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HYERS-ULAM-RASSIAS STABILITY OF A QUADRATIC-CUBIC-QUARTIC FUNCTIONAL EQUATION

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Abstract. In this paper, we investigate Hyers-Ulam-Rassias stability of a functional equation

$$f(x+ky)+f(x-ky)-k^2f(x+y)-k^2f(x-y)$$
$$+2(k^2-1)f(x)+(k^2+k^3)f(y)+(k^2-k^3)f(-y)-2f(ky)=0.$$

1. Introduction

Let V and W be real normed spaces, Y a real Banach space, and k a fixed real number with $|k|\neq 1$. In this paper, the following abbreviations are used for a given mapping $f: V\to W$:

$$\begin{aligned}
Qf(x,y) :=& f(x+y)+f(x-y)-2f(x)-2!f(y),\\
Cf(x,y) :=& f(x+2y)-3f(x+y)+3f(x)-f(x-y)-3!f(y),\\
Q'f(x,y) :=& f(x+2y)-4f(x+y)+6f(x)-4f(x-y)+f(x-2y)\\
&-4!f(y),\\
D_kf(x,y) :=& f(x+ky)+f(x-ky)-k^2f(x+y)-k^2f(x-y)\\
&+2(k^2-1)f(x)+(k^2+k^3)f(y)+(k^2-k^3)f(-y)-2f(ky)
\end{aligned}$$

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for all $x, y \in V$. All solutions of the functional equations $Qf(x,y) = 0$, $Cf(x,y) = 0$, and $Q'f(x,y) = 0$ are called a quadratic mapping, a cubic mapping, and a quartic mapping, respectively. If a mapping can be represented by the sum of a quadratic mapping, a cubic mapping and a quartic mapping, we call the mapping a quadratic-cubic-quartic mapping. When each solution of a functional equation is a quadratic-cubic-quartic mapping and all quadratic-cubic-quartic mapping is a solution of that equation, the functional equation is called a quadratic-cubic-quartic functional equation. Gordji *et al.* [4] investigated the stability of the quadratic-cubic-quartic functional equation

$$\begin{aligned} &f(x+ny) + f(x-ny) - n^2 f(x+y) - n^2 f(x-y) \\ &\qquad - 2(1-n^2)f(x) - \frac{n^2(n^2-1)}{6}(f(2y) + 2f(-y) - 6f(y)) = 0 \end{aligned}$$

in non-Archimedean normed spaces, when n is a fixed integer.

In 1940, Ulam [6] questioned the stability of group homomorphisms, and in 1941 Hyers [3] showed the stability of the Cauchy additive functional equation as a partial answer to that question. In 1978, Rassias [5] made Hyers' result generalized and Găvruta [2] more generalized Rassias' result. The concept of stability shown by Rassias is called 'Hyers-Ulam-Rassias stability'.

In this paper, we will show that the functional equation $D_r f(x,y) = 0$ is a quadratic-cubic-quartic functional equation when r is a rational number. And also we prove the Hyers-Ulam-Rassias stability of the functional equation $D_k f(x,y) = 0$ when k is a real number.

2. Main results

The following theorem is a special case of Baker's theorem [1].

THEOREM 2.1. (Theorem 1 in [1]) *Suppose that V and W are vector spaces over $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$ and $\alpha_0, \beta_0, \ldots, \alpha_m, \beta_m$ are scalar such that $\alpha_j\beta_l - \alpha_l\beta_j \neq 0$ whenever $0 \leq j < l \leq m$. If $f_l : V \to W$ for $0 \leq l \leq m$ and*

$$\sum_{l=0}^{m} f_l(\alpha_l x + \beta_l y) = 0$$

for all $x, y \in V$, then each f_l is a generalized polynomial mapping of degree at most $m-1$.

Baker [1] stated that if f is a generalized polynomial mapping of degree at most $m-1$, then f is expressed as $f(x) = x_0 + \sum_{l=1}^{m-1} a_l^*(x)$ for $x \in V$, where a_l^* is a monomial mapping of degree l and a_l^* has a property $a_l^*(rx) = r^l a_l^*(x)$ for $x \in V$ and $r \in \mathbb{Q}$.

