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ON k SLOPE DIAGONAL SUMS OF q-COMMUTING TABLE AND NONZERO PAULI TABLE

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Abstract. We explore the Pauli table $C^{(-1)}$ and nonzero Pauli table W. Recurrence rules and interrelationships of any k slope diagonal sums over $C^{(-1)}$ and W are studied in connection with diagonal sums of the Pascal table $C^{(1)}$. Since diagonal sums of $C^{(1)}$ are Fibonacci numbers, any k slope diagonal sums over $C^{(-1)}$ and W are explained by Fibonacci numbers.

1. Introduction

The Pascal table is an arithmetic table (abbr. AT) of $(x+y)^m$ $(m \geq 0)$, in which commutativity $xy = yx$ was assumed implicitly. In the area of quantum theory, theoretical physicists P. Dirac and W. Pauli [5] have studied noncommuting variables x and y, in particular the AT of $(x+y)^m$ with $xy = -yx$ is called the Pauli Pascal table (Pauli table, short) ([1], [2]). As a generalization of noncommutativity, x and y are called q-commuting variables if $yx = qxy$ with $q \in \mathbb{Z}^*$, and the AT of $(x+y)^m$ with q-commuting x, y is called the q-commuting table $C^{(q)}$ [4]. Hence $C^{(1)}$ and $C^{(-1)}$ are the Pascal and Pauli tables, respectively. When expanding $(x+y)^n$ with $yx = -xy$, lots of coefficients are zeros, for instance $(x+y)^2 = x^2 + y^2$. We denote by W the AT of $(x+y)^n$ with $yx = -xy$ having no zero entries and call it the nonzero Pauli table.

In this work we study the Pauli table $C^{(-1)}$ and nonzero Pauli table W. By considering any k slope diagonal sums over the tables, we investigate recurrence rules and interrelationships in connection with the diagonal sums of the Pascal table $C^{(1)}$. Since diagonal sums in $C^{(1)}$ are Fibonacci numbers and those in $C^{(-1)}$ are the duplicated Fibonacci sequence, any k slope diagonal sums over $C^{(-1)}$ and W are explained by Fibonacci numbers.

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2. q-commuting arithmetic table $C^{(q)}$

We begin with general q-commuting AT $C^{(q)}$ of $(x+y)^n$ with $yx = qxy$ for $q \in \mathbb{Z}$. The followings are AT $C^{(q)}$ with $q = -1, -2, -3$.

$C^{(-1)}$

	0	1	2	3	4	5
0	1					
1	1	1				
2	1	0	1			
3	1	1	1	1		
4	1	0	2	0	1	
5	1	1	2	2	1	1

$C^{(-2)}$

0	1	2	3	4	5
1					
1	1				
1	−1	1			
1	3	3	1		
1	−5	15	−5	1	
1	11	55	55	11	1

$C^{(-3)}$

0	1	2	3	4	5
1					
1	1				
1	−2	1			
1	7	7	1		
1	−20	70	−20	1	
1	61	610	610	61	1

For example when $yx = -2xy$, we have $(x+y)^5 = x^5 + 11x^4y + 55x^3y^2 + 55x^2y^3 + 11xy^4 + y^5$ by $C^{(-2)}$. A recurrence in $C^{(q)} = [e^{(q)}_{i,j}]$ $(i, j \geq 0)$ is as follows.

Theorem 2.1. *For any q, $e^{(q)}_{i+1,j+1} = e^{(q)}_{i,j} + q^{j+1}e^{(q)}_{i,j+1} = q^{i-j}e^{(q)}_{i,j} + e^{(q)}_{i,j+1}$. In particular in $C^{(-1)}$, $e^{(-1)}_{i,j} = e^{(-1)}_{i-2,j-2} + e^{(-1)}_{i-2,j}$.*

Proof. Comparing the expansions of $(x+y)^{i-1}$ and $(x+y)^i$ with $yx = qxy$, it is not hard to see the first identity. Moreover

$$\begin{aligned} e^{(-1)}_{i,j} &= (-1)^{i-j}e^{(-1)}_{i-1,j-1} + e^{(-1)}_{i-1,j} \\ &= (-1)^{i-j}((-1)^{i-j}e^{(-1)}_{i-2,j-2} + e^{(-1)}_{i-2,j-1}) + ((-1)^{i-j-1}e^{(-1)}_{i-2,j-1} + e^{(-1)}_{i-2,j}) \\ &= e^{(-1)}_{i-2,j-2} + e^{(-1)}_{i-2,j}. \end{aligned}$$ □

Theorem 2.2. *The 1, 2th columns in $C^{(q)}$ hold recurrences $e^{(q)}_{i,1}e^{(q)}_{i+1,1} = (q+1)e^{(q)}_{i+1,2}$ and $e^{(q)}_{i,1}e^{(q)}_{i-1,2} = e^{(q)}_{i,2}e^{(q)}_{i-2,1}$.*

Proof. We observe the first identity from the above tables, for instance $(-20)(61) = (-3+1)610$ in $C^{(-3)}$. Assume $e^{(q)}_{i-1,1}e^{(q)}_{i,1} = (q+1)e^{(q)}_{i,2}$ for some i. Then

$$\begin{aligned} e^{(q)}_{i,1}e^{(q)}_{i+1,1} &= (1+qe^{(q)}_{i-1,1})(1+qe^{(q)}_{i,1}) = e^{(q)}_{i,1} + qe^{(q)}_{i,1} + q^2e^{(q)}_{i-1,1}e^{(q)}_{i,1} \\ &= (q+1)e^{(q)}_{i,1} + q^2(q+1)e^{(q)}_{i,2} = (q+1)(e^{(q)}_{i,1} + q^2e^{(q)}_{i,2}) = (q+1)e^{(q)}_{i+1,2}. \end{aligned}$$

The second identity is also true from tables. As an induction hypothesis, we assume $e^{(q)}_{i,1}e^{(q)}_{i-1,2} = e^{(q)}_{i,2}e^{(q)}_{i-2,1}$ for some i. Then Theorem 2.1 implies

$$e^{(q)}_{i+1,1}e^{(q)}_{i,2} = (1+qe^{(q)}_{i,1})(e^{(q)}_{i-1,1}+q^2e_{i-1,2}) = e^{(q)}_{i+1,1}e^{(q)}_{i-1,1}+q^2e^{(q)}_{i-1,2}+q^3e^{(q)}_{i,1}e^{(q)}_{i-1,2}.$$

On the other hand, we also have

$$e^{(q)}_{i+1,2}e^{(q)}_{i-1,1} = (e^{(q)}_{i,1} + q^2e^{(q)}_{i,2})(1+qe^{(q)}_{i-2,1}) = e^{(q)}_{i,1}e^{(q)}_{i-1,1} + q^2e^{(q)}_{i,2} + q^3e^{(q)}_{i,1}e^{(q)}_{i-1,2}.$$

So comparing $e^{(q)}_{i+1,1}e^{(q)}_{i,2}$ and $e^{(q)}_{i+1,2}e^{(q)}_{i-1,1}$, it is enough to prove $e^{(q)}_{i,1}e^{(q)}_{i-1,1} + q^2e^{(q)}_{i,2} = e^{(q)}_{i+1,1}e^{(q)}_{i-1,1} + q^2e^{(q)}_{i-1,2}$. In fact, Theorem 2.1 immediately yields

$$\begin{aligned} e^{(q)}_{i,1}e^{(q)}_{i-1,1} + q^2e^{(q)}_{i,2} &= e^{(q)}_{i,1}e^{(q)}_{i-1,1} + q^2(q^{i-2}e^{(q)}_{i-1,1} + e^{(q)}_{i-1,2}) \\ &= (e^{(q)}_{i,1} + q^i)e^{(q)}_{i-1,1} + q^2e^{(q)}_{i-1,2} = e^{(q)}_{i+1,1}e^{(q)}_{i-1,1} + q^2e^{(q)}_{i-1,2}. \end{aligned}$$ □

