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REMARKS ON WEAK REVERSIBILITY-OVER-CENTER

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Abstract. Huang et al. proved that the n by n upper triangular matrix ring over a domain is weakly reversible-over-center by using the property of regular matrices. In this article we provide a concrete proof which is able to be available in the related study of centers. Next we extend an example of weakly reversible-over-center, which was argued by Huang et al., to the general case.

Throughout this note every ring is an associative ring with identity unless otherwise stated. Let R be a ring. We denote the center and the set of all idempotents of R by $Z(R)$ and $I(R)$, respectively. Denote the n by n ($n \geq 2$) full (resp., upper triangular) matrix ring over R by $Mat_n(R)$ (resp., $T_n(R)$). I_n denotes the identity matrix of both $Mat_n(R)$ and $T_n(R)$. Write $D_n(R) = \{(a_{ij}) \in T_n(R) \mid a_{11} = \cdots = a_{nn}\}$. Use E_{ij} for the matrix with (i, j)-entry 1 and zeros elsewhere. The following definitions are due to the literature. An element u of R is right regular if $ur = 0$ implies $r = 0$ for $r \in R$. Similarly, left regular elements can be defined. An element is regular if it is both left and right regular (and hence not a zero divisor). R is called *Abelian* if $I(R) \subseteq Z(R)$, and R is called *reduced* if $N(R) = 0$. Reduced rings are easily shown to be Abelian. R is said to be *directly finite* if $ab = 1$ for $a, b \in R$ implies $ba = 1$. Abelian rings are easily shown to be directly finite.

1. Weakly reversible-over-center rings

Following Choi et al. [2], a ring R is called *reversible-over-center* if $ab \in Z(R)$ for $a, b \in R$ implies $ba \in Z(R)$. Reduced rings are reversible-over-center and reversible-over-center rings are Abelian by [2, Theorem 1.1] and [2, Proposition 1.3(1)], respectively. In this article, we consider a generalization of reversible-over-center rings, concentrating upon the nonzero case of $ab \in Z(R)$. Following [3], a ring R is called *weakly reversible-over-center* if $0 \neq ab \in Z(R)$ for $a, b \in R$ implies $ba \in Z(R)$. Every reversible-over-center ring is clearly

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weakly reversible-over-center, but the converse need not hold by [3, Theorem 1.3]. In fact, $T_n(R)$, over any domain R, is weakly reversible-over-center by [3, Theorem 1.3] but it is non-Abelian (hence not reversible-over-center) when $n \geq 2$.

It is well-known that $Z(Mat_n(R)) = \{(a_{ij}) \in Mat_n(R) \mid a_{11} = \cdots = a_{nn} \in Z(R)$ and $a_{ij} = 0$ for all i, j with $i \neq j\}$, where R is a ring and $n \geq 2$.

Lemma 1.1. (1) [3, Lemma 1.2(1)] *Let R be a ring and $n \geq 2$. $Z(T_n(R)) = \{(a_{ij}) \in T_n(R) \mid a_{11} = \cdots = a_{nn} \in Z(R)$ and $a_{ij} = 0$ for all i, j with $i > j\}$.*

(2) [2, Theorem 1.1] *Let R be a reduced ring. If $ab \in Z(R)$ for $a, b \in R$, then $ab = ba$.*

Lemma 1.1 does important roles in the proof of the next result. Huang et al. [3] proved the following theorem by using only the form of regular matrices in $T_n(R)$ over a domain R. But one may need the concrete procedure to get the result in the study of centers. So we provide that here.

Theorem 1.2. *Let R be a domain. Then, for all $n \geq 1$, if $0 \neq AB \in Z(T_n(R))$ for $A, B \in T_n(R)$ then $AB = BA$.*

Proof. We use Lemma 1.1(1) freely.

(1) The case of $n = 1$ is proved by Lemma 1.1(2).

(i) Let $n = 2$ and $0 \neq AB \in Z(T_2(R))$ for $A = (a_{ij}), B = (b_{ij}) \in T_2(R)$. Then $AB = \alpha I_2$ for some $0 \neq \alpha \in Z(R)$, entailing $a_{11}b_{11} = \alpha = a_{22}b_{22}$ and $a_{11}b_{12} + a_{12}b_{22} = 0$. Moreover $b_{11}a_{11} = \alpha = b_{22}a_{22}$ by Lemma 1.1(2).

