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Technical Note

On the numerical solution of the point reactor kinetics equations

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ABSTRACT

The aim of this paper is to explore the 8th-order Adams-Bashforth-Moulton (ABM8) method in the solution of the point reactor kinetics equations. The numerical experiment considers feedback reactivity by Doppler effects, and insertions of reactivity. The Doppler effects is approximated with an adiabatic nuclear reactor that is a typical approximation. The numerical results were compared and discussed with several solution methods. The CATS method was used as a *benchmark* method. According with the numerical experiments results, the ABM8 method can be considered as one of the main solution method for changes reactivity relatively large.

1. Introduction

The point reactor kinetics equations can be considered as the most famous reducer-order model of reactor physics that describe the dynamics behavior of neutron density and concentration of delayed neutron precursors. The reactivity is the crucial variable and constitutes a non-linear term with the product of the neutron density that has represented a challenge in the solution which has been widely studied in innumerable works, e.g. Refs. [1–16]. The reactivity is the input variable of the point equations and has been modelled as: step reactivity, ramp reactivity, sinusoidal reactivity, and temperature feedback reactivity. The previous works represented important advances in the solution and analysis of the point equations of the reactor. A brief description of some works that are fundamental in this study are presented below.

Aboanber and Hamada [17] developed the PWS (Power Series Solution) method to solve the point reactor kinetics with six groups of delayed neutron and fuel temperature feedback. The PWS method is based on straightforward recurrence relation of a power series. Kinard and Allen [18] developed the Piecewise Constant Approximation (PCA) method that represent an efficient of numerical solution. The PCA method to calculate the solution to the point reactor kinetics equations is based in piecewise constant approximations of the reactivity and neutron source functions. Nahla [19] proposes an analytical solution based on the roots of the *inhour* equation which are the eigenvalues of coefficient matrix. The analytical solution for solve the point equations was named the new analytical method (NAM). The point reactor kinetics equations with fuel temperature feedback do not have an exact analytical solution: However, Nahla [20] developed an efficient technique that is based in changing the non-linear system to a linear system to estimate the reactivity value. The efficient technique is based on backward Euler (ETBE) and Crank Nicholson (ETCN) approximations. The best estimate method was developed by Ganapol [8] for accurate solutions of the point equations inclusive with nonlinear reactivity, which was called CATS (Convergent Accelerated Taylor Series) method, additionally their work is the *benchmark* method [21]. The CATS method is based on non-linear and linear convergence accelerations which are employed for the evaluation of Taylor series expansions and extrapolated values of neutron density and concentration of delayed neutron precursors. Picca et al. [22] solved the point kinetic equations with a new technique called the enhanced piecewise constant approximation (EPCA) which corrects the error in the source term iteratively with good accuracy, which is an improvement to the Kinard and Allen [18] method. Finally, Leite et al. [23] presents the ITS2 method which is based on low-order Taylor series expansions of the neutron density and reactivity functions to obtain an explicit solution of the point equations.

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In this work we explored and evaluated the 8th-order Adams-Bashforth-Moulton (ABM8) method to solve the point reactor kinetics with adiabatic Doppler effects and compensated response of ramp reactivity. The ABM8 method is based on *predictor–corrector* methods, the overall computation in a step consists of a preliminary prediction (Adams-Bashforth) of the answer, followed by a correction (Adams-Moulton) of this first predicted value. The results obtained of the numerical experiments were compared with different works given by Refs. [17–23], where CATS is the *benchmark* method [8,21].

2. Theoretical considerations

The point reactor kinetics equations are obtained with the P_1 approximation of the transport theory to describe the distribution of neutrons in a nuclear reactor, whose simplifications are: one-speed approximation v that is characterized by single neutron energy, the angular flux is linearly anisotropic, and the neutron source term $S(\mathbf{r},t)$ is isotropic. Under these simplifications the neutron flux $\varphi(\mathbf{r},t)$ and the current vector $\mathbf{J}(\mathbf{r},t)$, known as *Fick's law*. According with *Fick's law*, the collision frequency $(v\Sigma_t)$ must be much greater in cooperation with $((1/|\mathbf{J}|)(d|\mathbf{J}|/dt)$ [24]. Then, the point reactor kinetics equations with reactivity feedback with adiabatic approximation is given by

$$\frac{dn}{dt}=\frac{\rho(t)-\beta}{\Lambda}n(t)+\sum_{i=1}^{m}\lambda_i C_i(t) \tag{1}$$

$$\frac{dC_i}{dt}=\frac{\beta_i}{\Lambda}n(t)-\lambda_i C_i(t),\ i=1,2,\ldots,m \tag{2}$$