The diagrams $\begin{array}{c|c|c} & 1 & 2 \\ \hline i & \beta & \\ \hline i+1 & \alpha & \gamma \end{array}$ satisfying $\alpha\beta = (q+1)\gamma$ and $\begin{array}{c|c|c} & 1 & 2 \\ \hline i-1 & \delta & \\ \hline i & & \beta \\ \hline i+1 & \alpha & \gamma \end{array}$ satisfying $\alpha\beta = \gamma\delta$ explain Theorem 2.2. Let $D_n^{(q)}$ be the nth diagonal (abbr. diag.) sum of $C^{(q)}$, which is the sum of all entries over diagonal starting from $e_{n,0}^{(q)}$. Thus

$$D_n^{(q)} = e_{n,0}^{(q)} + e_{n-1,1}^{(q)} + \cdots + e_{0,n}^{(q)} = e_{n,0}^{(q)} + e_{n-1,1}^{(q)} + \cdots + e_{\lfloor \frac{n+1}{2} \rfloor, \lfloor \frac{n}{2} \rfloor}^{(q)}$$

since $e_{i,j}^{(q)} = 0$ for $i < j$. By a weighted nth diag. sum of $C^{(q)}$, we define

$$\begin{aligned} \check{D}_n^{(q)} &= e_{n,0}^{(q)} + qe_{n-1,1}^{(q)} + q^2 e_{n-2,2}^{(q)} + \cdots + q^{\lfloor \frac{n}{2} \rfloor} e_{\lfloor \frac{n+1}{2} \rfloor, \lfloor \frac{n}{2} \rfloor}^{(q)} \\ &= (e_{n,0}^{(q)}, e_{n-1,1}^{(q)}, e_{n-2,2}^{(q)}, \cdots, e_{\lfloor \frac{n+1}{2} \rfloor, \lfloor \frac{n}{2} \rfloor}^{(q)}) \circ (1, q, q^2, \cdots, q^{\lfloor \frac{n}{2} \rfloor}) \end{aligned}$$

by utilizing inner product operation $\circ$.

Theorem 2.3. *For any q, $D_n^{(q)} = D_{n-2}^{(q)} + \check{D}_{n-1}^{(q)}$ for all $n \geq 2$.*

Proof. Consider $C^{(q)} = [e_{i,j}^{(q)}]$. Due to Theorem 2.1, we have

$$\begin{aligned} D_n^{(q)} &= e_{n,0}^{(q)} + e_{n-1,1}^{(q)} + e_{n-2,2}^{(q)} + e_{n-3,3}^{(q)} + \cdots \\ &= e_{n,0}^{(q)} + (e_{n-2,0}^{(q)} + qe_{n-2,1}^{(q)}) + (e_{n-3,1}^{(q)} + q^2 e_{n-3,2}^{(q)}) + (e_{n-4,2}^{(q)} + q^3 e_{n-4,3}^{(q)}) + \cdots \\ &= (e_{n-2,0}^{(q)} + e_{n-3,1}^{(q)} + e_{n-4,2}^{(q)} + \cdots) + (e_{n-1,0}^{(q)} + qe_{n-2,1}^{(q)} + q^2 e_{n-3,2}^{(q)} + \cdots) \\ &= D_{n-2}^{(q)} + \check{D}_{n-1}^{(q)}. \end{aligned}$$

□

3. Generalized diagonal sums of $C^{(-1)}$

A diagonal usually means a line toward northeast direction that moves 1 steps rightward and upward. A generalized diagonal that moves s steps rightward and k steps upward is called a k/s slope diagonal for $k, s \geq 0$. In $C^{(q)}$, by considering $k/1$ slope and $1/k$ slopes, let $D_{<k/1>,i}^{(q)} = D_{<k>,i}^{(q)}$ and $D_{<1/k>,i}^{(q)}$ be the $k/1$ and $1/k$ slope diag. sums respectively, starting from $e_{i,0}^{(q)}$. So 1 slope diag. sum $D_{<1>,i}^{(q)}$ is an ordinary diagonal sum and 0 slope diag. sum $D_{<0>,i}^{(q)}$ is a row sum.

Over $C^{(-1)}$, clearly $\{D_{<1>,i}^{(-1)}\} = \{D_i^{(-1)}\} = \{1, 1, 2, 1, 3, 2, 5, 3, 8, \cdots\}$. Since 2 slope diagonals are $\{1, 1\}$, $\{1, 0\}$, $\{1, 1\}$, $\{1, 0, 1\}$, $\{1, 1, 1\}$, $\{1, 0, 2\}$, $\{1, 1, 2, 1\}$, $\{1, 0, 3, 0\}$, $\{1, 1, 3, 2\}$, $\{1, 0, 4, 0, 1\}$, $\{1, 1, 4, 3, 1\}$, etc., the diag. sums are

$$\{D_{<2>,i}^{(-1)}\} = \{1, 1, 1, 2, 1, 2, 2, 3, 3, 5, 4, 7, 6, 10, 9, 15, 13, 22, 19, 32, 28, \cdots\}$$

Also from 3 slope diagonals $\{1, 1, 1\}$, $\{1, 0, 1\}$, $\{1, 1, 2\}$, $\{1, 0, 2\}$, $\{1, 1, 3, 1\}$, etc, the diag. sums are $\{D_{<3>,i}^{(-1)}\} = \{1, 1, 1, 1, 2, 1, 2, 1, 3, 2, 4, 3, 6, 4, 8, 5, 11 \cdots\}$.

$k/1$ slope diag. sums $D_{<k>,i}^{(-1)}$ of $C^{(-1)}$ $(0 \leq k \leq 5)$

$k \backslash i$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
0	1	2	2	4	4	8	8	16	16	32	32	64	64	128	128	256	256	512						
1	1	1	2	1	3	2	5	3	8	5	13	8	21	13	34	21	55	34	89	55	144	89	233	144
2	1	1	1	2	1	2	2	3	3	5	4	7	6	10	9	15	13	22	19	32	28	47	41	69
3	1	1	1	1	2	1	2	1	3	2	4	3	6	4	8	5	11	7	15	10	21	14	29	19
4	1	1	1	1	1	2	1	2	1	2	2	3	3	4	4	6	5	8	6	10	8	13	11	17
5	1	1	1	1	1	1	2	1	2	1	2	1	3	2	4	3	5	4	7	5	9	6	11	7

From the above table, clearly $D^{(-1)}_{<k>,i} = 1$ $(0 \le i \le k)$ and $D^{(-1)}_{<1>,i}$ are Fibonacci numbers and duplicated, and $D^{(-1)}_{<0>,i}$ are powers of 2 and duplicated.

