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STRUCTURE JACOBI OPERATOR OF SEMI-INVARINAT SUBMANIFOLDS IN COMPLEX SPACE FORMS

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Abstract. Let M be a semi-invariant submanifold of codimension 3 with almost contact metric structure (ϕ, ξ, η, g) in a complex space form $M_{n+1}(c)$, $c \neq 0$. We denote by R_ξ and R'_X be the structure Jacobi operator with respect to the structure vector ξ and be $R'_X = (\nabla_X R)(\cdot, X)X$ for any unit vector field X on M, respectively. Suppose that the third fundamental form t satisfies $dt(X,Y) = 2\theta g(\phi X, Y)$ for a scalar $\theta(\neq 2c)$ and any vector fields X and Y on M. In this paper, we prove that if it satisfies $R_\xi \phi = \phi R_\xi$ and at the same time $R'_\xi = 0$, then M is a Hopf real hypersurfaces of type (A), provided that the scalar curvature $\bar{r}$ of M holds $\bar{r} - 2(n-1)c \leq 0$.

1. Introduction

Let $\tilde{M}$ be a Kaehlerian manifold with parallel complex structure J. Then a submanifold M of $\tilde{M}$ is called a *CR submanifold* if there exists a differentiable distribution $\triangle : p \to \triangle_p \subset T_p(M)$ on M such that $\triangle$ is J-invariant and the complementary orthogonal distribution $\triangle^\perp$ is totally real, where T_pM denotes the tangent space at each point p in M ([1], [27]). In particular, M is said to be a *semi-invariant submanifold* provided that $\dim\triangle^\perp = 1$. The unit normal in $J\triangle^\perp$ is called the *distinguished normal* to the semi-invariant submanifold ([4], [25]). In this case, M admits an almost contact metric structure (ϕ, ξ, η, g). A typical example of a semi-invariant submanfold is real hypersurfaces in a Kaehlerian manifold. And new examples of nontrivial semi-invariant submanifolds in a complex projective space $P_n\mathbb{C}$ are constructed in [15] and [22]. Accordingly, we may expect to generalize some results which are valid in a real hypersurface to a semi-invariant submanifold.

An n-dimensional complex space form $\tilde{M}_n(c)$ is a Kaehlerian manifold of constant holomorphic sectional curvature $4c$. As is well known, complete and simply connected complex space forms are isometric to a complex projective space $P_n\mathbb{C}$, or a complex hyperbolic space $H_n\mathbb{C}$ according as $c > 0$ or $c < 0$.

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For the real hypersurface of $\tilde{M}_n(c), c \neq 0$, many results are known. One of them, Takagi([23], [24]) classified all the homogeneous real hypersurfaces of $P_n\mathbb{C}$ as six model spaces which are said to be A_1, A_2, B, C, D and E, and Cecil-Ryan ([5]) and Kimura ([17]) proved that they are realized as the tubes of constant radius over Kaehlerian submanifolds when the structure vector field ξ is principal.

On the other hand, real hypersurfaces in $H_n\mathbb{C}$ have been investigated by Berndt [2], Montiel and Romero [18] and so on. Berndt [2] classified all homogeneous real hypersurfaces in $H_n\mathbb{C}$ and showed that they are realized as the tubes of constant radius over certain submanifolds. Also such kinds of tubes are said to be real hypersurfaces of type A_0, A_1, A_2 or type B.

Let M be a real hypersurface of type A_1 or type A_2 in a complex projective space $P_n\mathbb{C}$ or that of type A_0, A_1 or A_2 in a complex hyperbolic space $H_n\mathbb{C}$. Now, hereafter unless otherwise stated, such hypersurfaces are said to be of *type* (A) for our convenience sake.

Characterization problems for a real hypersurface of type (A) in a complex space form were studied by many authors ([7], [8], [11], [16], [18], [20] etc.).

We remark that, in particular, a homogeneous real hypersurface of type (A) in $\tilde{M}_n(c)$ has a lot of nice geometric properties. For example, Okumura ([20]) or Montiel and Romero ([18]) showed respectively that a real hypersurface of type (A) in $P_n\mathbb{C}$ or in $H_n\mathbb{C}$ if and only if the structure tensor ϕ commutes with the shape operator A ($\phi A = A\phi$).