The initial conditions are $n(0)=n_0$ and $C_i(0)=C_{0i}=\frac{\beta_i}{\Lambda\lambda_i}n_0$. In this equations β_i is the precursors fraction of delayed neutrons of i-th group, λ_i is the decay constant of i-th group, and C_i is the precursors concentration of delayed neutrons of i-th group, and Λ is the mean neutron generation time. The numerical analysis of the reactivity ρ by Doppler temperature and ramp insertions of reactivity are of the objective of this work.

2.1. Temperature feedback reactivity

In thermal reactors, the Doppler feedback is due primarily to epithermal capture resonances in the non-fissionable fuel isotopes as well as Th^{232}, U^{238}, and Pu^{240}. The Doppler temperature coefficient of reactivity is calculated by Ref. [24]:

$$\alpha_{T_f}^{D}=\frac{\partial\rho}{\partial T_f} \tag{3}$$

Considered some point at the equilibrium (ρ_0 and T_0), the reactivity by Doppler effect is given by:

$$\rho(t)=\rho_0-\alpha_{T_f}^{D}\left(T_f(t)-T_0\right) \tag{4}$$

where T_f is the fuel temperature defined by the adiabatic approximation, i.e., without removal of generated heat in the core of the nuclear reactor:

$$\frac{dT_f}{dt}=H\,n(t) \tag{5}$$

where $H=P_0/(\rho_f\,Cp_f)$ is the reciprocal of the reactor heat capacity. Eqs. (4) and (5) are widely applied in nuclear reactor dynamics analysis with Doppler feedback, this equation was presented more than 40 years ago by Hetrick [25]. The adiabatic approximation for Doppler feedback represents more limitative case in nuclear reactors relative to an insertion of positive reactivity, because it considers the hypothetical scenarios such as a power excursion without heat removal, where the Doppler effects generate a reactor self-control keeping satisfactory safety margins. Now, we derive Eq. (4) with respect to time and when Eq. (5) is applied, leads to:

$$\frac{d\rho}{dt}=-\ \alpha_{T_f}^{D}H\,n(t) \tag{6}$$

This equation represents the step change with temperature feedback.

2.2. Ramp reactivity insertions

Now, the compensated response to ramp reactivity insertions can be represented by:

$$\rho(t)=a(t-t_0)-b\int_{t_0}^{t}n(t')dt' \tag{7}$$

or

$$\frac{d\rho}{dt}=a-b\,n(t) \tag{8}$$

The first term on the right of this equation represents the impressed reactivity variation whereas the last term, with b the shutdown coefficient, represents the reactivity feedback.

The initial condition for Eqs. (6) and (8) is $\rho(0)=\rho_0$.

3. Adams-Bashforth- Moulton method

In this section we present the numerical method for obtaining the high level of precision required. Based on the generalized formula for the integration of a differential equation of the type:

$$y_{k+1}=\sum_{i=1}^{p}\alpha_i y_{k+1-i}+h\sum_{i=0}^{p}\beta_i y'_{k+1-i}+E \tag{9}$$

where E is the error. The constants α_i and β_i are chosen to give the highest possible order. It is important to note that when $\beta_0=0$ correspond to an explicit formula or Adams-Bashforth method, i.e., explicit equations or predictors, but if $\beta_0\neq 0$ correspond to Adams-Moulton method, in this case it refers to an implicit or corrector equation.

The eighth order predictor is obtained:

$$P_{k+1}=\sum_{i=1}^{8}\alpha_i y_{k+1-i}+h\sum_{i=1}^{8}\beta_i y'_{k+1-i}+E_p \tag{10}$$

where E_p is the error in the predictor.

The differential equation with constant coefficients for the eighth order corrector is given by:

$$y_{k+1}=\sum_{i=1}^{7}\alpha_i y_{k+1-i}+h\sum_{i=0}^{7}\beta_i y'_{k+1-i}+E_c \tag{11}$$

where E_c is the error in the corrector.