Lemma 3.1. *In $C^{(-1)} = [e^{(-1)}_{i,j}]$, $e^{(-1)}_{i,1} = 1$ if i is odd, and 0 if i is even. Moreover $e^{(-1)}_{i,j} = 0$ when i is even and j is odd.*

Proof. Clearly $e^{(-1)}_{2,1} = e^{(-1)}_{1,0} - e^{(-1)}_{1,1} = 0$ and $e^{(-1)}_{3,1} = e^{(-1)}_{2,0} - e^{(-1)}_{2,1} = 1$, for $e^{(-1)}_{1,1} = 1$. Continuing we have $e^{(-1)}_{i,1} = 1$ if i is odd and $e^{(-1)}_{i,1} = 0$ if i is even.

With even i and odd j, we assume $e^{(-1)}_{i,j} = 0$. Then

$$\begin{aligned} e^{(-1)}_{i+2,j} &= e^{(-1)}_{i+1,j-1} + (-1)^j e^{(-1)}_{i+1,j} \\ &= (e^{(-1)}_{i,j-2} + (-1)^{j-1} e^{(-1)}_{i,j-1}) - (e^{(-1)}_{i,j-1} + (-1)^j e^{(-1)}_{i,j}) = 0. \end{aligned}$$ □

Theorem 3.2. *The sequence $\{D^{(-1)}_{<k>,i}\}$ with $k = 1, 2$ satisfy the followings.*

(1) $D^{(-1)}_{<1>,i} = D^{(-1)}_{<1>,i-1} + (-1)^i D^{(-1)}_{<1>,i-2}$ with two initials $1, 1$. Also $D^{(-1)}_{<1>,i} = D^{(-1)}_{<1>,i-2} + D^{(-1)}_{<1>,i-4}$ with four initials $1, 1, 2, 1$.

(2) $D^{(-1)}_{<2>,i} = D^{(-1)}_{<2>,i-2} + D^{(-1)}_{<2>,i-6}$ with 6 initials $1, 1, 1, 2, 1, 2$.

Proof. The $\{D^{(-1)}_{<1>,i}\} = \{1, 1, 2, 1, 3, 2, 5, 3, 8, \cdots\}$ shows $D^{(-1)}_{<1>,i} = D^{(-1)}_{<1>,i-1} + (-1)^i D^{(-1)}_{<1>,i-2}$ for some $i > 0$. We simply write $e^{(-1)}_{i,j}$ by $e_{i,j}$. Then

$$\begin{aligned} D^{(-1)}_{<1>,i} &= e_{i,0} + e_{i-1,1} + e_{i-2,2} + e_{i-3,3} + e_{i-4,4} + \cdots \\ &= e_{i,0} + ((-1)^{i-2} e_{i-2,0} + e_{i-2,1}) + ((-1)^{i-4} e_{i-3,1} + e_{i-3,2}) \\ &\quad + ((-1)^{i-6} e_{i-4,2} + e_{i-4,3}) + \cdots \\ &= (e_{i-1,0} + e_{i-2,1} + e_{i-3,2} + \cdots) + (-1)^i (e_{i-2,0} + e_{i-3,1} + e_{i-4,2} + \cdots) \\ &= D^{(-1)}_{<1>,i-1} + (-1)^i D^{(-1)}_{<1>,i-2}, \end{aligned}$$

by Theorem 2.1. Furthermore we also have

$$\begin{aligned} D^{(-1)}_{<1>,i} &= D^{(-1)}_{<1>,i-1} + (-1)^i D^{(-1)}_{<1>,i-2} \\ &= (D^{(-1)}_{<1>,i-2} + (-1)^{i-1} D^{(-1)}_{<1>,i-3}) + (-1)^i (D^{(-1)}_{<1>,i-3} + (-1)^{i-2} D^{(-1)}_{<1>,i-4}) \\ &= D^{(-1)}_{<1>,i-2} + D^{(-1)}_{<1>,i-4}. \end{aligned}$$

Now for (2), $\{D^{(-1)}_{<2>,i}\} = \{1, 1, 1, 2, 1, 2, 2, 3, 3, 5, 4, 7, 6, 10, 9, 15, \cdots\}$ shows $D^{(-1)}_{<2>,i} = D^{(-1)}_{<2>,i-2} + D^{(-1)}_{<2>,i-6}$ for some i. Then

$$\begin{aligned} D^{(-1)}_{<2>,i} &= e_{i,0} + e_{i-2,1} + e_{i-4,2} + e_{i-6,3} + e_{i-8,4} + e_{i-10,5} + \cdots \\ &= e_{i,0} + ((-1)^{i-3} e_{i-3,0} + e_{i-3,1}) + ((-1)^{i-6} e_{i-5,1} + e_{i-5,2}) \\ &\quad + ((-1)^{i-9} e_{i-7,2} + e_{i-7,3}) + ((-1)^{i-12} e_{i-9,3} + e_{i-9,4}) + \cdots \\ &= (e_{i-1,0} + e_{i-3,1} + e_{i-5,2} + e_{i-7,3} + e_{i-9,4} + \cdots) \\ &\quad + ((-1)^{i-1} e_{i-3,0} + (-1)^i e_{i-5,1} + (-1)^{i-1} e_{i-7,2} + (-1)^i e_{i-9,3} + \cdots). \end{aligned}$$

Let $A = (-1)^{i-1} e_{i-3,0} + (-1)^i e_{i-5,1} + (-1)^{i-1} e_{i-7,2} + \cdots$. Then

$$\begin{aligned} A &= (-1)^{i-1} e_{i-3,0} + (-1)^i ((-1)^{i-6} e_{i-6,0} + e_{i-6,1}) \\ &\quad + (-1)^{i-1} ((-1)^{i-9} e_{i-8,1} + e_{i-8,2}) + (-1)^i ((-1)^{i-12} e_{i-10,2} + e_{i-10,3}) + \cdots \\ &= (-1)^{i-1} e_{i-3,0} + (e_{i-6,0} + (-1)^i e_{i-6,1}) + (e_{i-8,1} + (-1)^{i-1} e_{i-8,2}) + \cdots \end{aligned}$$

$$= (e_{i-6,0} + e_{i-8,1} + e_{i-10,2} + \cdots) + ((-1)^{i-1}e_{i-3,0} + (-1)^{i}e_{i-6,1} + \cdots)$$
$$= D^{(-1)}_{<2>,i-6} + (-1)^{i-1}e_{i-4,0} + (-1)^{i}e_{i-6,1} + (-1)^{i-1}e_{i-8,2} + \cdots.$$

And let $B = e_{i-1,0} + e_{i-3,1} + e_{i-5,2} + e_{i-7,3} + e_{i-9,4} + \cdots$. Then

$$\begin{aligned} B &= e_{i-1,0} + ((-1)^{i-4}e_{i-4,0} + e_{i-4,1}) + ((-1)^{i-7}e_{i-6,1} + e_{i-6,2}) \\ &\quad + ((-1)^{i-10}e_{i-8,2} + e_{i-8,3}) + ((-1)^{i-13}e_{i-10,3} + e_{i-10,4}) + \cdots \\ &= (e_{i-2,0} + e_{i-4,1} + e_{i-6,2} + e_{i-8,3} + e_{i-10,4} + \cdots) \\ &\quad + ((-1)^{i}e_{i-4,0} + (-1)^{i-1}e_{i-6,1} + (-1)^{i}e_{i-8,2} + (-1)^{i-1}e_{i-10,3} + \cdots) \\ &= D^{-1}_{<2>,i-2} + ((-1)^{i}e_{i-4,0} + (-1)^{i-1}e_{i-6,1} + (-1)^{i}e_{i-8,2} + \cdots). \end{aligned}$$

Therefore $D^{(-1)}_{<2>,i} = A + B$ equals

$$\begin{aligned} D^{(-1)}_{<2>,i} &= D^{-1}_{<2>,i-2} + ((-1)^{i}e_{i-4,0} + (-1)^{i-1}e_{i-6,1} + (-1)^{i}e_{i-8,2} + \cdots) \\ &\quad + D^{(-1)}_{<2>,i-6} + ((-1)^{i-1}e_{i-4,0} + (-1)^{i}e_{i-6,1} + (-1)^{i-1}e_{i-8,2} + \cdots) \\ &= D^{-1}_{<2>,i-2} + D^{(-1)}_{<2>,i-6}. \end{aligned}$$

□

Now any k slope diag. sums $\{D^{(-1)}_{<k>,i}\}$ in $C^{(-1)}$ satisfy the followings.