So, multiplying $a_{11}b_{12} + a_{12}b_{22} = 0$ by b_{11} on the left, we obtain
$0 = (b_{11}a_{11})b_{12} + b_{11}a_{12}b_{22} = b_{12}(a_{11}b_{11}) + b_{11}a_{12}b_{22} = b_{12}(a_{22}b_{22}) + b_{11}a_{12}b_{22} = (b_{12}a_{22} + b_{11}a_{12})b_{22}$. But $b_{22} \neq 0$, and so we have $b_{12}a_{22} + b_{11}a_{12} = 0$ because R is a domain. This yields $BA = \alpha I_2 = AB$.

(ii) Let $n = 3$. Suppose that $0 \neq AB \in Z(T_3(R))$ for $A = (a_{ij}), B = (b_{ij}) \in T_3(R)$. Then $AB = \alpha I_3$ for some $0 \neq \alpha \in Z(R)$. This yields $a_{ii}b_{ii} = \alpha$ (hence $b_{ii}a_{ii} = \alpha$ by Lemma 1.1(2)), $a_{11}b_{12} + a_{12}b_{22} = 0$, $a_{22}b_{23} + a_{23}b_{33} = 0$, and $a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} = 0$.

Moreover $b_{11}a_{12} + b_{12}a_{22} = 0$ and $b_{22}a_{23} + b_{23}a_{33} = 0$ by the result of the case of $n = 2$ because $A'B' = A''B'' = \alpha I_2 \in Z(T_2(R))$ (hence $B'A' = B''A'' = \alpha I_2$) for $A' = (a'_{ij}), A'' = (a''_{st}), B' = (b'_{ij}), B'' = (b''_{st}) \in T_2(R)$, where $a'_{ij} = a_{ij}$, $b'_{ij} = b_{ij}$ for all $i, j = 1, 2$, and $a''_{st} = a_{st}$, $b''_{st} = b_{st}$ for all $s, t = 2, 3$. We will show $b_{11}a_{13} + b_{12}a_{23} + b_{13}a_{33} = 0$.

From $b_{11}a_{12} + b_{12}a_{22} = 0$ and $a_{22}b_{23} + a_{23}b_{33} = 0$, we obtain
$(b_{11}a_{12})b_{23} = (-b_{12}a_{22})b_{23} = -b_{12}(a_{22}b_{23}) = -b_{12}(-a_{23}b_{33}) = b_{12}a_{23}b_{33}$.

So, multiplying $a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} = 0$ by b_{11} on the left, we get

$$\begin{aligned}0 &= (b_{11}a_{11})b_{13} + b_{11}a_{12}b_{23} + b_{11}a_{13}b_{33} = (a_{33}b_{33})b_{13} + b_{11}a_{12}b_{23} + b_{11}a_{13}b_{33} \\ &= b_{13}(a_{33}b_{33}) + b_{11}a_{12}b_{23} + b_{11}a_{13}b_{33} = b_{13}a_{33}b_{33} + b_{12}a_{23}b_{33} + b_{11}a_{13}b_{33} \\ &= (b_{11}a_{13} + b_{12}a_{23} + b_{13}a_{33})b_{33}.\end{aligned}$$

But $b_{33} \neq 0$ and R is a domain; hence we obtain $b_{13}a_{33} + b_{12}a_{23} + b_{11}a_{13} = 0$. Therefore $BA = \alpha I_3 = AB$.

(iii) Let $n = 4$. Suppose that $0 \neq AB \in Z(T_4(R))$ for $A = (a_{ij}), B = (b_{ij}) \in T_4(R)$. Then $AB = \alpha I_4$ for some $0 \neq \alpha \in Z(R)$. This yields $a_{ii}b_{ii} = \alpha$ (hence $b_{ii}a_{ii} = \alpha$ by Lemma 1.1(2)), $a_{11}b_{12} + a_{12}b_{22} = 0$, $a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} = 0$, $a_{22}b_{23} + a_{23}b_{33} = 0$, $a_{22}b_{24} + a_{23}b_{34} + a_{24}b_{44} = 0$, $a_{33}b_{34} + a_{34}b_{44} = 0$, and $a_{11}b_{14} + a_{12}b_{24} + a_{13}b_{34} + a_{14}b_{44} = 0$.