The Adams-Bashforth-Moulton method can solve differential equations of the form $y'=f(t_k,y_k)$ in two steps. The first step is to approximate $f(t,\mathbf{y})$ with the Lagrange polynomial of degree 8 with initial conditions that can be obtained with the 4th-order Runge Kutta method. In this work we take these previous points with time

step $(h/10)$. Using Eq. (9), the coefficients α_i, β_i for $i = 1, \ldots, 8$ are calculated with a Taylor series expansion of $y_{k-1}, y_{k-2}, \ldots, y_{k-7}, y'_{k-1}, y'_{k-2}, \ldots, y'_{k-7}$ in order to obtain the eighth-order predictor of the Adams-Bashforth method [26], which is represented by the following equation:

$$p_{k+1} = y_k + \frac{h}{120960}(434241y'_k - 1152169y'_{k-1} + 2183877y'_{k-2} - 2664477y'_{k-3} + 2102243y'_{k-4} - 1041723y'_{k-5} + 295767y'_{k-6} - 36799y'_{k-7}) \tag{12}$$

The error estimation must be taken into account that $E_p = \frac{E}{9!}h^9 y_k^{(9)}(\xi)$, by equaling the coefficients given by Eq. (12) with the Taylor series expansions. Then the predictive error obtained is given by:

$$E_p = \frac{1070017}{3628800}h^9 y_k^{(9)}(\xi_1) \tag{13}$$

Similarly, the second step is to repeat the same procedure with the updated eight points. The corrector of the Adams-Moulton method [26] can be obtained from the fundamental theorem of calculus, and the Lagrange's interpolator polynomial using Eq. (11), we obtain:

$$y_{k+1} = y_k + \frac{h}{120960}(36799y'_{k+1} + 139849y'_k - 121797y'_{k-1} + 123133y'_{k-2} - 88547y'_{k-3} + 41499y'_{k-4} - 11351y'_{k-5} + 1375y'_{k-6}) \tag{14}$$

and the corrector error obtained is given by:

$$E_c = -\frac{33953}{3628800}h^9 y_k^{(9)}(\xi_2) \tag{15}$$

In order to obtain better results under the assumption that $y_k^{(9)}(\xi_1) \approx y_k^{(9)}(\xi_2) = K$, where K is a constant, we can use Eqs. (12)-(15) to find the predictor's modifier and the corrector's modifier:

$$mp_{k+1} = p_{k+1} + \frac{1070017}{1103970}(y_k - p_k) \tag{16}$$

and

$$my_{k+1} = y_{k+1} - \frac{33953}{1103970}(y_{k+1} - p_{k+1}) \tag{17}$$

respectively.

It ss possible to demonstrate using Eqs. (12) and (14) that the convergence of the method determines the time step as follows:

$$h < \frac{120960}{36799\left|f_y(\xi)\right|} \tag{18}$$

where $\left|f_y(\xi)\right|$ indicates the maximum value that the derivative takes at the point ξ. Then, applying Eq. (18) to Eqs. (1) and (2) we obtain:

$$h < \frac{120960}{36799\lambda_i} \tag{19}$$

$$h < \frac{120960\Lambda}{36799|\rho(t) - \beta|} \tag{20}$$

In the next section, the ABM8 method will be presented with constant time step size, although it is different from the methods that have been used for approximately 45 years with variable step [27].

4. Numerical experiments

The numerical experiments presented in this section have as main objective to analyze the point reactor kinetics equations with six groups of delayed neutron precursors. Besides, the exactitude of the 8th-order Adams-Bashforth-Moulton (ABM8) predictor-corrector method is discussed and compared with previous

Table 1
Comparison of the neutron density peak for different step change of reactivity.

Method	$\rho_0 = 1.0$		$\rho_0 = 1.5$		$\rho_0 = 2.0$		[a]Max. Rel. Error (%)
	Peak	Time (s)	Peak	Time (s)	Peak	Time (s)	
NAM [19]	807.8666	0.953	43025.93	0.168	167856.6	0.098	0.0065
ETBE [20]	805.1905	0.955	41183.77	0.168	153735.3	0.098	8.4068
ETCN [20]	806.5291	0.954	42093.39	0.168	160612.7	0.098	4.3093
CATS [8,21]	807.8681	0.953	43024.61	0.168	167845.7	0.098	0
ITS2 [23]	807.8681	0.953	43024.60	0.168	167845.6	0.098	1.131×10^{-5}
ABM8 ($h = 0.001s$)	807.8676	0.954	43020.82	0.168	167738.9	0.098	0.0636
ABM8 ($h = 0.0001s$)	807.8681	0.954	43024.60	0.168	167845.6	0.098	1.131×10^{-5}

[a] The maximum relative error was found in $\rho_0 = 2.0$.