Suppose that g, f', h are generalized polynomial mappings of degree at most 4 and r is a rational number such that $r \neq 0, \pm 1$. Baker [1] also stated that if the equalities $g(rx) = r^2 g(x)$, $f'(rx) = r^3 f'(x)$ and $h(rx) = r^4 h(x)$ hold for all $x \in V$, then g, f' and h are a quadratic mapping, a cubic mapping and a quartic mapping, respectively.

Now we will show that the functional equation $D_r f(x,y) = 0$ is a quadratic-cubic-quartic functional equation when r is a rational number such that $r \neq 0, \pm 1$.

The following abbreviations are used in this section for convenience.

$$
\begin{aligned}
f_o(x) &:= \frac{f(x) - f(-x)}{2}, \quad f_e(x) := \frac{f(x) + f(-x)}{2},\\
\Delta f(x) &:= \frac{1}{k^4 - k^2}[-D_k f_e((k+2)x, x) - D_k f_e((k-2)x, x)\\
&\quad - 4D_k f_e((k+1)x, x) - 4D_k f_e((k-1)x, x) + 10 D_k f_e(kx, x)\\
&\quad + D_k f_e(2x, 2x) + 4D_k f_e(x, 2x) - k^2 D_k f_e(3x, x)\\
&\quad - 2(k^2+1) D_k f_e(2x, x) + (17k^2 - 8) D_k f_e(x, x)]\\
&\quad + \frac{(17k^2 + 10) D_k f(0,0)}{2k^2(k^2-1)}
\end{aligned}
$$

for all $x, y \in V$.

Theorem 2.2. *Let r be a rational number such that $r \neq 0, \pm 1$. A mapping f satisfies the functional equation $D_r f(x,y) = 0$ for all $x, y \in V$ if and only if f is a quadratic-cubic-quartic mapping.*

Proof. Assume that the mapping $f : V \to W$ satisfies the functional equation $D_r f(x,y) = 0$ for all $x, y \in V$, and g, h are the mappings defined as $g(x) = \frac{-f_e(2x) + 16 f_e(x)}{12}$ and $h(x) = \frac{f_e(2x) - 4 f_e(x)}{12}$. Then the equalities $f(0) = \frac{D_r f(0,0)}{2(r^2-1)} = 0$, $\Delta f(x) = 0$, $D_r f_o(x,y) = 0$, $D_r g(x,y) = 0$ and $D_r h(x,y) = 0$ hold for all $x, y \in V$, and f_o, g and h are generalized polynomial mappings of degree at most 4 by Theorem 2.1. We can see that the mappings f_o, g and h satisfy the properties $g(2x) = 4g(x)$,

$h(2x) = 2^4h(x)$ and $f_o(rx) - r^3f_o(x) = 0$ for all $x \in V$, since the equalities

$$f_e(4x) - 20f_e(2x) + 64f_e(x) = \Delta f(x), \tag{1}$$
$$f_o(rx) - r^3f_o(x) = \frac{-D_rf(0,x)}{2}$$

hold for all $x \in V$. Therefore, according to Baker's comment before this theorem, g, f_o and h are a quadratic mapping, a cubic mapping and a quartic mapping, respectively. From $f = f_o + g + h$, f is a quadratic-cubic-quartic mapping.

Conversely, assume that f is a quadratic-cubic-quartic mapping, i.e., there exist a quadratic mapping g, a cubic mapping f' and a quartic mapping h such that $f = f' + g + h$. Notice that the equalities $f'(rx) = r^3f'(x)$, $f'(x) = -f'(-x)$, $g(rx) = r^2g(x)$, $g(x) = g(-x)$, $h(rx) = r^4h(x)$, and $h(x) = h(-x)$ hold for all $x \in V$ and $r \in \mathbb{Q}$.

