{(a_k,\cdots,a_0)}$. To be precise, for a three digit integer $cba = c10^2 + b10 + a$ $(0 \le a, b, c \le 9,\ 0 \ne c)$, consider $(cx^2 + bx + a)^m$ and its AT $T^{(c,b,a)}$. We first express $T^{(c,b,a)}$ by means of $T^{(1,1)}$ and discuss powers of cba. This consideration is extended to powering integers having negative exponent $(cba)^{-t}$ $(t > 0)$. Finally for powers of any $(k+1)$ digit integers $a_k \cdots a_1a_0$, the AT of

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$(a_kx^k + \cdots + a_1x + a_0)^m$ and its relationships with the AT $T^{(1,1)}$ are investigated. A feature of the work is to provide an easy way to have, for example 357^4, 11112^5 and even $111^{-3} \approx (0.7311913813)10^{-6}$.

Throughout the work, $r_n^{(a_1,a_0)}$ and $e_{n,j}^{(a_1,a_0)}$ denote the nth row and the (n,j)th entry of AT $T^{(a_1,a_0)}$ of $(a_1x + a_0)^m$, respectively $(n, j \geq 0)$. We simply write $T^{(1,1)} = T$ the Pascal triangle, $r_n^{(1,1)} = r_n$ and $e_{n,j}^{(1,1)} = e_{n,j}$. Moreover $r_n^{(a_k,\cdots,a_0)}$ and $e_{n,j}^{(a_k,\cdots,a_0)}$ denote the corresponding ones in $T^{(a_k,\cdots,a_0)}$ of $(a_kx^k + \cdots + a_0)^m$. Let $[a^i]_{i\geq 0}$ be a diagonal matrix $\mathrm{diag}(1, a, a^2, \cdots)$ having entries a^i, and a multiplication of ordered tuples is defined by $(u_1, \cdots, u_k)(v_1, \cdots, v_k) = (u_1v_1, \cdots, u_kv_k)$.

2. Powering of any integers

For $2 \leq n \leq 4$, $11^n = 121$, 1331 and 14641 follow from nth row r_n of T. And $11^5 = 161051$ is due to $r_5 = (1, 5, 10, 10, 5, 1)$ by Table 1 or Table 2.

Table 1

	1					
		5				
		①	⓪			
			1	**0**		
					5	1
$S_{10}^{r_5}$	1	6	1	0	5	1

Table 2

	1	5	⓪	**0**	5	1
		①	**1**			
$S_{10}^{r_5}$	1	6	1	0	5	1

Table 3

	1	0	0								
			5	0	0						
				1	0	0	0				
						1	0	0	0		
									5	0	0
											1
$S_{10^2}^{r_5}$	1	0	5	1	0	1	0	0	5	0	1

The last row $S_{10}^{r_5} = 161051$ is the sum of entries of each columns, that equals 11^5. The both Table 1 and 2 will be called column sum tables of r_5 with base 10. As a generalization, powering of integers beginning and ending with 1 where all intermediate digits are 0s, such as 101, 1001, 10001, etc, are obtained by the column sum tables with base 10^k $(k > 0)$. In fact, $101^5 = 10510100501$ follows from the column sum table of r_5 with base 10^2 (Table 3).

In order to obtain powers of two digit integers $ba = b \cdot 10 + a$ $(0 \leq a, b \leq 9)$, we begin to consider AT $T^{(b,a)}$ of $(bx + a)^m$.

Theorem 1. *$T^{(b,a)} = T^b\ [a^i]_{i\geq 0} = T^{b-1}\ T^{(1,a)} = T^{b-s}\ T^{(s,a)}$ for any $0 \leq s \leq b$, and $r_n^{(b,a)} = r_n(b^{n-1}, \cdots, b, 1)(1, a, \cdots, a^{n-1})$.*

Proof. When $b = 1$, the AT of $(x + a)^m$ forms

$$T^{(1,a)} = \begin{bmatrix} 1 & & & \\ 1 & a & & \\ 1 & 2a & a^2 & \\ 1 & 3a & 3a^2 & a^3 \cdots \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 2 & 1 & \\ 1 & 3 & 3 & 1 \cdots \end{bmatrix} \begin{bmatrix} 1 & & & \\ & a & & \\ & & a^2 & \\ & & & a^3 \cdots \end{bmatrix} = T[a^i]_{i\geq 0}$$

so $r_n^{(1,a)} = r_n(1, \cdots, a^n)$ and $e_{n,j}^{(1,a)} = e_{n,j}a^j$. Let $q = \frac{a}{b}$ $(b \neq 0)$. Then

$$(bx+a)^n = b^n(x+q)^n = b^n r_n^{(1,q)} \begin{bmatrix} x^n \\ \cdots \\ 1 \end{bmatrix}$$

implies

$$\begin{aligned} r_n^{(b,a)} &= b^n r_n^{(1,q)} = b^n r_n(1, q, \cdots, q^n) \\ &= r_n(b^n, b^{n-1}a, \cdots, a^n) = r_n(b^n, \cdots, 1)(1, \cdots, a^n), \end{aligned}$$

hence we have

$$T^{(b,a)} = \begin{bmatrix} r_0^{(b,a)} \\ r_1^{(b,a)} \\ r_2^{(b,a)} \\ r_3^{(b,a)} \end{bmatrix} = \begin{bmatrix} r_0 \\ r_1(b,1)(1,a) \\ r_2(b^2,b,1)(1,a,a^2) \\ r_3(b^3,\cdots,1)(1,\cdots,a^3) \end{bmatrix} = \begin{bmatrix} 1 & & & \\ b & 1 & & \\ b^2 & 2b & 1 & \\ b^3 & 3b^2 & 3b & 1 \end{bmatrix} [a^i]_{i\geq 0}.$$