Table 2
Neutron density for step change reactivity of $\rho_0 = 1.0$.

Time (s)	PCA [18]	NAM [19]	ETBE [20]	ETCN [20]	CATS[a] [8,21]	ITS2 [23]	ABM8(This work)
0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
10	132.038654	132.0391	132.1052	132.0679	132.0385964	132.0385964	132.0385964
20	51.6998767	51.69971	51.72712	51.71622	51.69986095	51.69986095	51.69986095
30	28.1746922	28.17450	28.18947	28.18784	28.17468536	28.17468536	28.17468536
40	18.1463339	18.14629	18.15454	18.15552	18.14633000	18.14633000	18.14633000
50	12.7795795	12.77951	12.78886	12.78617	12.77957703	12.77957703	12.77957703
60	9.47493414	9.475042	9.482830	9.489881	9.474932501	9.474932501	9.474932501
70	7.24447861	7.244338	7.248503	7.232454	7.244477494	7.244477494	7.244477494
80	5.64629045	5.646299	5.646767	5.653475	5.646289700	5.646289700	5.646289700
90	4.45683475	4.456941	4.458845	4.468407	4.456834255	4.456834255	4.456834255
100	3.55010308	3.550194	3.550773	3.553165	3.550102766	3.550102766	3.550102766
Max. Error (%)	0.00576	0.0504	6.6604	2.9304	0	0	0

[a] BEFD method.

Table 3
Neutron density for step change reactivity $\rho_0 = 1.5$.

Time(s)	PCA [18]	NAM [19]	ETBE [20]	ETCN [20]	CATS[a] [8,21]	ITS2 [23]	ABM8(This work)
0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
10	107.880817	107.9093	108.5460	108.2318	107.9116832	107.9116832	107.9116832
20	41.5931518	41.60327	41.83258	41.72055	41.60428128	41.60428128	41.60428128
30	23.2933257	23.29860	23.41236	23.35934	23.29893150	23.29893150	23.29893150
40	15.2999964	15.30321	15.36867	15.33605	15.30342749	15.30342749	15.30342749
50	10.8877981	10.89008	10.93202	10.90965	10.89014315	10.89014315	10.89014315
60	8.09931761	8.101000	8.130170	8.115094	8.101031859	8.101031859	8.101031859
70	6.18138304	6.182614	6.204616	6.193371	6.182690459	6.182690459	6.182690459
80	4.79228358	4.793221	4.812251	4.810045	4.793307820	4.793307820	4.793307820
90	3.75479837	3.755480	3.769579	3.765779	3.755614629	3.755614629	3.755614629
100	2.96541706	2.966149	2.977983	2.965561	2.966074952	2.966074952	2.966074952
Max. Error (%)	3.09	0.24	63.43	32.01	0	0	0

[a] BEFD method.

works [17–23]. Where CATS is the benchmark method [8,21], considered as one of the best and exact methods applied to solve the point reactor kinetics equations. The parameters for typical of a U^{235} graphite reactor are: $\lambda_1 = 0.0124s^{-1}$, $\lambda_2 = 0.0305s^{-1}$, $\lambda_3 = 0.111s^{-1}$, $\lambda_4 = 0.301s^{-1}$, $\lambda_5 = 1.13s^{-1}$, $\lambda_6 = 3.0s^{-1}$, $\beta_1 = 0.00021$, $\beta_2 = 0.00141$, $\beta_3 = 0.00127$, $\beta_4 = 0.00255$, $\beta_5 = 0.00074$, $\beta_6 = 0.00027$, and $\Lambda = 5.0 \times 10^{-5}$s. After defining these values, we can determine the time step, using Eq. (19), taking the maximum value, i.e., $\lambda_6 = 3.0s^{-1}$ we obtain $h < 1.09s$. Now, if we use Eq. (20), which depend on the reactivity value, assuming that radioactivity can vary between $-25\beta \leq \rho \leq 25\beta$, we obtain $h \leq 0.001s$. These values for the time step are consistent with the values used as starters of the variable steps methods in the literature.