The equality $D_rg(x,y) = 0$ is deduced from the equality

$$D_rg(x,y) = Qg(x,ry) - r^2Qg(x,y)$$

for all $x, y \in V$. In order to prove that $D_rf'(x,y) = 0$ and $D_rh(x,y) = 0$ when r is a rational number, let us first see that $D_rf'(x,y) = 0$ and $D_nh(x,y) = 0$ when n is a natural number. Using mathematical induction, the equalities $D_rf'(x,y) = 0$ and $D_nh(x,y) = 0$ are obtained from the equalities

$$\begin{aligned}
&D_1f'(x,y) = 0, \qquad\qquad\qquad D_1h(x,y) = 0,\\
&D_2f'(x,y) = Cf'(x,y) - Cf'(x-y,y),\ D_2h(x,y) = Q'h(x,y),\\
&D_nf'(x,y) = D_{n-1}f'(x+y,y) + D_{n-1}f'(x-y,y) - D_{n-2}f'(x,y)\\
&\ + (n-1)^2D_2f'(x,y),\\
&D_nh(x,y) = D_{n-1}h(x+y,y) + D_{n-1}h(x-y,y) - D_{n-2}h(x,y)\\
&\ + (n-1)^2D_2h(x,y)
\end{aligned}$$

for all $x, y \in V$ and all $n \in \mathbb{N}$. Let us now see that $D_rf'(x,y) = 0$ and $D_rh(x,y) = 0$ hold when r is a rational number such that $r \neq 0, \pm 1$. Notice that if $r \in \mathbb{Q}\backslash\{0\}$, then there exist $m, n \in \mathbb{N}$ such that $r = \frac{n}{m}$ or $r = \frac{-n}{m}$. Since the equalities $D_{\frac{n}{m}}f'(x,y) = 0$, $D_{\frac{-n}{m}}f'(x,y) = 0$,

$D_{\frac{n}{m}}h(x,y)=0$ and $D_{\frac{-n}{m}}h(x,y)=0$ are deduced from the equalities

$$\begin{aligned}
D_{\frac{n}{m}}f'(x,y) =& D_n f'\left(x,\frac{y}{m}\right)-\frac{n^2}{m^2}D_m f'\left(x,\frac{y}{m}\right),\\
D_{\frac{-n}{m}}f'(x,y) =& D_{\frac{n}{m}}f'(x,y),\\
D_{\frac{n}{m}}h(x,y) =& D_n h\left(x,\frac{y}{m}\right)-\frac{n^2}{m^2}D_m h\left(x,\frac{y}{m}\right),\\
D_{\frac{-n}{m}}h(x,y) =& D_{\frac{n}{m}}h(x,y)
\end{aligned}$$

for all $x,y\in V$ and $n,m\in\mathbb{N}$, we conclude that $D_r f'(x,y)=0$ and $D_r h(x,y)=0$ hold for all $x,y\in V$. □

For a given mapping $f:V\to W$ and a real number $p\neq 2,3,4$, let $J_n f: V\to W$ be the mappings defined as $J_n f(x):=$

$$\begin{cases}
k^{3n}f_o(k^{-n}x)+\frac{4^{2n+1}-4^n}{3}f_e(2^{-n}x)-\frac{4^{2n+2}-4^{n+2}}{3}f_e(2^{-n-1}x) & \text{if } p>4,\\
k^{3n}f_o(k^{-n}x)-\frac{4^{n-1}}{3}\big(f_e(2^{-n+1}x)-16f_e(2^{-n}x)\big) & \\
+\frac{f_e(2^{n+1}x)-4f_e(2^n x)}{12\cdot 16^n} & \text{if } 3<p<4,\\
\frac{f_o(k^n x)}{k^{3n}}+\frac{16f_e(2^n x)-f_e(2^{n+1}x)}{12\cdot 4^n}+\frac{f_e(2^{n+1}x)-4f_e(2^n x)}{12\cdot 16^n} & \text{if } 2<p<3,\\
\frac{f_o(k^n x)}{k^{3n}}+\frac{16f_e(2^n x)-f_e(2^{n+1}x)}{12\cdot 4^n}+\frac{f_e(2^{n+1}x)-4f_e(2^n x)}{12\cdot 16^n} & \text{if } p<2
\end{cases}$$