Theorem 3.3. *For $i \geq 2(k+1)$, $D^{(-1)}_{<k>,i} = D^{(-1)}_{<k>,i-2} + D^{(-1)}_{<k>,i-2(k+1)}$ with $2(k+1)$ initials $\{\underbrace{1,\cdots,1}_{k+1},\underbrace{2,1,2,1,2,1,\cdots}_{k+1}\}$.*

Proof. When $k = 3, 4$, we list first few k slope diag. sums of $C^{(-1)}$. Clearly

$$\{D^{(-1)}_{<3>,i} \mid i \geq 0\} = \{\cdots, 4, 3, 6, 4, 8, 5, 11, 7, 15, 10, 21, 14, 29, 19, \cdots\},$$

and notice $21 = 15 + 6$, so $D^{(-1)}_{<3>,i} = D^{(-1)}_{<3>,i-2} + D^{(-1)}_{<3>,i-8}$. Similarly

$$\{D^{(-1)}_{<4>,i} \mid i \geq 0\} = \{\cdots, 6, 5, 8, 5, 6, 10, 8, 13, 11, 17, 15, 23, 20, 31, \cdots\}$$

and notice $20 = 15 + 5$, so $D^{(-1)}_{<4>,i} = D^{(-1)}_{<4>,i-2} + D^{(-1)}_{<4>,i-10}$.

Let us consider $D^{(-1)}_{<k>,i}$. If $0 \leq i \leq k$ then $D^{(-1)}_{<k>,i} = e^{(-1)}_{i,0} = 1$. If $k < i \leq 2k+1$ then $D^{(-1)}_{<k>,i} = e^{(-1)}_{i,0} + e^{(-1)}_{i-k,1} = \begin{cases} 1+1 & \text{if } i-k \text{ odd} \\ 1+0 & \text{if } i-k \text{ even} \end{cases}$ by Lemma 3.1. Hence the first $2(k+1)$ elements are $\{\underbrace{1,\cdots,1}_{k+1},\underbrace{2,1,2,1,2,1,\cdots}_{k+1}\}$. And

$$\begin{aligned} D^{(-1)}_{<k>,2k+2} &= e^{(-1)}_{2k+2,0} + e^{(-1)}_{k+2,1} + e^{(-1)}_{2,2} \\ &= \begin{cases} 1+1+1 & \text{if } k \text{ odd} \\ 1+0+1 & \text{if } k \text{ even} \end{cases} = D^{(-1)}_{<k>,2k} + D^{(-1)}_{<k>,0}. \end{aligned}$$

Now we write $e^{(-1)}_{i,j}$ by $e_{i,j}$ for convenient. Then for any i, we have

$$\begin{aligned} D^{(-1)}_{<k>,i} &= e_{i,0} + e_{i-k,1} + e_{i-2k,2} + e_{i-3k,3} + e_{i-4k,4} + \cdots \\ &= e_{i,0} + ((-1)^{i-k-1}e_{i-k-1,0} + e_{i-k-1,1}) + ((-1)^{i-2k-2}e_{i-2k-1,1} + e_{i-2k-1,2}) \\ &\quad + ((-1)^{i-3k-3}e_{i-3k-1,2} + e_{i-3k-1,3}) + ((-1)^{i-4k-4}e_{i-4k-1,3} + e_{i-4k-1,4}) + \cdots \\ &= (e_{i-1,0} + e_{i-k-1,1} + e_{i-2k-1,2} + e_{i-3k-1,3} + e_{i-4k-1,4} + \cdots) \\ &\quad + (-1)^{i-k-1}e_{i-k-1,0} + (-1)^{i}e_{i-2k-1,1} + (-1)^{i-k-1}e_{i-3k-1,2} + \cdots. \end{aligned}$$

Let $A = (-1)^{i-k-1}e_{i-k-1,0} + (-1)^{i}e_{i-2k-1,1} + (-1)^{i-k-1}e_{i-3k-1,2} + \cdots$. Then

$$
\begin{aligned}
A &= (-1)^{i-k-1}e_{i-k-1,0} + (-1)^i((-1)^{i-2k-2}e_{i-2k-2,0} + e_{i-2k-2,1}) \\
&\quad + (-1)^{i-k-1}((-1)^{i-3k-3}e_{i-3k-2,1} + e_{i-3k-2,2}) + \cdots \\
&= (-1)^{i-k-1}e_{i-k-1,0} + (-1)^i((-1)^i e_{i-2k-2,0} + e_{i-2k-2,1}) \\
&\quad + (-1)^{i-k-1}((-1)^{i-k-1}e_{i-3k-2,1} + e_{i-3k-2,2}) + \cdots \\
&= (e_{i-2k-2,0} + e_{i-3k-2,1} + e_{i-4k-2,2} + \cdots) \\
&\quad + (-1)^{i-k-1}e_{i-k-1,0} + (-1)^i e_{i-2k-2,1} + (-1)^{i-k-1}e_{i-3k-2,2} + \cdots \\
&= D_{<k>,i-2(k+1)} + (-1)^{i-k-1}e_{i-k-1,0} \\
&\quad + (-1)^i e_{i-2k-2,1} + (-1)^{i-k-1}e_{i-3k-2,2} + \cdots
\end{aligned}
$$

And let $B = e_{i-1,0} + e_{i-k-1,1} + e_{i-2k-1,2} + e_{i-3k-1,3} + \cdots$. Then

$$
\begin{aligned}
B &= e_{i-1,0} + ((-1)^{i-k-2}e_{i-k-2,0} + e_{i-k-2,1}) + ((-1)^{i-2k-3}e_{i-2k-2,1} + e_{i-2k-2,2}) \\
&\quad + ((-1)^{i-3k-4}e_{i-3k-2,2} + e_{i-3k-2,3}) + \cdots \\
&= e_{i-2,0} + e_{i-k-2,1} + e_{i-2k-2,2} + e_{i-3k-2,3} + e_{i-4k-2,4} + \cdots \\
&\quad + (-1)^{i-k}e_{i-k-2,0} + (-1)^{i-1}e_{i-2k-2,1} + (-1)^{i-k}e_{i-3k-2,2} + \cdots \\
&= D^{-1}_{<k>,i-2} + (-1)^{i-k}e_{i-k-2,0} + (-1)^{i-1}e_{i-2k-2,1} + (-1)^{i-k}e_{i-3k-2,2} + \cdots.
\end{aligned}
$$