The numerical experiments started from initial conditions given by $n_0 = 1\ ns/cm^3$ (normalized value), and $C_{0i}\ (ns/cm^3)$: $C_{01} = 338.7097$, $C_{02} = 924.5902$, $C_{03} = 228.8288$, $C_{04} = 169.4352$, $C_{05} = 13.0973$, and $C_{06} = 1.8$.

4.1. Step change with temperature feedback

The numerical experiments are analyzed with three different initial reactivity ρ_0 of 1.0$, 1.5 $ and 2.0 $. The parameters $H = 0.05\ K\ cm^3 s^{-1}$ and $\alpha^D_{T_f} = 5 \times 10^{-5} K^{-1}$ were used in the temperature and reactivity equations for feedback effects. In these numerical experiments the time step $h = 0.001s$ was used.

In this work the CATS [8,21] method is considered as the *benchmark* in order to assess the ABM8 method with respect to PCA [18], NAM [19], ETBE [20], -ETCN [20], and ITS2 [23] methods.

The performance of the ABM8 method and the comparison with other methods is presented in Table 1. In this table it can be observed the magnitude of the neutron density peak as a function of time. Then, considering the CATS method as the best estimation

Table 4
Neutron density for step change reactivity of $\rho_0 = 2.0$.

Time (s)	PCA [18]	NAM [19]	ETBE [20]	ETCN [20]	CATS[a] [8,21]	ITS2 [23]	ABM8 (this work)
0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
10	103.007499	103.5110	105.1118	104.2632	103.3808535	103.3808536	103.3808535
20	39.0022317	39.17932	39.77467	39.46449	39.13886903	39.13886903	39.13886903
30	21.9322404	22.02499	22.33538	22.17502	22.00377721	22.00377721	22.00377721
40	14.4485886	14.50649	14.69613	14.59655	14.49367193	14.49367193	14.49367193
50	10.2872358	10.32771	10.45320	10.38715	10.31861108	10.31861108	10.31861108
60	7.64017390	7.669895	7.761803	7.706731	7.663319203	7.663319203	7.663319203
70	5.81170208	5.834524	5.904826	5.868982	5.829395378	5.829395378	5.829395378
80	4.48559768	4.502759	4.559800	4.535230	4.499427073	4.499427073	4.499427073
90	3.49645747	3.509922	3.555554	3.532518	3.507422663	3.507422663	3.507422663
100	2.74634756	2.757519	2.795546	2.766554	2.755126886	2.755126886	2.755126886
Max. Error. (%)	37.34	13.01	173.09	88.23	0	0	0

[a] BEFD method.

Table 5a
Neutron density in ramp changes of reactivity for $a = 0.003s^{-1}$.

Time (s)	EPCA [22]	CATS* [28]	ITS2 [23]	ABM8 ($h = 0.001s$)
0	1.000000000	1.000000000	1.000000000	1.000000000
0.1	1.045371667	1.0453716665	1.0453716665	1.0453716665
0.5	1.324661986	1.3246619862	1.3246619862	1.3246619862
**5	3.215676113	3.2156761131	3.2156761131	3.2156761131
**7.5	3.210205182	3.2102051821	3.2102051821	3.2102051821
**10	3.145614687	3.1456146867	3.1456146867	3.1456146867
Max. Error (%) at $t = 0.1s$	**5** $\times 10^{-8}$	0	0	0

*BEFD method; ** $\times 10^{10}$.

Table 5b
Neutron density in ramp changes of reactivity for $a = 0.01\ s^{-1}$.

Time (s)	EPCA [22]	CATS (see Ref. [23])	ITS2 [23]	ABM8 ($h = 0.001s$)
0	1.000000000	1.0000000000	1.000000000	1.000000000
0.1	1.167210838	1.1672108379	1.1672108379	1.1672108379
0.5	4.269952865	4.2699528644	4.2699528644	4.2699528644
*5	1.033889665	1.0338896655	1.0338896655	1.0338896655
*7.5	1.019499913	1.0194999125	1.0194999125	1.0194999125
*10	1.012434883	1.0124348832	1.0124348832	1.0124348832
Max. Error (%) at $t = 0.1s$	4.90 $\times 10^{-8}$	0	0	0

* BEFD method; ** $\times 10^{11}$.