for all $x\in V$ and all nonnegative integers n when $1<|k|$, and $J_n f(x):=$

$$\begin{cases}
\frac{f_o(k^n x)}{k^{3n}}+\frac{4^{2n+1}-4^n}{3}f_e(2^{-n}x)-\frac{4^{2n+2}-4^{n+2}}{3}f_e(2^{-n-1}x) & \text{if } p>4,\\
\frac{f_o(k^n x)}{k^{3n}}-\frac{4^{n-1}}{3}\big(f_e(2^{-n+1}x)-16f_e(2^{-n}x)\big) & \\
+\frac{f_e(2^{n+1}x)-4f_e(2^n x)}{12\cdot 16^n} & \text{if } 3<p<4,\\
k^{3n}f_o(k^{-n}x)+\frac{16f_e(2^n x)-f_e(2^{n+1}x)}{12\cdot 4^n}+\frac{f_e(2^{n+1}x)-4f_e(2^n x)}{12\cdot 16^n} & \text{if } 2<p<3,\\
k^{3n}f_o(k^{-n}x)+\frac{16f_e(2^n x)-f_e(2^{n+1}x)}{12\cdot 4^n}+\frac{f_e(2^{n+1}x)-4f_e(2^n x)}{12\cdot 16^n} & \text{if } p<2
\end{cases}$$

for all $x\in V$ and all nonnegative integers n when $0<|k|<1$. By the definition of $J_n f$ and (1), we can calculate that $J_n f(x)-J_{n+1}f(x)=$

$$(2)\qquad \begin{cases}
\frac{-k^{3n}}{2}D_k f(0,\frac{x}{k^{n+1}})+\frac{4^n(4^{n+1}-1)}{3}\Delta f\big(\frac{x}{2^{n+2}}\big) & \text{if } p>4,\\
\frac{-k^{3n}}{2}D_k f(0,\frac{x}{k^{n+1}})-\frac{1}{192\cdot 16^n}\Delta f\big(2^n x\big)-\frac{4^{n-1}}{3}\Delta f\big(\frac{x}{2^{n+1}}x\big) & \\
 & \text{if } 3<p<4,\\
\frac{D_k f(0,k^n x)}{2k^{3n+3}}+\frac{4^{n+1}-1}{3\cdot 4^{2n+3}}\Delta f\big(2^n x\big) & \text{if } 2<p<3,\\
\frac{D_k f(0,k^n x)}{2k^{3n+3}}+\frac{4^{n+1}-1}{3\cdot 4^{2n+3}}\Delta f\big(2^n x\big) & \text{if } p<2
\end{cases}$$

for all $x \in V$ and all nonnegative integers n when $1 < |k|$, and $J_n f(x) - J_{n+1} f(x) =$

$$(3) \qquad \begin{cases} \frac{D_k f(0,k^n x)}{2k^{3n+3}} + \frac{4^n(4^{n+1}-1)}{3}\Delta f\big(2^{-n-2}x\big) & \text{if } p > 4, \\ \frac{D_k f(0,k^n x)}{2k^{3n+3}} - \frac{1}{192\cdot 16^n}\Delta f\big(2^n x\big) - \frac{4^{n-1}}{3}\Delta f\big(2^{-n-1}x\big) & \\ & \text{if } 3 < p < 4, \\ \frac{-k^{3n}}{2} D_k f(0, \frac{x}{k^{n+1}}) + \frac{4^{n+1}-1}{3\cdot 4^{2n+3}}\Delta f\big(2^n x\big) & \text{if } 2 < p < 3, \\ \frac{-k^{3n}}{2} D_k f(0, \frac{x}{k^{n+1}}) + \frac{4^{n+1}-1}{3\cdot 4^{2n+3}}\Delta f\big(2^n x\big) & \text{if } p < 2 \end{cases}$$

for all $x \in V$ and all nonnegative integers n when $0 < |k| < 1$. Therefore, together with the equality $f(x) - J_n f(x) = \sum_{i=0}^{n-1}(J_i f(x) - J_{i+1} f(x))$ for all $x \in V$, we obtain the following lemma.