Denoting by R the curvature tensor of the submanifold, we define the Jacobi operator $R_\xi = R(\cdot, \xi)\xi$ with respect to the structure vector ξ. Then R_ξ is a self adjoint endomorphism on the tangent space of a CR submanifold.

Using several conditions on the structure Jacobi operator R_ξ, characterization problems for real hypersurfaces of type (A) have recently studied (cf. [7], [8], [16]). In the provious paper ([7]), Cho and one of the present authors gave another characterization of real hypersurface of type (A) in a complex projective space $P_n\mathbb{C}$. Namely they prove the following :

Theorem CK([7]). *Let M be a connected real hypersurface of $P_n\mathbb{C}$ if it satisfies* (1) $R_\xi A\phi = \phi A R_\xi$ *or* (2) $R_\xi\phi = \phi R_\xi, R_\xi A = A R_\xi$, *then M is of type* (A), *where A denotes the shape operator of M.*

For each point p in a real hypersurface M and each unit tangent vector $X \in T_pM$, we define R'_X by $R'_X = (\nabla_X R)(\cdot, X)X$. If $\nabla_\xi R_\xi = 0$, then we have $R'_\xi = 0$. If the structure vector ξ is a geodesic vector field, then $R'_\xi = 0$ has a nice geometric meaning (cf. [3]).

On the other hand, semi-invariant submanifolds of codimension 3 in a complex space form $M_{n+1}(c)$ have been studied in [12] $\sim$ [15] and so on by using properties of induced almost contact metric structure and those of the third fundamental form of the submanifold. In the preceding work, Ki, Song and Takagi ([15]) assert the following :

Theorem KST([15]). *Let M be a real $(2n-1)$-dimensional semi-invariant submanifold of codimension 3 in a complex projective space $P_{n+1}\mathbb{C}$ with constant holomorphic sectional curvature $4c$. If the structure vector ξ is an eigenvector for the shape operator in the direction of the distinguished normal and the third fundamental form t satisfies $dt = 2\theta\omega$ for a certain scalar $\theta(< 2c)$, where $\omega(X,Y) = g(\phi X, Y)$ for any vectors X and Y on M, then M is a Hopf real hypersurface in a complex projective space $P_n\mathbb{C}$.*

In this paper, we consider a semi-invariant submanifold M of codimension 3 in a complex space form $M_{n+1}(c), c \neq 0$ which satisfies $R_\xi\phi = \phi R_\xi$ and at the same time $R'_\xi = 0$ such that the third funamental form t satisfies $dt = 2\theta\omega$ for a certain scalar $\theta(\neq 2c)$. If the scalar curvature $\bar{r}$ of M satisfies $\bar{r} - 2c(n-1) \leq 0$, then we prove that M is a real hypersurface is of type (A) in $M_n(c)$.

All manifolds in the present paper are assumed to be connected and of class C^∞ and the semi-invariant are supposed to be orientable.

2. Preliminares

Let $\tilde{M}$ be a real $2(n+1)$-dimensional Kaehlerian manifold with parallel almost complex structure J and a Riemannian metric tensor G. Let M be a real $(2n-1)$-dimensional Riemannian manifold immersed isometrically in $\tilde{M}$ by the immersion $i : M \to \tilde{M}$. In the sequel, we identify $i(M)$ with M itself. We denote by g the Riemmanian metric tensor on M from that of $\tilde{M}$.

If we denote by $\tilde{\nabla}$ the operator of covariant differentiation with respect to the metric tensor G on $\tilde{M}$ and by ∇ the one on M, then the Gauss and Weingarten formulas are respectively given by

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX,Y)C + g(KX,Y)D + g(LX,Y)E, \tag{2.1}$$

$$\begin{aligned} \tilde{\nabla}_X C &= -AX + l(X)D + m(X)E, \\ \tilde{\nabla}_X D &= -KX - l(X)C + t(X)E, \\ \tilde{\nabla}_X E &= -LX - m(X)C - t(X)D \end{aligned} \tag{2.2}$$

for any vector fields tangent to X and Y on M and any unit vector field C, D and E normal to M, because we take C, D and E are mutually orthogonal, where A, K, L are called the *second fundamental forms* and l, m and t *third fundamental forms.*

As is well-known, a submanifold M of a Kaehlerian manifold $\tilde{M}$ is said to be a $CR\ submanifold$ ([1], [27]) if it is endowed with a pair of mutually orthogonal and complementary differentiable distribution (Δ, Δ^{-1}) such that for any point

p in M we have $J\Delta_p = T_pM$, $JT_p^{\perp} \subset T_p^{\perp}M$, where $T_p^{\perp}M$ denote the normal space of M at p. In particular, M is said to be *semi-invariant submanifold*([4], [25]) provided that $dim\Delta^{\perp} = 1$ or to be a $CR\ submanifold$ with $CR\ dimension$ $n-1$([21]).