Therefore, $D^{(-1)}_{<k>,i} = A + B$ equals

$$
\begin{aligned}
D^{(-1)}_{<k>,i} &= D^{-1}_{<k>,i-2} + D^{-1}_{<k>,i-2(k+1)} \\
&\qquad + (-1)^{i-k}e_{i-k-2,0} + (-1)^{i-1}e_{i-2k-2,1} + (-1)^{i-k}e_{i-3k-2,2} + \cdots \\
&\qquad + (-1)^{i-k-1}e_{i-k-1,0} + (-1)^i e_{i-2k-2,1} + (-1)^{i-k-1}e_{i-3k-2,2} + \cdots \\
&= D^{-1}_{<k>,i-2} + D^{-1}_{<k>,i-2(k+1)}.
\end{aligned}
$$

□

4. Subsequences of k slope diagonal sums

From the sequence $\{D^{(-1)}_{<1>,i}\} = \{\cdots, 1, 2, 1, 3, 2, 5, 3, 8, 5, 13, 8, 21, 13, \cdots\}$, let $\{D^{ev}_{<1>,j}\} = \{1, 2, 3, 5, 8, 13, 21, \cdots\}$ and $\{D^{od}_{<1>,j}\} = \{1, 1, 2, 3, 5, 8, 13, \cdots\}$ be subsequences consisting of eventh and oddth terms of $\{D^{(-1)}_{<1>,i}\}$, respectively. Then

$$\{D^{(-1)}_{<1>,i} \mid i \geq 0\} = \{D^{ev}_{<1>,j} \mid j \geq 0\} \cup \{D^{od}_{<1>,j} \mid j \geq 0\}.$$

Similarly we consider

$$
\begin{aligned}
\{D^{(-1)}_{<2>,i} \mid i \geq 0\} &= \{1, 1, 1, 2, 1, 2, 2, 3, 3, 5, 4, 7, 6, 10, 9, 15, 13, 22, 19, 32, \cdots\} \\
&= \{D^{ev}_{<2>,j} \mid j \geq 0\} \cup \{D^{od}_{<2>,j} \mid j \geq 0\}
\end{aligned}
$$

with $\{D^{ev}_{<2>,i}\} = \{1, 1, 1, 2, 3, 4, 6, 9, 13, 19, \cdots\}$ having eventh, and $\{D^{od}_{<2>,j}\} = \{1, 2, 2, 3, 5, 7, 10, 15, 22, 32, \cdots\}$ having oddth terms. Moreover we have

$$
\begin{aligned}
\{D^{(-1)}_{<3>,i} \mid i \geq 0\} &= \{1, 1, 1, 1, 2, 1, 2, 1, 3, 2, 4, 3, 6, 4, 8, 5, 11, 7, 15, 10, 21, \cdots\} \\
&= \{D^{ev}_{<3>,j} \mid j \geq 0\} \cup \{D^{od}_{<3>,j} \mid j \geq 0\}
\end{aligned}
$$

with $\{D^{ev}_{<3>,i}\} = \{1, 1, 2, 2, 3, 4, 6, 8, 11, 15, 21, 29, \cdots\}$ of eventh, and $\{D^{od}_{<3>,j}\} = \{1, 1, 1, 1, 2, 3, 4, 5, 7, 10, 14, 19, \cdots\}$ of oddth terms.

A sequence $\{F_{p,n}\}$ is called a Fibonacci p-sequence if it satisfies $F_{p,n+1} = F_{p,n} + F_{k,n-p}$ with $p+1$ initials $F_{p,i}$ $(0 \leq i \leq p)$ ([6], [3]). If $p = 0$ then $F_{0,n+1} = F_{0,n} + F_{0,n} = 2F_{0,n}$ with initial 1, so $\{F_{0,n}\} = \{1, 2, 2^2, \cdots\}$. If $p = 1$ then $F_{1,n+1} = F_{1,n} + F_{1,n-1}$ with two initials $1, 1$, so $\{F_{1,n}\}$ is the Fibonacci.

Theorem 4.1. *Consider eventh and oddth subsequences of* $\{D^{(-1)}_{<k>,i}\}$.

(1) $\{D^{ev}_{<1>,j}\}$ *and* $\{D^{od}_{<1>,j}\}$ *are Fibonacci sequences with initials* $1,2$ *and* $1,1$.

(2) $\{D^{ev}_{<2>,j}\}$ *and* $\{D^{od}_{<2>,j}\}$ *are Fibonacci* 2*-sequences with initials* $1,1,1$ *and* $1,2,2$*, respectively. And both* $\{D^{ev}_{<3>,j}\}$ *and* $\{D^{od}_{<3>,j}\}$ *are Fibonacci* 3*-sequences with initials* $1,1,2,2$ *and* $1,1,1,1$*, respectively.*

Proof. Clearly $D^{ev}_{<1>,j} = D^{ev}_{<1>,j-1} + D^{ev}_{<1>,j-2}$ and $D^{od}_{<1>,j} = D^{od}_{<1>,j-1} + D^{od}_{<1>,j-2}$ prove (1). In $\{D^{ev}_{<2>,j}\}$ and $\{D^{od}_{<2>,j}\}$, the first few entries satisfy $D^{ev}_{<2>,j} = D^{ev}_{<2>,j-1} + D^{ev}_{<2>,j-3}$ and $D^{od}_{<2>,j} = D^{od}_{<2>,j-1} + D^{od}_{<2>,j-3}$ with 3 initials $1,1,1$ and $1,2,2$, respectively. Now for convenience write $e^{(-1)}_{i,j} = e_{i,j}$.

By considering even number $j = 2i$, $D^{ev}_{<2>,j} = D^{(-1)}_{<2>,2i}$ satisfies

$$\begin{aligned} D^{ev}_{<2>,j} &= e_{2i,0} + e_{2i-2,1} + e_{2i-4,2} + e_{2i-6,3} + e_{2i-8,4} + e_{2i-10,5} + \cdots \\ &= e_{2i-2,0} + e_{2i-4,2} + e_{2i-8,4} + e_{2i-12,6} + \cdots \end{aligned}$$

since $e_{2i-2,1} = e_{2i-6,3} = e_{2i-10,5} = \cdots = 0$ by Lemma 3.1. Moreover since

$$\begin{aligned} &e_{2i-4,2} = e_{2i-5,1} + e_{2i-5,2} = 1 + (e_{2i-6,1} + e_{2i-6,2}) = e_{2i-6,0} + e_{2i-6,2}, \\ &e_{2i-8,4} = (e_{2i-10,2} - e_{2i-10,3}) + (e_{2i-10,3} + e_{2i-10,4}) = e_{2i-10,2} + e_{2i-10,4}, \\ &e_{2i-12,6} = (e_{2i-14,4} - e_{2i-14,5}) + (e_{2i-14,5} + e_{2i-14,6}) = e_{2i-14,4} + e_{2i-14,6}, \end{aligned}$$

and so on, we have

$$\begin{aligned} D^{ev}_{<2>,j} &= e_{2i-2,0} + (e_{2i-6,0} + e_{2i-6,2}) + (e_{2i-10,2} + e_{2i-10,4}) + \cdots \\ &= (e_{2i-2,0} + e_{2i-6,2} + e_{2i-10,4} + \cdots) + (e_{2i-6,0} + e_{2i-10,2} + e_{2i-14,4} + \cdots) \\ &= (e_{2i-2,0} + e_{2i-4,1} + e_{2i-6,2} + e_{2i-8,3} + e_{2i-10,4} + e_{2i-12,5} + \cdots) \\ &\quad + (e_{2i-6,0} + e_{2i-8,1} + e_{2i-10,2} + e_{2i-12,3} + e_{2i-14,4} + e_{2i-16,5} + \cdots) \\ &= D^{(-1)}_{<2>,2i-2} + D^{(-1)}_{<2>,2i-6} = D^{ev}_{<2>,j-1} + D^{ev}_{<2>,j-3}. \end{aligned}$$

The rest regarding $\{D^{od}_{<2>,j}\}$ can be proved similarly.