Moreover $b_{11}a_{12}+b_{12}a_{22} = 0$, $b_{11}a_{13}+b_{12}a_{23}+b_{13}a_{33} = 0$, $b_{22}a_{23}+b_{23}a_{33} = 0$, $b_{22}a_{24} + b_{23}a_{34} + b_{24}a_{44} = 0$, $b_{33}a_{34} + b_{34}a_{44} = 0$ by the result of the case of $n = 3$ because $A'B' = A''B'' = \alpha I_3 \in Z(T_3(R))$ (hence $B'A' = B''A'' = \alpha I_3$) for $A' = (a'_{ij}), A'' = (a''_{st}), B' = (b'_{ij}), B'' = (b''_{st}) \in T_3(R)$, where $a'_{ij} = a_{ij}$, $b'_{ij} = b_{ij}$ for all $i, j = 1, 2, 3$, and $a''_{st} = a_{st}$, $b''_{st} = b_{st}$ for all $s, t = 2, 3, 4$. We will show $b_{11}a_{14} + b_{12}a_{24} + b_{13}a_{34} + b_{14}a_{44} = 0$.

From $b_{11}a_{12} + b_{12}a_{22} = 0$, $a_{22}b_{24} + a_{23}b_{34} + a_{24}b_{44} = 0$, $b_{11}a_{13} + b_{12}a_{23} + b_{13}a_{33} = 0$, and $a_{33}b_{34} + a_{34}b_{44} = 0$, we obtain $(b_{11}a_{12})b_{24} = (-b_{12}a_{22})b_{24} = -b_{12}(a_{22}b_{24}) = -b_{12}(-a_{23}b_{34}-a_{24}b_{44}) = b_{12}a_{23}b_{34}+b_{12}a_{24}b_{44}$ and $(b_{11}a_{13})b_{34} = (-b_{12}a_{23} - b_{13}a_{33})b_{34} = -b_{12}a_{23}b_{34} - b_{13}(a_{33}b_{34}) = -b_{12}a_{23}b_{34} - b_{13}(a_{33}b_{34}) = -b_{12}a_{23}b_{34} - b_{13}(-a_{34}b_{44}) = -b_{12}a_{23}b_{34} + b_{13}a_{34}b_{44}$.

So, multiplying $a_{11}b_{14} + a_{12}b_{24} + a_{13}b_{34} + a_{14}b_{44} = 0$ by b_{11} on the left, we obtain

$$\begin{aligned}0 &= (b_{11}a_{11})b_{14} + b_{11}a_{12}b_{24} + b_{11}a_{13}b_{34} + b_{11}a_{14}b_{44} \\ &= (a_{44}b_{44})b_{14} + b_{11}a_{12}b_{24} + b_{11}a_{13}b_{34} + b_{11}a_{14}b_{44} \\ &= b_{14}(a_{44}b_{44}) + (b_{12}a_{23}b_{34} + b_{12}a_{24}b_{44}) + (-b_{12}a_{23}b_{34} + b_{13}a_{34}b_{44}) + b_{11}a_{14}b_{44} \\ &= b_{14}a_{44}b_{44} + b_{12}a_{24}b_{44} + b_{13}a_{34}b_{44} + b_{11}a_{14}b_{44} \\ &= (b_{11}a_{14} + b_{12}a_{24} + b_{13}a_{34} + b_{14}a_{44})b_{44}.\end{aligned}$$

But $b_{44} \neq 0$ and R is a domain; hence we obtain $b_{11}a_{14} + b_{12}a_{24} + b_{13}a_{34} + b_{14}a_{44} = 0$. Therefore $BA = \alpha I_4 = AB$.