But since

$$\begin{bmatrix} 1 & & & \\ b & 1 & & \\ b^2 & 2b & 1 & \\ b^3 & 3b^2 & 3b & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ b-1 & 1 & & \\ (b-1)^2 & 2(b-1) & 1 & \\ (b-1)^3 & 3(b-1)^2 & 3(b-1) & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 2 & 1 & \\ 1 & 3 & 3 & 1 \end{bmatrix},$$

it is inductively equal to

$$\begin{bmatrix} 1 & & & \\ b-2 & 1 & & \\ (b-2)^2 & 2(b-2) & 1 & \\ (b-2)^3 & 3(b-2)^2 & 3(b-2) & 1 \end{bmatrix} T^2 = \cdots = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 2 & 1 & \\ 1 & 3 & 3 & 1 \end{bmatrix} T^{b-1} = T^b.$$

Thus we conclude that, for any $s \leq b$,

$$T^{(b,a)} = T^b\ [a^i]_{i\geq 0} = T^{b-1}\ T[a^i]_{i\geq 0} = T^{b-1}\ T^{(1,a)} = T^{b-s}\ T^{(s,a)}. \qquad \square$$

Clearly $T^{(b,a)} = T^{b-a}\ T^{(a,a)} = T^{b-a}[a^i]_{i\geq 0}T$ if $b \leq a$. So $T^{(2,2)} = \begin{bmatrix} 1 & & \\ 2 & 2 & \\ 4 & 8 & 4 \end{bmatrix} = [2^i]_{i\geq 0}T$ and $\begin{bmatrix} 1 \\ (1,1)(3,1)(1,7) \\ (1,2,1)(3^2,3,1)(1,7,7^2) \end{bmatrix} = \begin{bmatrix} 1 & & \\ 3 & 7 & \\ 9 & 42 & 49 \end{bmatrix} = T^{(3,7)}$.

Thus

$$r_5^{(3,7)} = r_5(3^5, \cdots, 1)(1, \cdots, 7^5) = (243, 2835, 13230, 30870, 36015, 16807),$$

which is the set of coefficients of $(3x+7)^5$. Moreover its column sum table with base 10 gives rise to the column sum $S_{10}^{r_5^{(3,7)}} = 69343957 = 37^5$ (Table 4).

Table 4

	3 5 0 0 5 7
	4 3 3 7 1 0
	2 8 2 8 0 8
	2 3 0 6 6
	1 3 3 1
$S_{10}^{r_5^{(3,7)}}$	6 9 3 4 3 9 5 7

Table 5

	2 4 3 0 0
	2 8 3 5 0 0
	1 3 2 3 0 0 0
	3 0 8 7 0 0 0
	3 6 0 1 5 0 0
	1 6 8 0 7
$S_{10^3}^{r_5^{(3,7)}}$	2 4 5 8 4 8 2 6 0 9 0 6 0 3 1 8 0 7

The next theorem is about powers of integers $10^k b + a$ ($k > 0$, $1 \leq a, b \leq 9$) filled with zeros except the first and last digits.

THEOREM 2. *Let $U = 10^k b + a$ ($k > 0$). Then any power U^n ($n > 0$) can be obtained by the column sum table of $r_n^{(b,a)}$ with base 10^k.*

Proof. Considering $U^n = (X + a)^n$ with $X = 10^k b$, we have

$$U^n = r_n^{(1,a)} \begin{bmatrix} b^n 10^{kn} \\ \cdots \\ b10^k \\ 1 \end{bmatrix} = r_n^{(1,a)} (b^n, \cdots, b, 1) \begin{bmatrix} 10^{kn} \\ \cdots \\ 10^k \\ 1 \end{bmatrix} = r_n^{(b,a)} \begin{bmatrix} 10^{kn} \\ \cdots \\ 10^k \\ 1 \end{bmatrix}$$

by Theorem 1, so the column sum table of $r_n^{(b,a)}$ with base 10^k yields U^n. □

For example, $3007^5 = 245848260906031807$ is obtained easily from the column sum table of $r_5^{(3,7)}$ with base 10^3 (Table 5).

For three digit integers, consider the AT $T^{(a_2,a_1,a_0)}$ of $(a_2x^2 + a_1x + a_0)^m$. Then its ith row $r_i^{(a_2,a_1,a_0)}$ is the set of coefficients of $(a_2x^2 + a_1x + a_0)^i$ and can be regarded as a $1 \times (2i + 1)$ row matrix. If we write the k tuple $(0, \cdots, 0)$ by $0^{[k]}$ then $(0^{[k]}; r_i^{(a_2,a_1,a_0)}; 0^{[j]})$ is a $1 \times (2i + k + j + 1)$ row matrix consisting of k zeros followed by $r_i^{(a_2,a_1,a_0)}$ and then j zeros.