Now consider $\{D^{ev}_{<3>,j} | j \geq 0\}$ and $\{D^{od}_{<3>,j} | j \geq 0\}$ in $\{D^{(-1)}_{<3>,i}\}$. We easily see the first few numbers satisfy $D^{ev}_{<3>,j} = D^{ev}_{<3>,j-1} + D^{ev}_{<3>,j-4}$ and $D^{od}_{<3>,j} = D^{od}_{<3>,j-1} + D^{od}_{<3>,j-4}$ with 4 initials $1,1,2,2$ and $1,1,1,1$, respectively for some j. Then for any odd integer $j = 2i+1$, we have

$$\begin{aligned} &D^{od}_{<3>,j-1} + D^{od}_{<3>,j-4} \\ &= (e_{2i,0} + e_{2i-3,1} + e_{2i-6,2} + \cdots) + (e_{2i-3,0} + e_{2i-6,1} + e_{2i-9,2} + \cdots) \\ &= e_{2i,0} + (e_{2i-3,1} + e_{2i-3,0}) + (e_{2i-6,2} + e_{2i-6,1}) + (e_{2i-9,3} + e_{2i-9,2}) + \cdots \\ &= e_{2i+1,0} + e_{2i-2,1} + e_{2i-5,2} + e_{2i-8,3} + \cdots \\ &= D^{od}_{<3>,2i+1} = D^{od}_{<3>,j}. \end{aligned}$$

The rest can be proved analogously. □

Now let us look at subsequences of k slope diag. sums $\{D^{(-1)}_{<k>,i}\}$ in $C^{(-1)}$.

Theorem 4.2. *Let* $\{D^{ev}_{<k>,j} \mid j \geq 0\}$ *and* $\{D^{od}_{<k>,j} \mid j \geq 0\}$ *be subsequences of* $\{D^{(-1)}_{<k>,i} \mid i \geq 0\}$ *consisting of eventh and oddth terms, respectively. Then* $\{D^{ev}_{<k>,j}\}$ *and* $\{D^{od}_{<k>,j}\}$ *are both Fibonacci* k*-sequences having initials*

$$\begin{cases} \{1,\cdots,1\}_{(k+1)tuples} & 2\mid k \\ \{\underbrace{1,\cdots,1}_{\frac{k+1}{2}},\underbrace{2,\cdots,2}_{\frac{k+1}{2}}\} & 2\nmid k \end{cases} \text{ and } \begin{cases} \{\underbrace{1,\cdots,1}_{\frac{k}{2}},\underbrace{2,\cdots,2}_{\frac{k}{2}+1}\} & 2\mid k \\ \{1,\cdots,1\}_{(k+1)tuples} & 2\nmid k \end{cases}, \textit{ respectively.}$$

Proof. Clearly $D^{(-1)}_{<k>,i} = e^{(-1)}_{i,0} = 1$ $(0 \le i \le k)$ and $D^{(-1)}_{<k>,i} = e^{(-1)}_{i,0} + e^{(-1)}_{i-k,1}$ $(k < i \le 2k+1)$ equals 2 or 1 according to $i-k$ is odd or even by Lemma 3.1. So $\{D^{ev}_{<k>,j}\}$ has $k+1$ initials $\{1,\cdots,1\}$ if $2 \mid k$, otherwise $\{\underbrace{1,\cdots,1}_{\frac{k+1}{2}},\underbrace{2,\cdots,2}_{\frac{k+1}{2}}\}$, while $\{D^{od}_{<k>,j}\}$ has initials $\{\underbrace{1,\cdots,1}_{\frac{k}{2}},\underbrace{2,\cdots,2}_{\frac{k}{2}+1}\}$ if $2 \mid k$, otherwise $\{1,\cdots,1\}$.

Write $e^{(-1)}_{i,j} = e_{i,j}$. Then with respect to $D^{ev}_{<k>,j}$ with $j = 2i$, we have

$D^{ev}_{<k>,j} = D^{(-1)}_{<k>,2i} = e_{2i,0} + e_{2i-k,1} + e_{2i-2k,2} + e_{2i-3k,3} + e_{2i-4k,4} + \cdots$

If k is even then by Lemma 3.1 we have

$e_{2i,0} = 1 = e_{2i-k,0}$ and $e_{2i-k,1} = e_{2i-3k,3} = e_{2i-5k,5} = \cdots = 0$, and

$e_{2i-2k,2} = e_{2i-2k-1,1} + (e_{2i-2k-2,1} + e_{2i-2k-2,2}) = e_{2i-2k-2,0} + e_{2i-2k-2,2}$,

and so on. So similar to the proof of Theorem 4.1, it follows

$$\begin{aligned} D^{ev}_{<k>,j} &= e_{2i,0} + e_{2i-2k,2} + e_{2i-4k,4} + e_{2i-6k,6} + \cdots \\ &= e_{2i-k,0} + (e_{2i-2k-2,0} + e_{2i-2k-2,2}) + (e_{2i-4k-2,2} + e_{2i-4k-2,4}) \\ &\quad + (e_{2i-4k-2,2} + e_{2i-4k-2,4}) + \cdots \\ &= (e_{2i-2,0} + e_{2i-2-2k,2} + e_{2i-2-4k,4} + e_{2i-2-6k,6} + \cdots) \\ &\quad + (e_{2i-2-2k,0} + e_{2i-2-4k,2} + e_{2i-2-6k,4} + \cdots) \\ &= (e_{2i-2,0} + e_{2i-2-k,1} + e_{2i-2-2k,2} + e_{2i-2-k,3} + e_{2i-2-4k,4} + \cdots) \\ &\quad + (e_{2i-2-2k,0} + e_{2i-2-3k,1} + e_{2i-2-4k,2} + e_{2i-2-5k,3} + e_{2i-2-6k,4} + \cdots), \end{aligned}$$

because $0 = e_{2i-2-k,1} = e_{2i-2-k,3} = \cdots = e_{2i-2-3k,1} = e_{2i-2-5k,3} = \cdots$. Therefore we conclude

$D^{ev}_{<k>,j} = D^{(-1)}_{<k>,2i-2} + D^{(-1)}_{<k>,2i-2(k+1)} = D^{ev}_{<k>,j-1} + D^{ev}_{<k>,j-(k+1)}$.

The rest case $j = 2i+1$ can be proved analogously. Moreover the Fibonacci k-recurrence $D^{od}_{<k>,j+1} = D^{od}_{<k>,j} + D^{od}_{<k>,j-k}$ is also proved similarly. □

Corollary 4.3. *As n gets larger, the ratio $\frac{D^{(-1)}_{<k>,n}}{D^{(-1)}_{<k>,n-1}}$ is a real root of $x^{2(k+1)} - x^{2k} - 1 = 0$, while $\frac{D^{ev}_{<k>,n}}{D^{ev}_{<k>,n-1}} = \frac{D^{od}_{<k>,n}}{D^{od}_{<k>,n-1}}$ is a root of $x^{(k+1)} - x^k - 1 = 0$.*

The proof is clear from Theorem 4.1 and 4.2. A real root β_1 of $x^4 - x^2 - 1 = 0$ equals 0.786, and β_1^2 is a root of $x^2 - x - 1$ the ratio of Fibonacci numbers. Also a root β_2 of $x^6 - x^4 - 1 = 0$ equals 1.210, and β_2^2 is a root of $x^3 - x^2 - 1$ the ratio of Fibonacci 2-numbers. May refer to OEIS A053602 and A123231 for $\{D^{(-1)}_{<k>,i}\}$.