Lemma 2.3. *If $f : V \to W$ is a mapping such that*

$$D_k f(x, y) = 0$$

for all $x, y \in V$, then

$$J_n f(x) = f(x)$$

for all $x \in V$ and all positive integers n.

From Lemma 2.3, we can prove the following stability theorem.

Theorem 2.4. *Let X be a real normed space, Y a real Banach space, and p a positive real number with $p \neq 2, 3, 4$. Suppose that $f : X \to Y$ is a mapping such that*

$$(4) \qquad \|D_k f(x, y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then there exists a unique solution mapping F of the functional equation $D_k F(x, y) = 0$ such that

(5)

$$\|f(x) - F(x)\| \leq \begin{cases} \frac{\theta\|x\|^p}{2||k|^3-|k|^p|} + \frac{K\theta\|x\|^p}{3\cdot 2^p}\big(\frac{4}{2^p-16} - \frac{1}{2^p-4}\big) & \text{if} \quad p > 4, \\ \frac{\theta\|x\|^p}{2||k|^3-|k|^p|} + \frac{K\theta\|x\|^p}{12}\big(\frac{1}{16-2^p} + \frac{1}{2^p-4}\big) & \text{if} \quad 3 < p < 4, \\ \frac{\theta\|x\|^p}{2||k|^3-|k|^p|} + \frac{K\theta\|x\|^p}{12}\big(\frac{1}{16-2^p} + \frac{1}{4-2^p}\big) & \text{if} \quad 2 < p < 3, \\ \frac{\theta\|x\|^p}{2||k|^3-|k|^p|} + \frac{K\theta\|x\|^p}{12}\big(\frac{1}{16-2^p} + \frac{1}{4-2^p}\big) & \text{if} \quad 0 < p < 2 \end{cases}$$

for all $x \in X$, *where*

$$K = \frac{37k^2 + 42 + (2k^2+8)2^p + k^2 3^p + 10|k|^p + 4|k-1|^p}{|k^4 - k^2|} + \frac{4|k+1|^p + |k-2|^p + |k+2|^p}{|k^4 - k^2|}.$$

Proof. We prove this theorem by dividing it into two cases, $|k| < 1$ and $1 < |k|$.

Let us first prove the case of $1 < |k|$. From the definition of Δf and (3), we have

$$\begin{aligned} \|\Delta f(x)\| = & \Big\| \frac{1}{k^4 - k^2}[-D_k f_e((k+2)x, x) - D_k f_e((k-2)x, x) \\ & - 4D_k f_e((k+1)x, x) - 4D_k f_e((k-1)x, x) + 10D_k f_e(kx, x) \\ & + D_k f_e(2x, 2x) + 4D_k f_e(x, 2x) - k^2 D_k f_e(3x, x) \\ & - 2(k^2+1)D_k f_e(2x, x) + (17k^2 - 8)D_k f_e(x, x)] \\ & + \frac{(17k^2+10)D_k f(0,0)}{2k^2(k^2-1)} \Big\| \\ \text{(6)} \qquad \le & K\|x\|^p \end{aligned}$$

for all $x \in X$. It follows from (2) and (4) that $\|J_n f(x) - J_{n+1} f(x)\| \le$

$$\begin{cases} \left(\frac{|k|^{3n}}{2\cdot|k|^{(n+1)p}} + \frac{4^n(4^{n+1}-1)K}{3\cdot 2^{(n+2)p}}\right)\theta\|x\|^p & \text{if } p > 4, \\ \left(\frac{|k|^{3n}}{2\cdot|k|^{(n+1)p}} + \frac{2^{np}K}{12\cdot 16^{n+1}} + \frac{4^{n-1}K}{3\cdot 2^{(n+1)p}}\right)\theta\|x\|^p & \text{if } 3 < p < 4, \\ \left(\frac{|k|^{np}}{2\cdot|k|^{3n+3}} + \frac{2^{np}K}{12\cdot 16^{n+1}} + \frac{4^{n-1}K}{3\cdot 2^{(n+1)p}}\right)\theta\|x\|^p & \text{if } 2 < p < 3, \\ \left(\frac{|k|^{np}}{2\cdot|k|^{3n+3}} + \frac{(4^{n+1}-1)2^{np}K}{3\cdot 4^{2n+1}}\right)\theta\|x\|^p & \text{if } 0 < p < 2 \end{cases}$$