THEOREM 3. $$r_n^{(a_2,a_1,a_0)} = r_n^{(1,a_0)} \begin{bmatrix} r_n^{(a_2,a_1)}; 0^{[n]} \\ \cdots \\ 0^{[2(n-i)]}; r_i^{(a_2,a_1)}; 0^{[i]} \\ \cdots \\ 0^{[2n]}; r_0^{(a_2,a_1)} \end{bmatrix} \quad \textit{for } n \geq 0.$$

Proof. Consider $$(a_2x^2 + a_1x + a_0)^n = r_n^{(a_2,a_1,a_0)} \begin{bmatrix} x^{2n} \\ \cdots \\ 1 \end{bmatrix}. \qquad (1)$$

Assume $a_2 = 1$ and let $X = x(x + a_1)$. Then

$$X^i = x^i\ r_i^{(1,a_1)} \begin{bmatrix} x^i \\ \cdots \\ 1 \end{bmatrix} = r_i^{(1,a_1)} \begin{bmatrix} x^{2i} \\ \cdots \\ x^i \end{bmatrix} = (0^{[2(n-i)]}; r_i^{(1,a_1)}; 0^{[i]}) \begin{bmatrix} x^{2n} \\ \cdots \\ x^{2i} \\ \cdots \\ x^i \\ \cdots \\ 1 \end{bmatrix}$$

for all $0 \le i \le n$, so we have

$$(x^2 + a_1 x + a_0)^n = (X + a_0)^n$$

$$= r_n^{(1,a_0)} \begin{bmatrix} X^n \\ \cdots \\ 1 \end{bmatrix} = r_n^{(1,a_0)} \begin{bmatrix} r_n^{(1,a_1)}; 0^{[n]} \\ \cdots \\ 0^{[2(n-i)]}; r_i^{(1,a_1)}; 0^{[i]} \\ \cdots \\ 0^{[2n]}; r_0^{(1,a_1)} \end{bmatrix} \begin{bmatrix} x^{2n} \\ \cdots \\ x^i \\ \cdots \\ 1 \end{bmatrix}. \qquad (2)$$

Hence we have $r_n^{(1,a_1,a_0)} = r_n^{(1,a_0)} \begin{bmatrix} r_n^{(1,a_1)}; 0^{[n]} \\ 0^{[2]}; r_{n-1}^{(1,a_1)}; 0^{[n-1]} \\ \cdots \\ 0^{[2n]}; r_0^{(1,a_1)} \end{bmatrix}$ by (1) and (2).

When $a_2 \neq 0$, let $X = a_2 x^2 + a_1 x = a_2 x(x+q)$ with $q = \frac{a_1}{a_2}$. Then

$$X^i = a_2^i x^i r_i^{(1,q)} \begin{bmatrix} x^i \\ \cdots \\ 1 \end{bmatrix} = a_2^i r_i^{(1,q)} \begin{bmatrix} x^{2i} \\ \cdots \\ x^i \end{bmatrix}$$

$$= (0^{[2(n-i)]}; a_2^i r_i^{(1,q)}; 0^{[i]}) \begin{bmatrix} x^{2n} \\ \cdots \\ x^{2i} \\ \cdots \\ 1 \end{bmatrix} = (0^{[2(n-i)]}; r_i^{(a_2,a_1)}; 0^{[i]}) \begin{bmatrix} x^{2n} \\ \cdots \\ 1 \end{bmatrix},$$

since $a_2^i r_i^{(1,q)} = r_i^{(a_2,a_1)}$ by Theorem 1. Thus

$$(a_2 x^2 + a_1 x + a_0)^n = (X + a_0)^n$$

$$= r_n^{(1,a_0)} \begin{bmatrix} X^n \\ \cdots \\ 1 \end{bmatrix} = r_n^{(1,a_0)} \begin{bmatrix} r_n^{(a_2,a_1)}; 0^{[n]} \\ 0^{[2]}; r_{n-1}^{(a_2,a_1)}; 0^{[n-1]} \\ \cdots \\ 0^{[2n]}; r_0^{(a_2,a_1)} \end{bmatrix} \begin{bmatrix} x^{2n} \\ \cdots \\ 1 \end{bmatrix}. \qquad (3)$$

So we have $r_n^{(a_2,a_1,a_0)} = r_n^{(1,a_0)} \begin{bmatrix} r_n^{(a_2,a_1)}; 0^{[n]} \\ \cdots \\ 0^{[2(n-i)]}; r_i^{(a_2,a_1)}; 0^{[i]} \\ \cdots \\ 0^{[2n]}; r_0^{(a_2,a_1)} \end{bmatrix}$ by (1) and (3). □

THEOREM 4. *For $1 \le a, b, c \le 9$, the column sum table of $r_n^{(c,b,a)}$ with base 10^k $(k > 0)$ yields the nth power of $10^{2k}c + 10^k b + a$. If $k = 1$, the column sum table of $r_n^{(c,b,a)}$ with base 10 gives $S_{10}^{r_n^{(c,b,a)}}$, the nth power of three digit integer cba.*