5. Pauli triangle without zero entries

When expanding $(x+y)^n$ with $yx = -xy$, lots of coefficients are zeros as in, for instance $(x+y)^2 = x^2 + y^2$ or $(x+y)^4 = x^4 + 2x^2y^2 + y^4$. The nonzero

Pauli table $W = [w_{i,j}]$ is an AT of $(x + y)^n$ with $yx = -xy$ without putting zero entries.

W

0	1
1	1 1
2	1 1
3	1 1 1 1
4	1 2 1
5	1 1 2 2 1 1
6	1 3 3 1

$C^{(-1)}$

1
1 1
1 0 1
1 1 1 1
1 0 2 0 1
1 1 2 2 1 1
1 0 3 0 3 0 1

and $C^{(1)}$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

A relationship of W and the Pascal $C^{(1)} = [e_{i,j}]$ is clear from the tabes.

Lemma 5.1. $w_{i,j} = \begin{cases} e_{\frac{i}{2},j} & \text{if } i \text{ even} \\ e_{\lfloor\frac{i}{2}\rfloor,\lfloor\frac{j}{2}\rfloor} & \text{if } i \text{ odd} \end{cases}$ *for* $i, j \geq 0$, *so* W *satisfies recurrences* $w_{2i,j} = w_{2i-2,j-1} + w_{2i-2,j}$ *and* $w_{2i+1,j} = w_{2i-1,j-2} + w_{2i-1,j}$.

Let d_i be the ith diag. sum of W. Let $D_{<1/2>,u}$ and $D^*_{<1/2>,u}$ be $\frac{1}{2}$ slope diag. sums of $C^{(1)}$ starting from $e_{u,0}$ and $e_{u,1}$, respectively.

Theorem 5.2. *The* d_i *is related to the diagonal sums of* $C^{(1)}$ *that* $d_i = D_{<1>,\lfloor\frac{i}{2}\rfloor} + D^*_{<1/2>,\lfloor\frac{i}{2}\rfloor}$ *if* $2 \nmid i$, *and* $d_i = D_{<1>,\lfloor\frac{i}{2}\rfloor-1} + D_{<1/2>,\lfloor\frac{i}{2}\rfloor}$ *otherwise.*

Proof. Note $\lfloor\frac{2t-p}{2}\rfloor = t - \lfloor\frac{p}{2}\rfloor - 1$ for $t, p > 0$. The d_i of $W = [w_{i,j}]$ is

$$d_i = w_{i,0} + w_{i-1,1} + w_{i-2,2} + w_{i-3,3} + w_{i-4,4} + w_{i-5,5} + \cdots$$
$$= (w_{i,0} + w_{i-2,2} + w_{i-4,4} + \cdots) + (w_{i-1,1} + w_{i-3,3} + w_{i-5,5} + \cdots).$$

Let $X = w_{i,0} + w_{i-2,2} + w_{i-4,4} + \cdots$ and $Y = w_{i-1,1} + w_{i-3,3} + w_{i-5,5} + \cdots$. When $i = 2t - 1$ is odd, due to Lemma 5.1 we have

$$X = e_{t,0} + e_{t-1,1} + e_{t-2,2} + \cdots = D_{<1>,t} = D_{<1>,\lfloor\frac{i}{2}\rfloor}$$

and $Y = e_{t-1,1} + e_{t-2,3} + e_{t-3,5} + \cdots = D^*_{<1/2>,t-1} = D^*_{<1/2>,\lfloor\frac{i}{2}\rfloor}$,
so it follows $d_i = X + Y = D_{<1>,\lfloor\frac{i}{2}\rfloor} + D^*_{<1/2>,\lfloor\frac{i}{2}\rfloor}$.

On the other hand if $i = 2t$ even then Lemma 5.1 again yields

$$X = e_{t,0} + e_{t-1,2} + e_{t-2,4} + \cdots = D_{<1/2>,t} = D_{<1/2>,\lfloor\frac{i}{2}\rfloor}$$

and $Y = e_{t-1,0} + e_{t-2,1} + e_{t-3,2} + \cdots = D_{<1>,t-1} = D_{<1>,\lfloor\frac{i}{2}\rfloor}$,
so we have $d_i = X + Y = D_{<1/2>,\lfloor\frac{n}{2}\rfloor} + D_{<1>,\lfloor\frac{n}{2}\rfloor-1}$. □

Lemma 5.3. $D_{<1/2>,t}$ *satisfies recurrences* $D_{<1/2>,t} + D^*_{<1/2>,t} = D^*_{<1/2>,t+1}$ *and* $D_{<1/2>,t} + D^*_{<1/2>,t-1} = D_{<1/2>,t+1}$.

Proof. $D_{<1/2>,t} + D^*_{<1/2>,t}$

$$= (e_{t,0} + e_{t-1,2} + e_{t-2,4} + \cdots) + (e_{t,1} + e_{t-1,3} + e_{t-2,5} + \cdots)$$
$$= (e_{t,1} + e_{t,0}) + (e_{t-1,3} + e_{t-1,2}) + e_{t-2,5} + e_{t-2,4}) + \cdots$$
$$= e_{t+1,1} + e_{t,3} + e_{t-1,5} + \cdots = D^*_{<1/2>,t+1}.$$

Similarly we have

$$D_{<1/2>,t} + D^*_{<1/2>,t-1} = e_{t+1,0} + e_{t,2} + e_{t-1,4} + e_{t-2,6} + \cdots = D_{<1/2>,t+1}.$$ □

Theorem 5.4. d_i *satisfies recurrences related to Fibonacci numbers.*

(1) $d_i = d_{i-2} + d_{i-3} - \Gamma_i$, *where* $\Gamma_i = F_{\frac{i}{2}-3}$ *if* i *is even, otherwise* $\Gamma_i = 0$.

(2) $d_i = d_{i-1} + (d_{i-2} + d_{i-4} + d_{i-8}) - (d_{i-3} + d_{i-5} + d_{i-9}) - \Gamma_i$, *where* $\Gamma_i = F_{\frac{i-11}{2}}$ *if* i *is odd, otherwise* $\Gamma_i = 0$.

Proof. We first assume odd $i = 2t-1$. By Theorem 5.2, we have

$d_{i-2} = D_{<1>,\lfloor\frac{2t-3}{2}\rfloor} + D^*_{<1/2>,\lfloor\frac{2t-3}{2}\rfloor} = D_{<1>,t-2} + D^*_{<1/2>,t-2}$,

$d_{i-3} = D_{<1>,\lfloor\frac{2(t-2)}{2}\rfloor-1} + D_{<1/2>,\lfloor\frac{2(t-2)}{2}\rfloor} = D_{<1>,t-3} + D_{<1/2>,t-2}$.