In this case the unit normal vector field in $J\Delta^{\perp}$ is called a *distinguished normal* to the semi-invariant submanifold and denote this by $\mathcal{C}$ ([25], [26]).

From now on we discuss that M is a real $(2n-1)$-dimensional semi-invariant submanifold of a codimension 3 in a Kaehlerian manifold $\tilde{M}$ of real $2(n+1)$-dimension . Then we can choose a local orthonormal frame field $\{e_1, \cdots, e_{n-1}, Je_1, \cdots, J_{e_{n-1}}, e_0 = \xi, C = J\xi, D = JE, E\}$ on the tangent space $T_p\tilde{M}$ of $\tilde{M}$ for any point p in M such that $e_1, \cdots, e_{n-1}, Je_1, \cdots, J_{e_{n-1}} \in T_pM$, $\xi \in T_p^{\perp}M$, and C, D, $E \in T_p^{\perp}M$.

Now, let ϕ be the restriction of J on M, then we have

$$JX = \phi X + \eta(X)\mathcal{C}, \quad \eta(X) = g(\xi, X), \quad J\mathcal{C} = -\xi \tag{2.3}$$

for any vector field X on M ([26]). From this it is, using Hermitian property of J, verified that the aggregate (ϕ, ξ, η, g) is an *almost contact metric structure* on M, that is, we have

$$\phi^2X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad g(\xi, X) = \eta(X),$$
$$\phi\xi = 0, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for any vector fields X and Y.

In the sequel, we denote the normal components of $\tilde{\nabla}_X C$ by $\nabla^{\perp}C$. The distinguished normal C is said to be *parallel* in the normal bundle if we have $\nabla^{\perp}C = 0$, that is, l and m vanish identically.

Using the Kaehler condition $\tilde{\nabla}J = 0$ and the Gauss and Weingarten formulas,we obtain from (2.3)

$$(\nabla_X\phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{2.4}$$

$$\nabla_X\xi = \phi AX, \tag{2.5}$$

$$KX = \phi LX - m(X)\xi, \tag{2.6}$$

$$LX = -\phi KX + l(X)\xi \tag{2.7}$$

for any vectors X and Y on M. From the last two equations, we have

$$g(K\xi, X) = -m(X), \tag{2.8}$$

$$g(L\xi, X) = l(X). \tag{2.9}$$

Using the frame field $\{e_0 = \xi, e_1, \cdots, e_{n-1}, \phi e_1, \cdots, \phi e_{n-1}\}$ on M it follows from (2.6) $\sim$ (2.9) that

$$\begin{aligned} T_rK &= \eta(K\xi) = -m(\xi) \\ T_rL &= \eta(L\xi) = l(\xi). \end{aligned} \tag{2.10}$$

By the way, there is no loss of generality such that we may assume $T_rL = 0$(cf. [15]). So we have

$$l(\xi) = 0. \tag{2.11}$$

In what follows, to write our formulas in a convention form, we denote by $\alpha = \eta(A\xi)$, $\beta = (A^2\xi)$, $T_rA = h$, $T_rK = k$, $T_r({}^tAA) = h_{(2)}$ and for a function f we denote by ∇f the gradient vector field of f.

From (2.10) we also have

$$m(\xi) = -k. \tag{2.12}$$

From (2.6) and (2.7) we get

$$\eta(X)l(\phi Y) - \eta(Y)l(\phi X) = m(Y)\eta(X) - m(X)\eta(Y),$$

which together with (2.12) gives

$$l(\phi X) = m(X) + k\eta(X). \tag{2.13}$$

Similarly, we have from (2.8)

$$m(\phi X) = -l(X), \tag{2.14}$$

where we have used (2.9) and (2.11).

Taking the inner product with LY to (2.6) and using (2.9), we get

$$g(KLX, Y) + g(LKX, Y) = -\{l(X)m(Y) + l(Y)m(X)\}. \tag{2.15}$$

Now, we put $\nabla_\xi \xi = U$ in the sequel. Then U is orthogonal to ξ be because of (2.5).