for all $x \in X$. Together with the equality $J_n f(x) - J_{n+m} f(x) = \sum_{i=n}^{n+m-1}(J_i f(x) - J_{i+1} f(x))$ for all $x \in X$, we get $\|J_n f(x) - J_{n+m} f(x)\| \le$

$$\text{(7)} \qquad \sum_{i=n}^{n+m-1} \begin{cases} \left(\frac{|k|^{3i}}{2\cdot|k|^{(i+1)p}} + \frac{4^i(4^{i+1}-1)K}{3\cdot 2^{(i+2)p}}\right)\theta\|x\|^p & \text{if } p > 4, \\ \left(\frac{|k|^{3i}}{2\cdot|k|^{(i+1)p}} + \frac{2^{ip}K}{12\cdot 16^{i+1}} + \frac{4^{i-1}K}{3\cdot 2^{(i+1)p}}\right)\theta\|x\|^p & \text{if } 3 < p < 4, \\ \left(\frac{|k|^{ip}}{2\cdot|k|^{3i+3}} + \frac{2^{ip}K}{12\cdot 16^{i+1}} + \frac{4^{i-1}K}{3\cdot 2^{(i+1)p}}\right)\theta\|x\|^p & \text{if } 2 < p < 3, \\ \left(\frac{|k|^{ip}}{2\cdot|k|^{3i+3}} + \frac{(4^{i+1}-1)2^{ip}K}{3\cdot 4^{2i+1}}\right)\theta\|x\|^p & \text{if } 0 < p < 2 \end{cases}$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. It follows from (7) that the sequence $\{J_n f(x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{J_n f(x)\}$ converges for all $x \in X$. Hence we can define a mapping $F : X \to Y$ by

$$F(x) := \lim_{n\to\infty} J_n f(x)$$

for all $x \in X$. Moreover, letting $n = 0$ and passing the limit $n \to \infty$ in (7) we get the inequality (5). For the case $2 < p < 3$, from the definition of F, we easily get

$$\begin{aligned}\|D_k F(x,y)\| = \lim_{n\to\infty} \Big\| & \frac{1}{2\cdot k^{3n}}\left(D_k f\left(k^n x, k^n y\right) - D_k f\left(-k^n x, -k^n y\right)\right) \\ & + \frac{4^n}{12}\left(-D_k f_e\left(\frac{2x}{2^n}, \frac{2y}{2^n}\right) + 16 D_k f_e\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\right) \\ & + \frac{D_k f_e\left(2^{n+1}x, 2^{n+1}y\right) - 4 D_k f_e\left(2^n x, 2^n y\right)}{12\cdot 16^n}\Big\| \\ \le \lim_{n\to\infty} & \left(\frac{k^{np}}{k^{3n}} + \frac{4^n(2^p+16)}{12\cdot 2^{np}} + \frac{2^{np}(2^p+4)}{12\cdot 16^n}\right)\theta(\|x\|^p + \|y\|^p) \\ =0 & \end{aligned}$$

for all $x, y \in X$. Also we easily show that $D_k F(x,y) = 0$ by the similar method for the other cases, either $0 < p < 2$ or $3 < p < 4$ or $4 < p$.

To prove the uniqueness of F, let $F' : X \to Y$ be another solution mapping satisfying (5). Instead of the condition (5), it is sufficient to show that there is a unique mapping that satisfies condition $\|f(x) - F(x)\| \le \frac{\theta\|x\|^p}{2||k|^3-|k|^p|} + \frac{K\theta\|x\|^p}{12}\left(\frac{1}{|16-2^p|} + \frac{1}{|4-2^p|}\right)$ simply. Notice that $\|f(x) - F(x)\| = \|f_e(x) - F_e(x)\| = \|f_o(x) - F_o(x)\|$ and $F'(x) = J_n F'(x)$ for all $n \in \mathbb{N}$ by Lemma 2.3.