The proof is similar to Theorem 2. For instance with $r_5 = (1, 5, 10, 10, 5, 1)$,

$$r_5^{(1,1,1)} = r_5 \begin{bmatrix} r_5; 0^{[5]} \\ 0^{[2]}; r_4; 0^{[4]} \\ \cdots \\ 0^{[10]}; r_0 \end{bmatrix} = r_5 \begin{bmatrix} 1&5&10&10&5&1&&&&& \\ &&1&4&6&4&1&&&& \\ &&&&1&3&3&1&&& \\ &&&&&&1&2&1&& \\ &&&&&&&&1&1& \\ &&&&&&&&&&1 \end{bmatrix}$$

$$= (1, 5, 15, 30, 45, 51, 45, 30, 15, 5, 1),$$

which is the set of coefficients of $(x^2 + x + 1)^5$. Moreover by Theorem 1 and 3, the $r_5^{(3,5,7)}$ of $T^{(3,5,7)}$ equals

$$r_5^{(1,7)} \begin{bmatrix} r_5^{(3,5)}; 0^{[5]} \\ 0^{[2]}; r_4^{(3,5)}; 0^{[3]} \\ \cdots \\ 0^{[8]}; r_1^{(3,5)}; 0^{[1]} \\ 0^{[10]}; r_0^{(3,5)} \end{bmatrix} = r_5(1, 7, \cdots, 7^5) \begin{bmatrix} r_5(3^5, \cdots, 1)(1, \cdots, 5^5); 0^{[5]} \\ 0^{[2]}; r_4(3^4, \cdots, 1)(1, \cdots, 5^4); 0^{[4]} \\ \cdots \\ 0^{[8]}; r_1(3, 1)(1, 5); 0^{[1]} \\ 0^{[10]}; r_0 \end{bmatrix}$$

$$= (1, 35, 490, 3430, 12005, 16807) \begin{bmatrix} 243&2025&6750&11250&9375&3125&&&&& \\ &&81&540&1350&1500&625&&&& \\ &&&&27&135&225&125&&& \\ &&&&&&9&30&25&& \\ &&&&&&&&3&5& \\ &&&&&&&&&&1 \end{bmatrix}$$

(4)

$$= (243, 2025, 9585, 30150, 69855, 121775, 162995, 164150, 121765, 60025, 16807),$$

which is the set of coefficients of $(3x^2+5x+7)^5$. So the column sum table of $r_5^{(3,5,7)}$ with base 10 shows $357^5 = 5798839393557$ (Table 6). Moreover $30507^5 = 26423936089214873669307$ is due to Table 7.

Table 6

	35505550557
	42855795620
	2051879170 8
	290912410 6
	36266261
	1111
$S_{10}^{r_5^{(3,5,7)}}$	5798839393557

Table 7

	0 0 0 0 0 0 0 0 0 07
	3 5 5 0 5 5 5 0 5 50
	4 2 8 5 5 7 9 5 6 28
	2 0 5 1 8 7 9 1 7 06
	2 9 0 9 1 2 4 1 01
	3 6 2 6 6 2 6
	1 1 1 1
$S_{10^2}^{r_5^{(3,5,7)}}$	26423936089214873669307

The table in (4) yields not only $r_5^{(3,5,7)}$ but also $r_i^{(3,5,7)}$ $(1 \le i \le 5)$. In fact,

$$r_4^{(3,5,7)} = r_4(1, \cdots, 7^4) \begin{bmatrix} 81&540&1350&1500&625&&&& \\ &&27&135&225&125&&& \\ &&&&9&30&25&& \\ &&&&&&3&5& \\ &&&&&&&&1 \end{bmatrix}$$

$$= (81, 540, 2106, 5280, 9571, 12320, 11466, 6860, 2401),$$

which is the coefficients set of $(3x^2 + 5x + 7)^4$. So $357^4 = S_{10}^{r_4^{(3,5,7)}} = 16243247601$ is obtained through the column sum table of $r_4^{(3,5,7)}$ with base 10. Now we are ready to have powers of any $k+1$ digit integer by AT $T^{(a_k,\cdots,a_0)}$ of $(a_kx^k+\cdots+a_0)^m$ and its ith row $r_i^{(a_k,\cdots,a_0)}$ for $k,i\geq 0$.

THEOREM 5. $r_n^{(a_k,\cdots,a_0)} = r_n^{(1,a_0)}\begin{bmatrix} r_n^{(a_k,\cdots,a_1)};0^{[n]} \\ \cdots \\ 0^{[(n-i)k]};r_i^{(a_k,\cdots,a_1)};0^{[i]} \\ \cdots \\ 0^{[nk]};r_0^{(a_k,\cdots,a_1)} \end{bmatrix}$. *So the column sum table of* $r_n^{(a_k,\cdots,a_0)}$ *with base* 10 *yields the* n*th power of* $k+1$ *digit integer* $a_k\cdots a_1a_0$ $(1\leq a_i\leq 9,\ 0\leq i\leq k)$.

Proof. Let $f(x)=(a_kx^k+\cdots+a_1x+a_0)^n=(X+a_0)^n$ with $X=x(a_kx^{k-1}+\cdots+a_1)$. Then $r_n^{(a_k,\cdots,a_0)}\begin{bmatrix} x^{nk} \\ \cdots \\ 1\end{bmatrix} = r_n^{(1,a_0)}\begin{bmatrix} X^n \\ \cdots \\ 1\end{bmatrix}$ and

$$X^i = r_i^{(a_k,\cdots,a_1)}\begin{bmatrix} x^{ik} \\ \cdots \\ x^i\end{bmatrix} = (0^{[(n-i)k]};r_i^{(a_k,\cdots,a_1)};0^{[i]})\begin{bmatrix} x^{nk} \\ \cdots \\ x^{ik} \\ \cdots \\ 1\end{bmatrix}.$$

Thus we have $r_n^{(a_k,\cdots,a_0)} = r_n^{(1,a_0)}\begin{bmatrix} r_n^{(a_k,\cdots,a_1)};0^{[n]} \\ \cdots \\ 0^{[(n-i)k]};r_i^{(a_k,\cdots,a_1)};0^{[i]} \\ \cdots \\ 0^{[nk]};r_0^{(a_k,\cdots,a_1)} \end{bmatrix}$.