We put

$$A\xi = \alpha\xi + \mu W, \tag{2.16}$$

where W is a unit vector orthogonal to ξ. Then we have

$$U = \mu\phi W \tag{2.17}$$

by virtue of (2.5). Thus, W is also orthogonal to U. Further, we have

$$\mu^2 = \beta - \alpha^2. \tag{2.18}$$

From (2.16) and (2.17) we have

$$\phi U = -A\xi + \alpha\xi. \tag{2.19}$$

If we take account of (2.5), (2.10) and (2.19), then we find

$$g(\nabla_X \xi, U) = \mu g(AW, X). \tag{2.20}$$

Since W is orthogonal to ξ, we can, using (2.5) and (2.17), see that

$$\mu g(\nabla_X W, \xi) = g(AU, X). \tag{2.21}$$

Differentiating (2.19) covariantly along M and using (2.4) and (2.5), we find

$$(\nabla_X A)\xi = -\phi\nabla_X U + g(AU + \nabla\alpha, X) - A\phi AX + \alpha\phi AX. \tag{2.22}$$

In the rest of this paper we shall suppose that $\tilde{M}$ is a Kaehlerian manifold of constant holomorphic sectional curvature $4c$, which is called a *complex space form* and denote by $M_{n+1}(c)$, that is, we have

$$\tilde{R}(\tilde{X},\tilde{Y})\tilde{Z} = c\{G(\tilde{Y},\tilde{Z})\tilde{X} - G(\tilde{X},\tilde{Z})\tilde{Y} + G(J\tilde{Y},\tilde{Z})J\tilde{X} - G(J\tilde{X},\tilde{Z})J\tilde{Y} - 2G(J\tilde{X},\tilde{Y})J\tilde{Z}\}$$

for any vectors $\tilde{X}$, $\tilde{Y}$ and $\tilde{Y}$ on $\tilde{M}$, where $\tilde{R}$ is the curvature tensor of $\tilde{M}$. Then equations of the Gauss and Codazzi are given by

$$\begin{aligned} R(X,Y)Z = c\{&g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X \\ &- g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z\} + g(AY,Z)AX - g(AX,Z)AY \\ &+ g(KY,Z)KX - g(KX,Z)KY + g(LY,Z)LX - g(LX,Z)LY, \end{aligned} \tag{2.23}$$

$$\begin{aligned} &(\nabla_X A)Y - (\nabla_Y A)X - l(X)KY + l(Y)KX - m(X)LY \\ &\quad + m(Y)LX = c\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi\}, \end{aligned} \tag{2.24}$$

$$(\nabla_X K)Y - (\nabla_Y K)X = -l(X)AY + l(Y)AX + t(X)LY - t(Y)LX, \tag{2.25}$$

$$(\nabla_X L)Y - (\nabla_Y L)X = -m(X)AY + m(Y)AX - t(X)KY + t(Y)KX, \tag{2.26}$$

where R is the Riemman Christoffel curvature tenser of M, and those of the Ricci tensor by

$$(\nabla_X l)Y - (\nabla_Y l)X + g((KA - AK)X,Y) = m(Y)t(X) - m(X)t(Y), \tag{2.27}$$

$$(\nabla_X m)Y - (\nabla_Y m)X + g((LA - AL)X,Y) + t(X)l(Y) - t(Y)l(X) = 0, \tag{2.28}$$

$$\begin{aligned} &(\nabla_X t)Y - (\nabla_Y t)X + g((LK - KL)X,Y) \\ &\qquad = l(Y)m(X) - l(X)m(Y) + 2cg(\phi X,Y). \end{aligned} \tag{2.29}$$

In the end of this section, we introduce the structure Jacobi operator R_ξ with respect to the structure vector field ξ which is defined by $R_\xi X = R(X,\xi)\xi$ for any vector field X. Then we have from (2.23)

$$R_\xi X = c(X - \eta(X)\xi) + \alpha AX - \eta(AX)A\xi + \eta(K\xi)KX - \eta(KX)K\xi + \eta(L\xi)LX - \eta(LX)L\xi.$$

Since l and m are dual 1-forms of $L\xi$ and $K\xi$ respectively because of (2.8) and (2.9), the last relationship is reformed as

$$R_\xi X = c(X - \eta(X)\xi) + \alpha AX - \eta(AX)A\xi + kKX + m(X)K\xi - l(X)L\xi, \quad (2.30)$$

where we have used (2.8) $\sim$ (2.12).