(iv) Next we will proceed by induction on n to show that the preceding result also holds in the case of $n \geq 5$. Suppose that $AB \in Z(T_n(R))$ for $A = (a_{ij}), B = (b_{ij}) \in T_n(R)$. Then $AB = \alpha I_n$ for some $0 \neq \alpha \in Z(R)$. This yields $a_{ii}b_{ii} = \alpha$ (hence $b_{ii}a_{ii} = \alpha$ by Lemma 1.1(2)). Moreover $\sum_{i=1}^{n} a_{1i}b_{in} = 0$ and $\sum_{u=s}^{t} a_{su}b_{ut} = 0$ for every $t \in \{2, 3, \ldots, n-1\}$ with $s < t$, where s runs over $\{1, 2, \ldots, n-1\}$.

Note that $A'B' = A''B'' = \alpha I_{n-1} \in Z(T_{n-1}(R))$ for $A' = (a'_{ij}), A'' = (a''_{st}), B' = (b'_{ij}), B'' = (b''_{st}) \in T_{n-1}(R)$, where $a'_{ij} = a_{ij}$, $b'_{ij} = b_{ij}$ for all

$i, j = 1, 2, \ldots, n-1$, and $a''_{st} = a_{st}$, $b''_{st} = b_{st}$ for all $s, t = 2, 3, n$. Then, by the induction hypothesis, we get $B'A' = B''A'' = \alpha I_{n-1}$ and this yields $\sum_{u=s}^{t} b_{su} a_{ut} = 0$ for every $t \in \{2, 3, \ldots, n-1\}$ with $s < t$, where s runs over $\{1, 2, \ldots, n-1\}$, because B. We will show $\sum_{i=1}^{n} b_{1i} a_{in} = 0$.

Multiplying $\sum_{i=1}^{n} a_{1i} b_{in} = 0$ by b_{11} on the left, we have

$$b_{11}a_{11}b_{1n} + b_{11}a_{12}b_{2n} + \cdots + b_{11}a_{1i}b_{in} + \cdots + b_{11}a_{1(n-1)}b_{(n-1)n} + b_{11}a_{1n}b_{nn} = 0.$$

We observe this equality term by term.

From $a_{ii}b_{ii} = b_{ii}a_{ii} = \alpha$, we get

$$(b_{11}a_{11})b_{1n} = (a_{nn}b_{nn})b_{1n} = b_{1n}(a_{nn}b_{nn}) = b_{1n}a_{nn}b_{nn}$$

.

From $\sum_{i=1}^{2} b_{1i} a_{i2} = 0$ and $\sum_{i=2}^{n} a_{2i} b_{in} = 0$, we get

$$\begin{aligned}(b_{11}a_{12})b_{2n} &= (-b_{12}a_{22})b_{2n} = -b_{12}(a_{22}b_{2n}) \\ &= -b_{12}(-a_{23}b_{3n} - a_{24}b_{4n} - \cdots - a_{2(n-1)}b_{(n-1)n} - a_{2n}b_{nn}) \\ &=[-0] + [b_{12}a_{23}b_{3n} + b_{12}a_{24}b_{4n} + \cdots + b_{12}a_{2(n-1)}b_{(n-1)n}] + b_{12}a_{2n}b_{nn}.\end{aligned}$$

From $\sum_{i=1}^{3} b_{1i} a_{i3} = 0$ and $\sum_{i=3}^{n} a_{3i} b_{in} = 0$, we get

$$\begin{aligned}&(b_{11}a_{13})b_{3n} = (-b_{12}a_{23} - b_{13}a_{33})b_{3n} = -b_{12}a_{23}b_{3n} - b_{13}(a_{33}b_{3n}) \\ &= -b_{12}a_{23}b_{3n} - b_{13}(-a_{34}b_{4n} - a_{35}b_{5n} - \cdots - a_{3(n-1)}b_{(n-1)n} - a_{3n}b_{nn}) \\ &=[-b_{12}a_{23}b_{3n}] + [b_{13}a_{34}b_{4n} + b_{13}a_{35}b_{5n} + \cdots + b_{13}a_{3(n-1)}b_{(n-1)n}] + b_{13}a_{3n}b_{nn}.\end{aligned}$$