Since $r_n^{(a_k,\cdots,a_0)}$ is the set of coefficients of $f(x)$, we have $f(10)=(a_k10^k+\cdots+a_110+a_0)^n=(\underbrace{a_k\cdots a_0}_{k+1\text{ digit}})^n$. So nth power of the $k+1$ digit integer $a_k\cdots\ a_1a_0$ is obtained through the column sum table of $r_n^{(a_k,\cdots,a_0)}$ with base 10. □

From $r_5^{(1,1,1,1)}=(1,5,15,35,65,101,135,155,155,135,101,65,35,15,5,1)$ by Theorem 5, the power $1111^5=1692662195786551$ is obtained by the column sum table of $r_5^{(1,1,1,1)}$ with base 10 (Table 8).

Moreover the coefficients of $(x^4+x^3+x^2+x+2)^5$ equals

$$r_5(1,1,1,1,2) = r_5^{(1,2)}\begin{bmatrix} r_5^{(1,1,1,1)},0^{(5)} \\ 0^{[4]};r_4^{(1,1,1,1)},0^{[4]} \\ \cdots \\ 0^{[20]};r_0^{(1,1,1,1)} \end{bmatrix}$$

$$= (1,5,10,10,5,1)(1,2,\cdots,2^5)\begin{bmatrix} 1&5&15&35&65&101&135&155&155&135&101&65&35&15&5&1&&\\ &1&4&10&20&31&40&44&40&31&20&10&4&1&&&&\\ &&&&&1&3&6&10&12&12&10&6&3&1&&&\\ &&&&&&&&&1&2&3&4&3&2&1&&\\ &&&&&&&&&&&&&1&1&1&1&\\ &&&&&&&&&&&&&&&&&1 \end{bmatrix}$$

$= (1,5,15,35,75,141,235,355,\cdots,855,745,601,450,280,160,80,32)$.
So the power $11112^5 = 16941862927485989683 2$ is obtained from the column sum table of $r_5^{(1,1,1,1,2)}$ with base 10 (Table 9).

Table 8

	1555515555155551
	136035530631
	111111
$S_{10}^{r_5^{(1,1,1,1)}}$	1692662195786551

Table 9

	155551555515555100002
	13743505860540586683
	12356789876421
$S_{10}^{r_5^{(1,1,1,1,2)}}$	16941862927485989683 2

3. Negative powering of integers

We turn our attention to powers of integers with negative exponent. Let $T'^{(b,a)}$ be the AT of $(bx+a)^{-m}$ for $m>0$, and $r_n'^{(b,a)}$ and $e_{n,j}'^{(b,a)}$ $(n \geq 1,\ j \geq 0)$ be the nth row and the (n,j)th component of $T'^{(b,a)}$, respectively. We write $T'^{(1,1)} = T'$, $r_n'^{(1,1)} = r_n'$ and $e_{n,j}'^{(1,1)} = e_{n,j}'$ for short. In $T' = [e_{n,j}']$, $(x+1)^{-m}$ is expanded in ascending powers of x, and the series $\sum_{j=0}^{\infty} e_{n,j}' x^j$ of $r_n' = (e_{n,0}', e_{n,1}', e_{n,2}', \cdots)$ converges if $|x|<1$.

m	AT T' of $(x+1)^{-m}$									
1	1	−1	1	−1	1	−1	1	−1	1	⋯
2	1	−2	3	−4	5	−6	7	−8	9	⋯
3	1	−3	6	−10	15	−21	28	−36	45	⋯
4	1	−4	10	−20	35	−56	84	−120	165	⋯
5	1	−5	15	−35	70	−126	210	−330	495	⋯

Let $r_{n,A}' = (e_{n,0}', \cdots, e_{n,2l}', \cdots)$ and $r_{n,B}' = (e_{n,1}', \cdots, e_{n,2l+1}', \cdots)$ be subsets consisting of eventh and oddth entries of r_n' respectively, and consider the column sum tables of each. Indeed, from $r_6' = (1,-6,21,-56,126,$ $-252,462,-792,1287,\cdots)$, the column sum subtables of subset $r_{6,A}' =$ $(1,21,126,462,1287,\cdots)$ and $r_{6,B}' = (-6,-56,\ -252,-792,\cdots)$ with base 10^{-2} are

	1.2	1	2	6	⑥	②	**8**	**7**
			1		④	**1**	**2**	
$S_{10^{-2}}^{r'_{6,A}}$	1.2	2	3	0	7	4	8	7

and

	0.6	5	6	⑤	②	**9**	**2**
				②		**7**	
$-S_{10^{-2}}^{r'_{6,B}}$	0.6	5	8	5	9	9	2

So the columns sums show

$S_{10^{-2}}^{r'_{6,A}} + S_{10^{-2}}^{r'_{6,B}} \approx (1.22307487) - (0.6585992) \approx 0.56447567 \approx \frac{10^6}{11^6}$, ($\approx$ denotes approximation), and $11^{-6} \approx (0.56447)10^{-6}$ is obtained. With a help of the example, we have an effective way for powering of 11 with negative exponent, which may not be obtained easily by algebraic manipulations.

THEOREM 6. *Let $S_{10^{-2}}^{r'_{n,A}}$ and $S_{10^{-2}}^{r'_{n,B}}$ be the column sums of $r'_{n,A}$ and $r'_{n,B}$ respectively. Then $11^{-n} \approx (S_{10^{-2}}^{r'_{n,A}} + S_{10^{-2}}^{r'_{n,B}})10^{-n}$.*

Proof. Let $f(x) = (x+1)^{-n}$. Then with $x = 0.1 < 1$, $f(0.1) = \frac{10^n}{11^n}$.