3. Structure equations satisfying $dt = 2\theta\omega$

In this section we will suppose that M is a semi-invariant submanifold of codimension 3 in a complex space form $M_{n+1}(c)$, $c \neq 0$. Further, suppose that the third fundamental form t satisfies

$$dt = 2\theta\omega, \quad \omega(X, Y) = g(\phi X, Y) \quad (3.1)$$

for any vector fields X and Y and a certain scalar θ, where d denotes the exterior differential operator. Then (2.29) is reduced to

$$g((LK - KL)X, Y) + l(X)m(Y) - l(Y)m(X) = -2(\theta - c)g(\phi X, Y),$$

which together with (2.15) yields

$$g(LKX, Y) + l(X)m(Y) = -(\theta - c)g(\phi X, Y), \quad (3.2)$$

From this and (2.8)$\sim$(2.12) we have

$$KL\xi = kL\xi, \quad LK\xi = 0. \quad (3.3)$$

for any vector X on M.

Differentiating (3.1) covariantly along M and making use of (2.4) and the first Bianchi identity, we find

$$(X\theta)\omega(Y, Z) + (Y\theta)\omega(Z, X) + (Z\theta)\omega(X, Y) = 0,$$

which implies $(n - 2)X\theta = 0$. Therefore, θ is a constant if $n > 2$.

For the case where $\theta = c$ in (3.1) we have $dt = 2c\omega$. In this case, the normal connection of M is said to be *L-flat*([21]).

By properties of the almost contact metric structure we have from (3.2)

$$T_r({}^tKK) - \|m\|^2 + \|l\|^2 = 2(n - 1)(\theta - c),$$

where we have used (2.6), (2.9) and (2.10), which connected to (2.8) gives

$$\|K - m \otimes \xi\|^2 + \|l\|^2 = 2(n - 1)(\theta - c), \quad (3.4)$$

where $\|T\|^2 = g(T, T)$ for any tensor field T on M. Hence $\theta - c$ is nonnegative.

In the same way, we can verify, using (2.7), (2.11), (2.14) and (3.2), that

$$\|m + k\xi\|^2 - T({}^tLL) = 2(n - 1)(\theta - c). \quad (3.5)$$

In the previous paper [15] we prove the following :

Lemma 3.1. *Let M be a semi-invariant submanifold with L-flat normal connection in $M_{n+1}(c)$, $c \neq 0$. If $A\xi = \alpha\xi$, then we have $\nabla^{\perp} C = 0$ and $K = L = 0$ on M.*

Transforming (3.2) by ϕ and using (2.6) and (2.14), we find

$$K^2 X + \eta(X) K^2 \xi + l(X) L\xi = (\theta - c)\{X - \eta(X)\xi\},$$

which shows $\eta(X)K^2\xi - g(K^2\xi, X)\xi = 0$. Thus, we have

$$K^2\xi = (\|K\xi\|^2)\xi$$

because of (2.8), where $g(K\xi, K\xi) = \|K\xi\|^2$. Combining above two equations, it follows that

$$K^2 X + l(X)L\xi + \|K\xi\|^2 \eta(X)\xi = (\theta - c)(X - \eta(X)\xi). \tag{3.6}$$

In the same way, we have from (3.2)

$$L^2\xi = kK\xi + (\|K\xi\|^2 + k^2)\xi, \tag{3.7}$$

where we have used (2.7), (2.13) and (3.3).

Since we have (2.14) and the second equation of (3.3), we see from (3.2)

$$(\theta - c - \|K\xi\|^2)L\xi = 0.$$

On the other hand we have from (3.2)

$$kl(LX) = (\theta - c - \|L\xi\|^2)m(X) + k(\theta - c)\eta(X)$$

because of (2.13) and (3.3), which together with (3.7) yields

$$(\theta - c - \|L\xi\|^2 - k^2)(\|K\xi\|^2 - k^2) = 0.$$

Now, let Ω_0 be a set of points such that $\|L\xi\| \neq 0$ and Ω_0 be nonvoid. Then we have

$$\|K\xi\|^2 = \theta - c, \quad \|L\xi\|^2 + k^2 = \theta - c \tag{3.8}$$

on Ω_0.