Next let $3 \leq k \leq n-1$. From $\sum_{i=1}^{k} b_{1i} a_{ik} = 0$ and $\sum_{i=k}^{n} a_{ki} b_{in} = 0$, we get

$$\begin{aligned}(b_{11}a_{1k})b_{kn} =&(-b_{12}a_{2k} - b_{13}a_{3k} - \cdots - b_{1(k-1)}a_{(k-1)k} - b_{1k}a_{kk})b_{kn} \\ =&- b_{12}a_{2k}b_{kn} - b_{13}a_{3k}b_{kn} - \cdots - b_{1(k-1)}a_{(k-1)k}b_{kn} - b_{1k}(a_{kk}b_{kn}) \\ =&- b_{12}a_{2k}b_{kn} - b_{13}a_{3k}b_{kn} - \cdots - b_{1(k-1)}a_{(k-1)k}b_{kn} \\ &- b_{1k}(-a_{k(k+1)}b_{(k+1)n} - \cdots - a_{k(n-1)}b_{(n-1)n} - a_{kn}b_{nn}) \\ =&[-b_{12}a_{2k}b_{kn} - b_{13}a_{3k}b_{kn} - \cdots - b_{1(k-1)}a_{(k-1)k}b_{kn}] \\ &+ [b_{1k}a_{k(k+1)}b_{(k+1)n} + \cdots + b_{1k}a_{k(n-1)}b_{(n-1)n}] + b_{1k}a_{kn}b_{nn}.\end{aligned}$$

Especially, from $\sum_{i=1}^{n-2} b_{1i}a_{i(n-2)} = 0, \sum_{i=n-2}^{n} a_{(n-2)i}b_{in} = 0,$ and $\sum_{i=1}^{n-2} b_{1i}a_{i(n-2)} = 0, \sum_{i=n-2}^{n} a_{(n-2)i}b_{in} = 0$, we get

$$
\begin{aligned}
(b_{11}a_{1(n-2)})b_{(n-2)n} &= (-b_{12}a_{2(n-2)} - b_{13}a_{3(n-2)} - \cdots - b_{1(n-3)}a_{(n-3)(n-2)} \\
&\quad - b_{1(n-2)}a_{(n-2)(n-2)})b_{(n-2)n} \\
&= - b_{12}a_{2(n-2)}b_{(n-2)n} - b_{13}a_{3(n-2)}b_{(n-2)n} - \cdots - b_{1(n-3)}a_{(n-3)(n-2)}b_{(n-2)n} \\
&\quad - b_{1(n-2)}(a_{(n-2)(n-2)}b_{(n-2)n}) \\
&=[-b_{12}a_{2(n-2)}b_{(n-2)n} - b_{13}a_{3(n-2)}b_{(n-2)n} - \cdots - b_{1(n-3)}a_{(n-3)(n-2)}b_{(n-2)n}] \\
&\quad - b_{1(n-2)}(-a_{(n-2)(n-1)}b_{(n-1)n} - a_{(n-2)n}b_{nn}) \\
&=[-b_{12}a_{2(n-2)}b_{(n-2)n} - b_{13}a_{3(n-2)}b_{(n-2)n} - \cdots - b_{1(n-3)}a_{(n-3)(n-2)}b_{(n-2)n}] \\
&\quad + [b_{1(n-2)}a_{(n-2)(n-1)}b_{(n-1)n}] + b_{1(n-2)}a_{(n-2)n}b_{nn};
\end{aligned}
$$

and

$$
\begin{aligned}
(b_{11}a_{1(n-1)})b_{(n-1)n} &= (-b_{12}a_{2(n-1)} - b_{13}a_{3(n-1)} - \cdots - b_{1(n-2)}a_{(n-2)(n-1)} \\
&\quad - b_{1(n-1)}a_{(n-1)(n-1)})b_{(n-1)n} \\
&= - b_{12}a_{2(n-1)}b_{(n-1)n} - b_{13}a_{3(n-1)}b_{(n-1)n} - \cdots - b_{1(n-2)}a_{(n-2)(n-1)}b_{(n-1)n} \\
&\quad - b_{1(n-1)}(a_{(n-1)(n-1)}b_{(n-1)n}) \\
&= - b_{12}a_{2(n-1)}b_{(n-1)n} - b_{13}a_{3(n-1)}b_{(n-1)n} - \cdots - b_{1(n-2)}a_{(n-2)(n-1)}b_{(n-1)n} \\
&\quad - b_{1(n-1)}(-a_{(n-1)n}b_{nn}) \\
&=[-b_{12}a_{2(n-1)}b_{(n-1)n} - b_{13}a_{3(n-1)}b_{(n-1)n} - \cdots - b_{1(n-2)}a_{(n-2)(n-1)}b_{(n-1)n}] \\
&\quad + b_{1(n-1)}a_{(n-1)n}b_{nn}.
\end{aligned}
$$