By Taylor series expansion, the coefficient of x^j in $f(x)$ equals $e'_{n,j} = \binom{-n}{j} = (-1)^j \frac{n(n+1)\cdots(n+j-1)}{j!}$. So $e'_{n,j} > 0$ if j is even, otherwise $e'_{n,j} < 0$. Hence the subsets $r'_{n,A} = (e'_{n,0}, \cdots, e'_{n,2l}, \cdots)$ and $r'_{n,B} = (e'_{n,1}, \cdots, e'_{n,2l+1}, \cdots)$ are classified according to the sign of $e'_{n,j}$. And the column sum subtable of each $r'_{n,A}$ and $r'_{n,B}$ with base 10^{-2} produce $S_{10^{-2}}^{r'_{n,A}}$ and $S_{10^{-2}}^{r'_{n,B}}$ respectively. It thus follows that

$$\frac{10^n}{11^n} = f(0.1) \approx S_{10^{-2}}^{r'_{n,A}} + S_{10^{-2}}^{r'_{n,B}}, \text{ i.e., } 11^{-n} \approx (S_{10^{-2}}^{r'_{n,A}} + S_{10^{-2}}^{r'_{n,B}})10^{-n}. \qquad \square$$

For instance, with $r'_1 = (1, -1, 1-1, 1, \cdots)$, the column sum table of $r'_{1,A} = (1, 1, 1, \cdots)$ and $r'_{1,B} = (-1, -1, -1, \cdots)$ with base 10^{-2} yield $S_{10^{-2}}^{r'_{1,A}} \approx 1.0101010$ and $S_{10^{-2}}^{r'_{1,B}} \approx -0.1010101$. Thus we have

$$S_{10^{-2}}^{r'_{1,A}} + S_{10^{-2}}^{r'_{1,B}} \approx 0.9090909 \approx \frac{10}{11}, \text{ and } 11^{-1} \approx (0.9090909)10^{-1}.$$

Moreover from $r'_{10} = (1, -10, 55, -220, 715, -2002, 5005, -11440, 24310, \cdots)$, $r'_{10,A} = (1, 55, 715, 5005, 24310, \cdots)$ and $r'_{10,B} = (-10, -220, -2002, -11440, \cdots)$, we have $S_{10^{-2}}^{r'_{10,A}} \approx 1.62674810$ and $S_{10^{-2}}^{r'_{10,B}} \approx -1.2411640$ by the tables

	1.55150510
	75043
	2
$S_{10^{-2}}^{r'_{10,A}}$	1.62674810

and

	1.0200240
	22014
	1
$-S_{10^{-2}}^{r'_{10,B}}$	1.2411640

Thus we immediately have

$$11^{-10} = (S_{10^{-2}}^{r'_{10,A}} + S_{10^{-2}}^{r'_{10,B}})10^{-10} \approx (0.3855)10^{-10}.$$

A generalization to two digit integer $ba = 10b + a$ $(0 \le a, b \le 9)$ with negative exponent is as follows. We need a lemma in [2].

LEMMA 7. *Let $T'^{(b,a)} = [e'^{(b,a)}_{i,j}]$ be an AT of $(bx+a)^{-m}$. Then $r'^{(b,a)}_n = r'_n(\frac{1}{a^n}, \frac{b}{a^{n+1}}, \frac{b^2}{a^{n+2}}, \cdots)$ and $e'^{(b,a)}_{n,j} = e'_{n,j}\frac{b^j}{a^{n+j}}$.*

Let $r'^{(b,a)}_{n,A} = (e'^{(b,a)}_{n,0}, \cdots, e'^{(b,a)}_{n,2l}, \cdots)$ and $r'^{(b,a)}_{n,B} = (e'^{(b,a)}_{n,1}, \cdots, e'^{(b,a)}_{n,2l+1}, \cdots)$ be subsets of the nth row $r'^{(b,a)}_n$ of $T'^{(b,a)}$ defined as above.

Theorem 8. *Let $S^{r'_{n,A}}_{10^{-2}}$ and $S^{r'_{n,B}}_{10^{-2}}$ be the sums of column sum tables of $r'^{(b,a)}_{n,A}$ and $r'^{(b,a)}_{n,B}$ with base 10^{-2}, respectively. Then the negative nth power of two digit integer ba equals $(ba)^{-n} \approx (S^{r'_{n,A}}_{10^{-2}} + S^{r'_{n,B}}_{10^{-2}})10^{-n}$.*

The proof follows immediately from Lemma 7 and Theorem 6. For example, in order to get 13^{-5}, consider $f(x) = (3x+1)^{-5}$. For $x = 0.1 < 1$, $f(x) = \frac{10^5}{13^5}$. And the coefficients of $f(x) = (3x+1)^{-5}$ equals

$$r'^{(3,1)}_5 = r'_5(1, 3, 3^2, \cdots)$$
$$= (1, -5, 15, -35, 70, -126, 210, -330, 495, -715, 1001, -1365, 1820, -2380, 3060, -3876, 4845, -5985, 7315, -8855, \cdots)(1, 3, 3^2, \cdots, 3^{19}, \cdots)$$
$$= (1, -15, 135, -945, 5670, -30618, 153090, -721710, 3247695, -14073345, 59108049, -241805655, 967222620, -3794488740, 14635885140, \cdots).$$