In fact, if not, then we have $m(X) = -k\eta(X)$, which connected to (2.13) gives $l(\phi X) = 0$ and hence $L\xi = 0$, a contradiction. Thus, the second equation of (3.8) is established.

We discuss our arguments on Ω_0. Using equations already obtained, we can find (for detail, see (2.22) and (2.24) of [15])

$$\nabla_X k = 2AL\xi, \tag{3.9}$$

$$\nabla_X L\xi = t(X)K\xi - AKX - kAX. \tag{3.10}$$

Differentiating (3.9) covariantly and taking the skew-symmetric part obtained, we get

$$(\theta - 2c)\{\eta(X)K\xi - m(X)\xi\} = 0,$$

where we have used (2.13), (2.24), (3.3) and (3.10), which shows

$$(\theta - 2c)(m(X) + k\eta(X)) = 0$$

and hence $(\theta - 2c)l(X) = 0$ by virtue of (2.13). Thus, $\theta - 2c = 0$ on Ω_1. Therefore we conclude that

Lemma 3.2. *Let M be a semi-invariant submanifold of codimension 3 in $M_{n+1}(c)$, $c \neq 0$ satisfying $\theta \neq 2c$. Then, we have $l = 0$.*

Throughout this paper, we assume that M satisfies (3.1) with $\theta \neq 2c$. Then, by Lemma 3.2 we have $l = 0$ and hence

$$m(X) = -k\eta(X) \tag{3.11}$$

because of (2.13). Hence (2.8) and (2.9) are reduced respectively to

$$K\xi = k\xi, \quad L\xi = 0. \tag{3.12}$$

Since $l = 0$ on M, (3.2) and (2.7) are reformed respectively as

$$g(LKX, Y) = -(\theta - c)g(\phi X, Y), \tag{3.13}$$

$$L = -\phi K. \tag{3.14}$$

From the last two relationships, we obtain

$$KL + LK = 0, \tag{3.15}$$

$$L^2 X = (\theta - c)(X - \eta(X)\xi). \tag{3.16}$$

It is clear, using (3.11), that (2.6) becomes

$$KX = \phi LX + k\eta(X). \tag{3.17}$$

If we take account of (3.11) and Lemma 3.2, then (2.24)$\sim$(2.28) are reformed respectively as

$$\begin{aligned}(\nabla_X A)Y - (\nabla_Y A)X = k\{\eta(Y)LX - \eta(X)LY\} \\ + c\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\},\end{aligned} \tag{3.18}$$

$$(\nabla_X K)Y - (\nabla_Y K)X = t(X)LY - t(Y)LX, \tag{3.19}$$

$$(\nabla_X L)Y - (\nabla_Y L)X = k\{\eta(X)AY - \eta(Y)AX\} - t(X)KY + t(Y)KX, \tag{3.20}$$

$$KAX - AKX = k\{\eta(X)t - t(X)\xi\}, \tag{3.21}$$

$$LAX - ALX = (Xk)\xi - \eta(X)\nabla k + k(\phi AX + A\phi X), \tag{3.22}$$

Putting $X = \xi$ in (3.21) and using (3.12), we find

$$KA\xi = kA\xi + k\{t - t(\xi)\xi\}. \tag{3.23}$$

If we apply this by ϕ and use (2.19), (3.12) and (3.14), then we get

$$g(KU, X) = k\{t(\phi X) - u(X)\}, \tag{3.24}$$

where $u(X) = g(U, X)$ for any vector X.

Replacing X by ξ in (3.22) and using (2.5), (3.12) and (3.14), we get

$$KU = (\xi k)\xi - \nabla k + kU. \tag{3.25}$$

which together with (3.24) gives

$$Xk = (\xi k)\eta(X) + k\{2u(X) - t(\phi X)\}. \tag{3.26}$$

This yields $\phi\nabla k = k\{2(A\xi - \alpha\xi) + t - t(\xi)\xi\}$.

If we apply (3.22) by ϕ and take account of (3.17) and the last equation, then we find

$$\begin{aligned}\phi ALX - KAX = k\{&(t - t(\xi)\xi)\eta(X) + 2\eta(X)(A\xi - \alpha\xi)\\ &+ 2g(A\xi, X)\xi - AX - \phi A\phi X\},\end{aligned}$$

or, using (3.21) we have $\phi AL + LA\phi = 0$.