For the case $3 < p < 4$, we have

$$\begin{aligned}
&\|J_n f(x) - F'(x)\| \\
=&\|J_n f(x) - J_n F'(x)\| \\
=&\bigg\| k^{3n} f_o(k^{-n}x) - \frac{4^{n-1}}{3}\big(f_e(2^{-n+1}x) - 16 f_e(2^{-n}x)\big) \\
&+ \frac{f_e(2^{n+1}x) - 4f_e(2^n x)}{12\cdot 16^n} - k^{3n}F'_o(k^{-n}x) \\
&+ \frac{4^{n-1}}{3}\big(F'_e(2^{-n+1}x) - 16F'_e(2^{-n}x)\big) - \frac{F'_e(2^{n+1}x) - 4F'_e(2^n x)}{12\cdot 16^n}\bigg\| \\
\le&|k|^{3n}\|(f_o - F'_o)(k^{-n}x)\| + \frac{\|(f_e - F'_e)(2^n x)\|}{3\cdot 16^n} + \frac{\|(f_e - F'_e)(2^{n+1}x)\|}{12\cdot 16^n} \\
&+ \frac{4^{n-1}}{3}\|(f_e - F'_e)(2^{-n+1}x)\| + \frac{4^{n+1}}{3}\|(f_e - F'_e)(2^{-n}x)\| \\
\le&\bigg(\frac{|k|^{3n}}{|k|^{np}} + \frac{2^{np}}{3\cdot 16^n} + \frac{4\cdot 2^{(n+1)p}}{3\cdot 16^{n+1}} + \frac{4^{n-1}}{3\cdot 2^{(n-1)p}} + \frac{4^{n+1}}{3\cdot 2^{np}}\bigg)\times \\
&\bigg(\frac{1}{2||k|^3 - |k|^p|} + \frac{K}{12|16 - 2^p|} + \frac{K}{12|4 - 2^p|}\bigg)\theta\|x\|^p
\end{aligned}$$

for all $x \in X$ and all positive integers n. Taking the limit in the above inequality as $n \to \infty$, we can conclude that $F'(x) = \lim_{n\to\infty} J_n f(x)$ for all $x \in X$. For the other cases, either $0 < p < 2$ or $2 < p < 3$ or $4 < p$, we also easily show that $F'(x) = \lim_{n\to\infty} J_n f(x)$ by the similar method. This means that $F(x) = F'(x)$ for all $x \in X$.

Now consider the case of $|k| < 1$, which has not yet been proven. From (3), (4), (6) and the definition of $J_n f$, we have $\|J_n f(x) - J_{n+m} f(x)\| \le$

$$\sum_{i=n}^{n+m-1}\begin{cases}
\big(\frac{|k|^{ip}}{2\cdot|k|^{3(i+1)}} + \frac{4^i(4^{i+1}-1)}{3\cdot 2^{(i+2)p}}K\big)\theta\|x\|^p & \text{if } p > 4,\\
\big(\frac{|k|^{ip}}{2\cdot|k|^{3(i+1)}} + \frac{2^{ip}K}{12\cdot 16^{i+1}} + \frac{4^{i-1}K}{3\cdot 2^{(i+1)p}}\big)\theta\|x\|^p & \text{if } 3 < p < 4,\\
\big(\frac{|k|^{3i}}{2\cdot|k|^{(i+1)p}} + \frac{(4^{i+1}-1)2^{ip}}{3\cdot 4^{2i+1}}K\big)\theta\|x\|^p & \text{if } 2 < p < 3,\\
\big(\frac{|k|^{3i}}{2\cdot|k|^{(i+1)p}} + \frac{(4^{i+1}-1)2^{ip}}{3\cdot 4^{2i+1}}K\big)\theta\|x\|^p & \text{if } 0 < p < 2
\end{cases}$$

for all $x \in X$ and $n, m \in \mathbb{N}\cup\{0\}$. The remainder of the proof in the case of $0 < |k| < 1$, derived from the above inequality, is omitted because it proceeds very similar to the case of $1 < |k|$. □

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