With two subsets $r'^{(3,1)}_{5,A}$ and $r'^{(3,1)}_{5,B}$ of $r'^{(3,1)}_5$, the column sum tables

	1.3570909549204045
	1.56307680265132
	152410228836
	359673561
	94685
	120
$S^{r'^{(3,1)}_{5,A}}_{10^{-2}}$	3.1096121925077245

and

	1.5451810455540325 5
	9061733568735555
	3720780483687
	1441941603
	2375629
	577
$-S^{r'^{(3,1)}_{5,B}}_{10^{-2}}$	2.8402851958262875 5

with base 10^{-2} yield

$$\frac{10^5}{13^5} \approx S^{r'^{(3,1)}_{5,A}}_{10^{-2}} + S^{r'^{(3,1)}_{5,B}}_{10^{-2}} \approx 0.269326997, \text{ so } 13^{-5} \approx (0.269326997)10^{-5}.$$

Theorem 9. *Consider $T'^{(1,1,a)}$ of $(x^2 + x + a)^{-m}$ and its nth row $r'^{(1,1,a)}_n$. Then* $r'^{(1,1,a)}_n = r'^{(1,a)}_n \begin{bmatrix} r_0; 0^{[\infty]} \\ \cdots \\ 0^{[i]}; r_i; 0^{[\infty]} \\ \cdots \end{bmatrix}$.

Proof. With $X = x(x+1)$, we clearly have

$$r'^{(1,1,a)}_n \begin{bmatrix} 1 \\ x \\ \cdots \end{bmatrix} = (x^2 + x + a)^{-n} = (X + a)^{-n} = r'^{(1,a)}_n \begin{bmatrix} 1 \\ X \\ \cdots \end{bmatrix},$$

and $X^i = x^i r_i \begin{bmatrix} 1 \\ \cdots \\ x^i \end{bmatrix} = r_i \begin{bmatrix} x^i \\ \cdots \\ x^{2i} \end{bmatrix} = (0^{[i]}; r_i; 0^{[\infty]}) \begin{bmatrix} 1 \\ \cdots \\ x^i \\ \cdots \end{bmatrix}$. Thus we have

$$r_n'^{(1,1,a)}\begin{bmatrix}1\\x\\\cdots\\x^i\\\cdots\end{bmatrix}=r_n'^{(1,a)}\begin{bmatrix}r_0,0^{[\infty]}\\0^{[1]};r_1;0^{[\infty]}\\\cdots\\0^{[i]};r_i;0^{[\infty]}\\\cdots\end{bmatrix}\begin{bmatrix}1\\x\\\cdots\\x^i\\\cdots\end{bmatrix}. \qquad \square$$

For example, with $r_1' = (1, -1, 1, -1, \cdots)$, Theorem 9 shows

$$r'^{(1,1,1)}_1 = r_1'\begin{bmatrix}r_0\\\cdots\\0^{[i]};r_i\\\cdots\end{bmatrix} = r_1'\begin{bmatrix}1&&&\\&11&&\\&&121&\\&&&1331\\&&&\cdots\end{bmatrix} = (1,-1,0,1,-1,0,1,-1,0,\cdots).$$

Similarly, $r_2'^{(1,1,1)}$ also follows from $r_2' = (1, -2, 3, -4, 5, -6, \cdots)$ that

$$r'^{(1,1,1)}_2 = r_2'\begin{bmatrix}1&&&&\\&11&&&\\&&121&&\\&&&1331&\\&&&&14641\\&&&\cdots&\end{bmatrix} = (1,-2,1,2,-4,2,3,-6,3,4,-8,\cdots).$$

Continuing with r_n', the AT $T'^{(1,1,1)}$ of $(x^2+x+1)^{-m}$ forms as follows.

m	$T'^{(1,1,1)}$ of $(x^2+x+1)^{-m}$															
1	1	−1	0	1	−1	0	1	−1	0	1	−1	0	1	−1	0	1 ⋯
2	1	−2	1	2	−4	2	3	−6	3	4	−8	4	5	10	5	6 ⋯
3	1	−3	3	2	−9	9	3	−18	18	4	−30	30	5	−45	45	6 ⋯
4	1	−4	6	0	−15	24	−6	−36	60	−20	−70	120	−45	−120	210	−84 ⋯
5	1	−5	10	−5	−20	49	−35	−50	145	−115	−100	335	−280	−175	665	−574 ⋯

Now for 111^{-2}, consider $f(x) = (x^2+x+1)^{-2}$. Then $f(0.1) = (1.11)^{-2} = 111^{-2}\cdot 10^4$. From $r_2'^{(1,1,1)} = (1,-2,1,2,-4,2,3,-6,3,4,\cdots)$, the expansion

$$(x^2+x+1)^{-2} = 1 - 2x + x^2 + 2x^4 - 4x^4 + 2x^5 + 3x^6 + \cdots$$

follows easily. Moreover we consider three subsets of $r_2'^{(1,1,1)}$ that

$r_{2,A}'^{(1,1,1)} = \{e_{2,j}'^{(1,1,1)} \mid j \equiv 0 \bmod 3\} = (1,2,3,4,5,\cdots)$,

$r_{2,B}'^{(1,1,1)} = \{e_{2,j}'^{(1,1,1)} \mid j \equiv 1 \bmod 3\} = (-2,-4,-6,-8,\cdots)$ and

$r_{2,C}'^{(1,1,1)} = \{e_{2,j}'^{(1,1,1)} \mid j \equiv 2 \bmod 3\} = (1,2,3,4,5,\cdots)$.