Now summarize the results above. Consider $[b_{12}a_{23}b_{3n} + b_{12}a_{24}b_{4n} + \cdots + b_{12}a_{2(n-1)}b_{(n-1)n}]$ in the right hand side of the equality of $b_{11}a_{12}b_{2n}$. Every $-b_{12}a_{2h}b_{hn}$ $(h = 3, 4, \ldots, n-1)$ occurs as a first term in the first brackets of the equality of $b_{11}a_{1k}b_{kn}$.

Consider $[b_{13}a_{34}b_{4n} + b_{13}a_{35}b_{5n} + \cdots + b_{13}a_{3(n-1)}b_{(n-1)n}]$ in the right hand side of the equality of $b_{11}a_{13}b_{3n}$. Every $-b_{13}a_{3h}b_{hn}$ $(h = 4, 5, \ldots, n-1)$ occurs as a second term in the first brackets of the equality of $b_{11}a_{1k}b_{kn}$.

Consider $[b_{1k}a_{k(k+1)}b_{(k+1)n} + \cdots + b_{1k}a_{k(n-1)}b_{(n-1)n}]$ in the right hand side of the equality of $b_{11}a_{1k}b_{kn}$. Every $-b_{1k}a_{kl}b_{ln}$ $(l = k+1, \ldots, n-1)$ occurs as a $(k-1)$-th term in the first brackets of the equality of $b_{11}a_{1m}b_{mn}$ $(m = k+1, \ldots, n-1)$.

Conversely, for every term $-b_{1w}a_{wk}b_{kn}$ $(w = 2, 3, \ldots, k-1)$ of the brackets $[-b_{12}a_{2k}b_{kn} - b_{13}a_{3k}b_{kn} - \cdots - b_{1(k-1)}a_{(k-1)k}b_{kn}]$ in the right hand side of the equality of $b_{11}a_{1k}b_{kn}$, there exists $b_{1w}a_{wk}b_{kn}$ in the second brackets of the equality of $b_{11}a_{1w}b_{wn}$.

Consequently we have

$$\begin{aligned}0 &= b_{11}a_{11}b_{1n} + b_{11}a_{12}b_{2n} + \cdots + b_{11}a_{1i}b_{in} + \cdots + b_{11}a_{1(n-1)}b_{(n-1)n} \\ &\quad + b_{11}a_{1n}b_{nn} \\ &= b_{1n}a_{nn}b_{nn} + b_{12}a_{2n}b_{nn} + b_{13}a_{3n}b_{nn} + \cdots + b_{1k}a_{kn}b_{nn} + \cdots \\ &\quad + b_{1(n-1)}a_{(n-1)n}b_{nn} + b_{11}a_{1n}b_{nn} \\ &= (b_{1n}a_{nn} + b_{12}a_{2n} + b_{13}a_{3n} + \cdots + b_{1k}a_{kn} + \cdots + b_{1(n-1)}a_{(n-1)n} + b_{11}a_{1n})b_{nn}.\end{aligned}$$

But $b_{nn} \neq 0$ and R is a domain; hence we obtain $\sum_{i=1}^{n} b_{1i}a_{in} = 0$. Therefore $BA = \alpha I_n = AB$. □

As noted above, Huang et al. proved that the n by n upper triangular matrix ring over a domain is weakly reversible-over-center by using the property of regular matrices, in [3, Theorem 1.3]. One can see concrete argument for regular upper triangular matrices in [4, Theorem 1.1].