In $C^{(1)}$, since diag. sums $D_{<1>,t-2} = F_{t-1}$ and $D_{<1>,t-3} = F_{t-2}$ are Fibonacci numbers, we have by Lemma 5.3 that

$$d_{i-2} + d_{i-3} = F_{t-1} + F_{t-2} + D^*_{<1/2>,t-2} + D_{<1/2>,t-2} = F_t + D^*_{<1/2>,t-1}$$
$$= D_{<1>,t-1} + D^*_{<1/2>,t-1} = D_{<1>,\lfloor\frac{2t-1}{2}\rfloor} + D^*_{<1/2>,\lfloor\frac{2t-1}{2}\rfloor} = d_i.$$

Assume even $i = 2t$. Then again by Theorem 5.2, we have

$d_{i-2} = D_{<1>,t-2} + D_{<1/2>,t-1} = F_{t-1} + D_{<1/2>,t-1}$,

$d_{i-3} = D_{<1>,t-2} + D^*_{<1/2>,t-2} = F_{t-1} + D^*_{<1/2>,t-2}$.

So Lemma 5.3 implies

$d_{i-2} + d_{i-3} = 2F_{t-1} + D_{<1/2>,t-1} + D^*_{<1/2>,t-2} = 2F_{t-1} + D_{<1/2>,t}$.

But since $d_i = D_{<1>,t-1} + D_{<1/2>,t} = F_t + D_{<1/2>,t}$, we have

$d_i - (d_{i-2} + d_{i-3}) = F_t + D_{<1/2>,t} - 2F_{t-1} - D_{<1/2>,t} = -F_{t-3} = -\Gamma_i$.

Now for (2), let $\Delta_i = d_{i-1} + (d_{i-2} + d_{i-4} + d_{i-8}) - (d_{i-3} + d_{i-5} + d_{i-9})$. Then the following is the table of d_i and Δ_i for some $i \geq 9$. By somewhat long computations of d_i as above that we shall omit here, the identity $d_i = \Delta_i - \Gamma_i$ follows.

i	d_i	Δ_i	$\Delta_i - d_i$
9	10	11	$1 = F_{-1} = F_{i-10}$
10	12	12	0
11	17	17	$0 = F_0 = F_{i-11}$
12	20	20	0
13	29	30	$1 = F_1 = F_{i-12}$

i	d_i	Δ_i	$\Delta_i - d_i$
14	34	34	0
15	49	50	$1 = F_2 = F_{i-13}$
16	58	58	0
17	83	85	$2 = F_3$
18	99	99	0

i	d_i	Δ_i	$\Delta_i - d_i$
19	141	144	$3 = F_4$
20	169	169	0
21	240	245	$5 = F_5$
22	289	289	0
23	409	417	$8 = F_6$

□

Clearly $d_{13} + d_{14} = 29 + 34 = 58 + 5 = d_{16} + F_5$ and $d_{14} + d_{15} = 83 = d_{17}$. Furthermore by letting subsequences $\{d_t^{od}\}$ and $\{d_t^{ev}\}$ consisting of oddth and eventh terms of $\{d_i\}$, we have more relations of these with Fibonacci numbers.

Theorem 5.5. *(1)* $d_t^{ev} + d_t^{od} = d_{t+1}^{od}$ *and* $d_{t-2}^{od} + d_{t-1}^{ev} = d_{t+1}^{ev} + F_{t-2}$.

(2) $d_{t-4}^{ev} = (d_{t+1}^{od} - d_{t+1}^{ev}) - 2(d_t^{od} - d_t^{ev}) + (d_{t-1}^{od} - d_{t-1}^{ev})$ *and* $d_{t-4}^{od} = (d_t^{od} - d_t^{ev}) - (d_{t-1}^{od} - d_{t-1}^{ev})$.

Now let $\overline{d_t^{ev}} = d_t^{ev} + d_{t-1}^{ev} + d_{t-3}^{ev}$ *and* $\overline{d_t^{od}} = d_t^{od} + d_{t-1}^{od} + d_{t-3}^{od}$. *Then* $d_{t+1}^{ev} = \overline{d_t^{ev}} - F_{t-3}$ *and* $d_{t+1}^{od} = \overline{d_t^{od}} - F_{t-2}$. *So* $\overline{d_t^{ev}} - d_{t+1}^{od} = \overline{d_{t-1}^{od}} - d_t^{od}$.

Proof. We first look at the below table of d_t^{ev}, d_t^{od}, $\overline{d_t^{ev}}$ and $\overline{d_t^{od}}$. The identities $d_t^{ev} + d_t^{od} = d_{t+1}^{od}$ and $d_{t-2}^{od} + d_{t-1}^{ev} = d_{t+1}^{ev} + F_{t-2}$ are equivalent to

$d_{2i} + d_{2i+1} = d_{2i+3}$ and $d_{2i-1} + d_{2i} = d_{2i+2} + F_{\lfloor\frac{t-1}{2}\rfloor}$.

Note that $\lfloor\frac{2i-1}{2}\rfloor = i - 1$, so by Theorem 5.2 we have

$d_{2i-1} = D_{<1>,i-1} + D^*_{<1/2>,i-1} = F_i + D^*_{<1/2>,i-1}$,

$d_{2i} = D_{<1>,i-1} + D_{<1/2>,i} = F_i + D_{<1/2>,i}$,

and $d_{2i+1} = D_{<1>,i} + D^*_{<1/2>,i} = F_{i+1} + D^*_{<1/2>,i}$. By Lemma 5.3, we have

$$
\begin{aligned}
d_{2i-1} + d_{2i} &= 2F_i + (D^*_{<1/2>,i-1} + D_{<1/2>,i}) \\
&= F_{i+1} + F_{i-2} + D_{<1/2>,i+1} = (D_{<1>,i} + D_{<1/2>,i+1}) + F_{i-2} \\
&= (D_{<1>,\lfloor \frac{2i+2}{2} \rfloor -1} + D_{<1/2>,\lfloor \frac{2i+2}{2} \rfloor}) + F_{i-2} = d_{2i+2} + F_{i-2}.
\end{aligned}
$$

t	d_t^{ev}	d_t^{od}	$\overline{d_t^{ev}}$	$\overline{d_t^{od}}$	$\overline{d_t^{ev}} - d_{t+1}^{ev}$	$\overline{d_t^{od}} - d_{t+1}^{od}$	$d_t^{ev} - d_{t-1}^{od}$	$\overline{d_t^{ev}} - \overline{d_{t-1}^{od}}$
0	1	1						
1	2	2					1	
2	2	4					0	
3	4	6	7	11	$0 = F_0$	$1 = F_1$	0	
4	7	10	13	18	$1 = F_1$	$1 = F_2$	1	2
5	12	17	21	31	$1 = F_2$	$2 = F_3$	2	3
6	20	29	36	52	$2 = F_3$	$3 = F_4$	3	5
7	34	49	61	88	$3 = F_4$	$5 = F_5$	5	9
8	58	83	104	149	$5 = F_5$	$8 = F_6$	9	16
9	99	141	177	253	8	13	16	28
10	169	240	302	430	13	21	28	49

Similarly it follows immediately that

$$
\begin{aligned}
d_{2i} + d_{2i+1} &= (F_i + D_{<1/2>,i}) + (F_{i+1} + D^*_{<1/2>,i}) = F_{i+2} + D^*_{<1/2>,i+1} \\
&= D_{<1>,i+2} + D^*_{<1/2>,i+1} = D_{<1>,\lfloor \frac{2i+3}{2} \rfloor -1} + D^*_{<1/2>,\lfloor \frac{2i+3}{2} \rfloor} = d_{2i+3}.
\end{aligned}
$$

And the rest can be proved analogously. □

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