Then each column sum subtables with base 10^{-3} give rise to sums $S_{10^{-3}}^{r_{2,A}'^{(1,1,1)}} \approx 1.002003004005$, $S_{10^{-3}}^{r_{2,B}'^{(1,1,1)}} \approx -0.2004006008$, and $S_{10^{-3}}^{r_{2,C}'^{(1,1,1)}} \approx 0.01002003004005$. Hence we have

$$f(0.1) = S_{10^{-3}}^{r_{2,A}'^{(1,1,1)}} + S_{10^{-3}}^{r_{2,B}'^{(1,1,1)}} + S_{10^{-3}}^{r_{2,C}'^{(1,1,1)}} \approx 0.8116224332,$$

so $111^{-2} \approx (0.8116224332)10^{-4}$ is obtained from the Pascal triangle.

Similarly for 111^{-3}, consider $r_3'^{(1,1,1)} = (1,-3,3,2,-9,9,3,-18,18,4,-30,\cdots)$ and three subsets $r_{3,A}'^{(1,1,1)} = (1,2,3,4,5,\cdots)$, $r_{3,B}'^{(1,1,1)} = (-3,-9,-18,-30,$

$-45,\cdots)$ and $r'^{(1,1,1)}_{3,C} = (3,9,18,30,45,\cdots)$. Then the corresponding column sums $S^{r'^{(1,1,1)}_{3,A}}_{10^{-3}}$, $S^{r'^{(1,1,1)}_{3,B}}_{10^{-3}}$ and $S^{r'^{(1,1,1)}_{3,C}}_{10^{-3}}$ follow from the column sums table with base 10^{-3} that

$$\begin{array}{c|l} & 1.002003004005 \\ \hline S^{r'^{(1,1,1)}_{3,A}}_{10^{-3}} & 1.002003004005 \end{array} \qquad \begin{array}{c|r} & 0.3009008000005 \\ & 1\quad 3\quad 4 \\ \hline -S^{r'^{(1,1,1)}_{3,B}}_{10^{-3}} & 0.3009018030045 \end{array}.$$

Hence we have

$$\begin{aligned} &S^{r'^{(1,1,1)}_{3,A}}_{10^{-3}} + S^{r'^{(1,1,1)}_{3,B}}_{10^{-3}} + S^{r'^{(1,1,1)}_{3,C}}_{10^{-3}} \\ &\approx 1.002003004005 - 0.3009018030045 + 0.03009018030045 \\ &\approx 0.7311913813. \end{aligned}$$

Since $(1.11)^{-3} = 111^{-3}10^6$, we derive $111^{-3} \approx (0.7311913813)10^{-6}$, where the calculations are basically based on Pascal triangle.

Note that, for 111^{-4}, we may consider 6 subsets $\{1,0\}$, $\{-4,-15\}$, $\{6,24\}$, $\{-6,-20,-45,-84\}$, $\{36,-70,-120\}$, and $\{60,120,210\}$ of $r'^{(1,1,1)}_4$. The column sum tables of the first three subsets are 1, $\frac{\begin{array}{r}0.4005\\ 1\end{array}}{0.4015}$, and $\frac{\begin{array}{r}0.06004\\ 2\end{array}}{0.06024}$. So with only these values, we have $1.0 - 0.4015 + 0.06024 = 0.65874$, where it already provides $(0.65874)10^{-8} \approx 111^{-4}$. For more precise value of 111^{-4}, we may use more column sum tables from the rest subsets that

$$\frac{\begin{array}{r}0.000006000005004\\ 2\quad 4\quad 8\end{array}}{0.000006020045084}, \frac{\begin{array}{r}0.0000006000000\\ 3\quad 7\ \ 120\end{array}}{0.0000036070120}, \frac{\begin{array}{r}0.00000000000000\\ 6\ \ 12\ \ 210\end{array}}{0.00000060120210}.$$

Then together with 0.65874 above, we have

$0.65874 - 0.000006020045084 - 0.000003607012 + 0.0000006012021$ equals 0.6587309742, that gives $111^{-4} \approx (0.6587309742)10^{-8}$.

References

[1] G. Farkas and G. Kallos, *Prime numbers in generalized Pascal triangles,* Acta Technica Jaurinensis **1** (2008), 109–117.

[2] J. Jo, Y. Oh, and E. Choi, *Arithmetic matrix of quadratic polynomial with negative exponent by Pascal matrix*, J. Alg and Appl. Math. **17** (2019), 67–90.

[3] G. Kallos, *The generalization of Pascal's triangle from algebraic point of view*, Acta Acad. Paedagogicae Agriensis, Sectio Mathematicae, 24 (1997), 11–18.

[4] G. Kallos, *A Generalization of Pascal's triangle using powers of base numbers*, Ann. Math. Blaise Pascal **13** (2006), 1–15.

[5] C. J. Lacke, *Powers of eleven in Pascal's triangle*, Mathematics and Computer Education (2002), 28–30.

[6] L. LOW, *Even more on Pascal's triangles and powers of* 11, Math. Teacher **59** (1966), 461–463.
[7] R.L. Morton, *Pascal's triangle and Powers of* 11, Math. Teacher **57** (1964), 392–394.
[8] F.J. Mueller, *More on Pascal's triangle and powers of* 11, Math. Teacher **58** (1965), 425–428.

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