Table 5c
Neutron density in ramp changes of reactivity with $a = 0.1s^{-1}$.

Time (s)	EPCA [22]	CATS* [28]	ITS2 [23]	ABM8 ($h = 0.001s$)	ABM8 ($h = 0.0001s$)
0	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.1	24.73365830	24.733658251	24.733658251	24.733658247	24.733658251
**0.5	15.43361754	15.433617863	15.433617863	15.433623754	15.433617863
**5	10.02974092	10.029740921	10.029740921	10.029740921	10.029740921
**7.5	10.01798437	10.017984372	10.017984372	10.017984372	10.017984372
**10	10.01188621	10.011886207	10.011886207	10.011886207	10.011886207
Max. Error (%) at $t = 0.5s$	3.23 $\times 10^{-5}$	0	0	5.89 $\times 10^{-4}$	0

*BEFD method; ** $\times 10^{11}$.

Table 6a
Peak of the neutron density in compensated ramp with $a = 0.003s^{-1}$ and $b = 10^{-11}\text{cm}^3/\text{s}$.

Method	Peak ($\times 10^9$)	Time (s)	Max. Rel. Error (%)
ITS2 [23]	5.11415988	2.910582	3.91 $\times 10^{-7}$
CATS* [8,21]	5.1141599	2.9105821	0
PWS [17]	5.0537	2.84999	1.1822
ABM8 ($h = 0.001s$)	5.11375187	2.911	0.0080
ABM8 ($h = 0.0001s$)	5.11415913	2.9106	1.51 $\times 10^{-5}$

*BEFD method.

method, it can be observed that for $\rho_0 = 1.0$, the ITS2 methods present best results, followed by the ABM8 method, i.e., ranks second in best results. Then, ABM8 method has best results than NAM, ETBE and ETCN methods. The ITS2 methods has greater accuracy in predicting the peak of neutron density, because the variable step length used by these methods is less than the neutron density overdraft. According with the results for $\rho_0 = 1.5$ and $\rho_0 = 2.0$, the ABM8 method ranks third in best results, now the NAM method is the best. It can be observed that the ABM8 and NAM methods present similar results, which is interesting because the NAM method is based in the analytical solution. The maximum relative error was found in $\rho_0 = 2.0$ which is presented in Table 1, where it can be observed that the ITS2 is close to the benchmark method (CATS), followed by the NAM and ABM8 methods. Regarding the methods ETCN and ETBE, they present errors of an order of magnitude greater than the ABM8 method. It is important to note that when $h = 0.0001s$, the ABM8 method has the same accuracy as the ITS2 method.

The transient behavior of neutron density with three cases of the step reactivity ρ_0 =1.0 $, 1.5 $ and 2.0 $, are presented in Tables 2–4. In order to evaluate the transient performance of the ABM8 method, two types of comparisons are made with previous solution methods. The first comparison is between the results of the ABM8 with PCA [18], NAM [19], ETBE [20], ETCN [20], ITS2 [23], and benchmark methods CATS [8,21].

The maximum error presented in Tables 2–4 is calculated respect to the benchmark method, which was found at 10s the elapsed time for the three cases analyzed. It can be observed that the ABM8 and ITS2 methods agree with the CATS method and confirms the extreme accuracy of the method for these numerical experiments. In these tables it can be observed that the ETBE and ETCN present greater error compared to other methods, and this increases as the step reactivity change increases. The EPCA [22] method that is not included in Table 2, but it was evaluated and compared with the CATS* method (CATS* mean BEFD method), and present a maximum error of 6.0 $\times 10^{-5}$% at 10s of elapsed time.

The second comparison is carried out by comparing the CPU

Table 6b
Peak of the neutron density in compensated ramp with $a = 0.1s^{-1}$ and $b = 10^{-11}\text{cm}^3/\text{s}$.