Since θ is constant if $n > 2$, differentiating (3.16) covariantly, we get

$$2L\nabla_X L = (c - \theta)\{\eta(X)\phi A + g(\phi A, X)\xi\},$$

or, using (3.13) and (3.20), it is verified that (see, [15])

$$\begin{aligned}2(\nabla_X L)LY =& (\theta - c)\{-2t(X)\phi Y + \eta(Y)(A\phi - \phi A)X - g((A\phi + \phi A)X, Y)\xi\\ &+ \eta(X)(\phi A + A\phi)Y\} + k\{\eta(Y)(AL + LA)X\\ &- g((AL + LA)X, Y)\xi - \eta(X)(LA - AL)Y\},\end{aligned}$$

which together with (3.12) and (3.26) yields

$$\begin{aligned}&(\theta - c)(A\phi - \phi A)X + (k^2 + \theta - c)(u(X)\xi + \eta(X)U)\\ &\quad + k\{(AL + LA)X + k\{-t(\phi X)\xi + \eta(X)\phi \circ t\} = 0.\end{aligned} \tag{3.28}$$

Taking the trace of this, we obtain

$$kTr(AL) = 0. \tag{3.29}$$

In the previous paper [15], the following proposition was proved for the case where $c > 0$.

Proposition 3.3. *If M satisfies $dt = 2\theta\omega$ for a scalar $\theta(\neq 2c)$ and $\mu = 0$ in $M_{n+1}(c)$, $c \neq 0$, then we have $k = 0$ on M.*

Proof. This fact was proved for $c > 0$ (see, Proposition 3.5 of [15]). But, regardless of the sign of c this one is established. However, only $\xi k = 0$ and $\xi\alpha = 0$ should be newly certified. We are now going to prove that $\xi k = 0$.

Differentiating (3.11) covariantly and using (2.5), we find

$$\nabla_X m = -(Xk)\xi + k\phi AX,$$

from which, taking the skew-symmetric part and using (2.28) with $l = 0$,

$$LAX - ALX - k(\phi A + A\phi)X = (Xk)\xi - \eta(X)\nabla k.$$

If we put $X = \xi$ in this and make use of (3.12), then we find

$$\nabla k = (\xi k)\xi \tag{3.30}$$

because $A\xi = \alpha\xi$ was assumed. From the last two equations, if follows that

$$LA - AL = k(\phi A + A\phi). \tag{3.31}$$

Differentiating (3.30) covariantly, and taking the skew-symmetric part obtained, we find

$$(\xi k)(A\phi + \phi A) = 0, \tag{3.32}$$

where we have used (2.5).

Since we have $A\xi = \alpha\xi$ because of (2.16), we can write (2.22) as

$$(\nabla_X A)\xi = -A\phi AX + \alpha\phi AX + (X\alpha)\xi,$$

which together with (3.12) and (3.18) gives

$$2A\phi AX + \alpha(A\phi + \phi A) + 2c\phi X = \eta(X)\nabla\alpha - (X\alpha)\xi. \tag{3.33}$$

Putting $X = \xi$ in this, we also find

$$\nabla\alpha = (\xi\alpha)\xi. \tag{3.34}$$

Using the quite same method as that used to (3.32) from (3.30), we can derive from the last equation the following :

$$(\xi\alpha)(\phi A + A\phi) = 0. \tag{3.35}$$

Now, if we suppose that $\xi k \neq 0$. Then we have

$$\phi A + A\phi = 0, \quad LA = AL$$

on this open subset because of (3.31) and (3.32). We discuss our arguments on such a place. By virtue of (3.34) and the last relationship, we can write (3.33) as

$$A^2\phi + c\phi = 0,$$

If we apply this by ϕ, then we obtain

$$A^2X + cX = (\alpha^2 + c)\eta(X)\xi, \tag{3.36}$$

where we have used $A\xi = \alpha\xi$.

Since we have $A\xi = \alpha\xi$, that is, $U = 0$ was assumed, (3.28) can be written as $(\theta - c)A\phi X + kALX = 0$ with the aid of (3.24), which together with (3.17) yields

$$(\theta - c)AX + kAKX = \alpha(\theta - c + k^2)\eta(X)\xi.$$

Combining this to (3.36), we find $kKX+(\theta-c)X = (\theta-c+k^2)\eta(X)\xi$, which shows $(n - 1)(\theta - c) = 0$. Thus we have $\theta - c = 0$ if $n > 2$, This contradicts Lemma 3.1. Thus $\xi k = 0$ is proved on M.