2. An extended example

In this section we extend the argument in [3, Example 2.1(2)] to the general situation for more application.

Example 2.1. We refer to the construction in [1, Example 4.8]. Let K be a field and $A = K\langle a, b\rangle$ be the free algebra generated by the noncommuting indeterminates a, b over K. Let I be the ideal of A generated by b^n and set $R = A/I$, where $n \geq 2$. Identify a, b with their images in R for simplicity. We will show that R is weakly reversible-over-center.

We extend the method in [3, Example 2.1(2)] to get $Z(R) = K$. Every element of R can be expressed by

$$k_0 + k_1 a f(a) + k_2 b g(b) + h \text{ with } f(x), g(x) \in K[x], h \in R, \text{ and } k_i \in K,$$

where the degree of $g(x)$ is equal to or less than $n-1$, and every term of h contains a, b when h is nonzero.

Let $F = k_0 + k_1 a f(a) + k_2 b g(b) + h \in Z(R)$. Then $aF - Fa = 0$ and this yields

$$k_2(abg(b) - bg(b)a) + (ah - ha) = 0.$$

Here if $k_2(abg(b) - bg(b)a) \neq 0$ then every term has only one a; and if $ah - ha \neq 0$, then every term has two or more a's. three or more. So, from $k_2(abg(b) - bg(b)a) = -(ah - ha)$, we must obtain

$$k_2(abg(b) - bg(b)a) = 0 \text{ and } ah - ha = 0,$$

i.e., $ak_2bg(b) = k_2bg(b)a$ and $ah = ha$. Assume $k_2bg(b) \neq 0$. Then $ak_2bg(b) - k_2bg(b)a$ cannot be zero, contrary to $k_2(abg(b) - bg(b)a) = 0$. So $k_2bg(b) = 0$ and we have $F = k_0 + k_1af(a) + h$.

Consider $bF - Fb = 0$. Then $k_1(baf(a) - af(a)b) = -(bh - hb)$. If $k_1(baf(a) - af(a)b) \neq 0$ then every term has only one b; and if $bh - hb \neq 0$ then every term has two or more b's. So, from $k_1(baf(a) - af(a)b) = -(bh - hb)$, we obtain

$$k_1(baf(a) - af(a)b) = 0 \text{ and } bh - hb = 0,$$

i.e., $bk_1af(a) = k_1af(a)b$ and $bh = hb$. Assume $k_1af(a) \neq 0$. Then $bk_1af(a) - k_1af(a)b$ cannot be zero, contrary to $k_1(baf(a) - af(a)b) = 0$. So $k_1af(a) = 0$ and therefore we have $F = k_0 + h$.

Now we get $ah = ha$ and $bh = hb$ from $aF = Fa$ and $bF = Fb$, respectively.

Next we can express h by $ah_1 + bh_2$ with $h_i \in R$, where the constant term of h_i is zero. From $ah = ha$, we obtain

$$aah_1 + abh_2 = a(ah_1 + bh_2) = ah = ha = (ah_1 + bh_2)a = ah_1a + bh_2a,$$

entailing

$$a(ah_1 + bh_2 - h_1a) = bh_2a.$$

So we must get

$$a(ah_1 + bh_2 - h_1a) = 0 \text{ and } bh_2a = 0.$$

Here $bh_2a = 0$ implies $bh_2 = 0$, and $h = ah_1 + bh_2 = ah_1$ follows.

Next, from $bh = hb$, we obtain $bah_1 = bh = hb = ah_1b$; hence we must have $ah_1b = 0$ and $bah_1 = 0$. This implies $ah_1 = 0$ and $h = 0$ follows.

Therefore $F = k_0$, and thus $Z(R) = K$.

Now let $0 \neq F_1F_2 \in Z(R)$ for $F_i \in R$. Then $F_1F_2 = k$ for some $k \in K$. Note $k^{-1}F_1F_2 = 1$. But R is Abelian and so directly finite; hence $(k^{-1}F_1)F_2 = 1$ implies $F_2(k^{-1}F_1) = k^{-1}F_2F_1 = 1$. So $F_2F_1 = k = F_1F_2$. Therefore R is weakly reversible-over-center.

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