Method	Number of peaks	Peak ($\times 10^{10}$)	Time (s)	Max. Rel. Error (%)
ITS2 [23]	1	24.20381495	0.2246634	2.0658×10^{-7}
	2	1.624467974	0.4642663	
	3	1.153627981	0.6065470	
CATS* [8,21]	1	24.203815	0.22466344	0
PWS [17]	1	22.144	0.224	8.5103
ABM8 ($h = 0.001s$)	1	24.14039	0.225	0.2621
	2	1.624398	0.464	
	3	1.153595	0.607	
ABM8 ($h = 0.0001s$)	1	24.20307	0.2247	3.1×10^{-3}
	2	1.624467	0.4643	
	3	1.153628	0.6065	
ABM8 ($h = 0.00001s$)	1	24.2038083	0.22466	2.7682×10^{-5}
	2	1.624467960	0.46427	
	3	1.153627979	0.60655	
ABM8 ($h = 0.000001s$)	1	24.20381483	0.224663	7.0237×10^{-7}
	2	1.624467974	0.464266	
	3	1.153627981	0.606547	

*BEFD method.

time. The ITS2 method reports for the reactivity values of 1.0 $, 1.5 $ and 2.0 $ a CPU time of 22 s, 45 s and 52 s, respectively. The NAM, ETBE and ETCN methods report a values of 3.04s, 2.04 s and 2.01 s, respectively. While with the ABM8 method the CPU time is about 0.9s in these numerical experiments, in an HP Pro 2.7 GHz AMD A12 computer using Fortran. No convergence accelerations and without any criteria were employed in the ABM8 algorithm to reduce that time.

4.2. Compensated ramp change

Now, the numerical experiment considers the case of compensated response to ramp insertions of reactivity that is given by Eq. (8) for different values of a, using a fixed shutdown coefficient $b = 10^{-11}\text{cm}^3/\text{s}$. In these numerical experiments the time step $h = 0.001s$ was also used.

The numerical results of the transient behavior are presented in Tables 5a, 5b and 5c for $a = 0.003s^{-1}$, $a = 0.01\ s^{-1}$ and $a = 0.1s^{-1}$, respectively with shutdown coefficient $b = 10^{-13}cm^3/s$ and initial reactivity $\rho_0 = 0$. In these numerical experiments the time step $h = 0.001s$ was used para $a = 0.003s^{-1}$ and $a = 0.01\ s^{-1}$. In order to improve the numerical approximation, the time step $h = 0.0001s$ was used for $a = 0.1s^{-1}$. The CPU time of the ABM8 method is about $0.12s$ for $a = 0.003s^{-1}$, $a = 0.01\ s^{-1}$ and $a = 0.1\ s^{-1}$ with $h = 0.001s$, but with $h = 0.0001s$ for $a = 0.1\ s^{-1}$ the CPU time is about 0.90s.

The neutron density peak and respective time to peak with shutdown coefficient $b = 10^{-11}\text{cm}^3/\text{s}$ is presented in Tables 6a and 6b for $a = 0.003s^{-1}$ and $a = 0.1s^{-1}$, respectively. For $a = 0.003s^{-1}$ it can be observed in Table 6a that the PWS method presents the largest relative error followed by three orders of magnitude more exact the ABM8 method respect to the CATS* method. However, when we reduce the time step, better results are obtained. The ITS2 is more exact respect to the CATS* method. For $a = 0.1s^{-1}$ the numerical results of AMB8 and ITS2 methods present three peaks whose behavior is damped oscillatory, and the CATS* and PWS report a single peak of neutron density. According with the maximum relative error the ITS2 method present lower relative error followed by the ABM8 method.

We must make an observation due to the difficulty in comparing the methods reported in the literature and the method proposed in this work. Traditional methods use variable pitch, with the characteristic that you must first select the precision or tolerance that you want to obtain and the algorithm changes the step size. Otherwise, the proposed method (ABM8) uses a constant step, the precision obtained is observed and if it is desired to improve, the time step must be reduced. To achieve rigorous comparisons with the constant step method, the minimum step size used by the variable step method should be used.

5. Conclusions

The 8th-order Adams-Bashforth-Moulton (ABM8) method constitutes an easy to implement algorithm that provides results with good accuracy for different applications and is being generally both conceptually and structurally simple. The ABM method was applied to point reactor kinetics problems with adiabatic Doppler effects with temperature feedback of reactivity. The method is based on the predictor-corrector methods and modifiers with six delayed neutron groups. According with the numerical results and the comparison with previous methods, the ABM8 is another alternative solution comparable even with such powerful methods as the CATS [21] that was considered as a benchmark in this work. Another property that should be considered of the ABM8 method over other methods is the CPU times that could be very low.

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Appendix A. Supplementary data

Supplementary data to this article can be found online at https://doi.org/10.1016/j.net.2019.11.034.

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