By the same as above we can prove $\xi\alpha = 0$ by virtue of (3.34) and (3.35). This completes the proof. □

We set $\Omega = \{p \in M : k(p) \neq 0\}$, and suppose that Ω is not empty, In the rest of this paper, we discuss our arguments on the open subset Ω of M. So, by Proposition 3.3, we see that $\mu \neq 0$ on Ω.

4. Jacobi operators of semi-invariant submanifolds

We will continue now, our arguments under the same hypotheses $dt = 2\theta\omega$ for a scalar $\theta(\neq 2c)$ as in section 3. Then, by virtue of (3.11) and (3.12) we can write (2.30) as

$$R_\xi X = c(X - \eta(X)\xi) + \alpha AX - \eta(AX)A\xi + kKX - k^2\eta(X)\xi, \tag{4.1}$$

which implies.

$$R_\xi KX = c(KX - k\eta(X)\xi) + \alpha AKX - \eta(AKX)A\xi + kK^2X - k^3\eta(X)\xi,$$

where we have used (3.12), from which taking the skew-symmetric part,

$$(R_\xi K - KR_\xi)X = \alpha(AK - KA)X + g(A\xi, X)KA\xi - g(KA\xi, X)A\xi,$$

which together with (2.16), (3.21) and (3.23) gives

$$(R_\xi K - KR_\xi)X = k\mu\{t(X)W - w(X)t - t(\xi)(\eta(X)W - w(X)\xi)\},$$

where $g(W, X) = w(X) =$ for any vector X.

According to Proposition 3.3, we then have

Lemma 4.1. *$R_\xi K = KR_\xi$ holds on Ω if and only if $t \in f(\xi, W)$, where $f(\xi, W)$ denoted a linear subspace spanned by ξ and W.*

Under the hypotheses of Lemma 4.1, we have

$$t = t(\xi)\xi + t(W)W. \tag{4.2}$$

From (2.17) and (4.2) we obtain $t(\phi X) = -\frac{1}{\mu}t(W)u(X)$, which together with (3.24) yields

$$KU = \tau U, \tag{4.3}$$

where τ is defined by $\mu\tau = -k(\mu + t(W))$, or using (3.14),

$$LU = \mu\tau W. \tag{4.4}$$

By virtue of (3.13) and the last two relationships, it follows that

$$\tau^2 = \theta - c. \tag{4.5}$$

Since $\theta - c \neq 0$ on Ω by Lemma 3.1, τ is a positive constant on Ω if $n > 2$.

In a direct consequcence of (3.14) and (4.3), we see that

$$\mu LW = \tau U. \tag{4.6}$$

Using (2.16) and (3.12), we can write (3.23) as

$$\mu KW = k\mu W + k(t - t(\xi)\xi),$$

which together with (4.2) implies that

$$KW = -\tau W \tag{4.7}$$

because of Proposition 3.3.

Now, by using (3.24) and (4.3) we see that

$$t(\phi X) = -\mu(1 + \frac{\tau}{k})u(X) \tag{4.8}$$

on Ω, which connected to the property of the almost contact metric structure implies that

$$t = t(\xi)\xi - \mu(1 + \frac{\tau}{k})W. \tag{4.9}$$

If we take account of (4.3), then (3.25) can be written as

$$\nabla k = (\xi k)\xi + (k - \tau)U. \tag{4.10}$$

On the other hand, if we use (2.19) and (2.24), then (2.22) implies that

$$(\nabla_\xi A)\xi = 2AU + \nabla\alpha + 2\eta(L\xi) - 2\eta(K\xi)L\xi.$$

Thus, it follows, using (3.12), that

$$(\nabla_\xi A)\xi = 2AU + \nabla\alpha. \tag{4.11}$$

Putting $X = \xi$ in (2.22) and using (2.16), (2.18) and (4.11), we get

$$\nabla_\xi U = 3\phi AU + \alpha A\xi - \beta\xi + \phi\nabla\alpha, \tag{4.12}$$

which together with (3.12), (3.14) and (4.7) gives

$$-K\nabla_\xi U = 3LAU + \alpha\mu W + \mu^2 k\xi + L\nabla\alpha. \tag{4.13}$$

In the following, we see, using (2.16) and (2.19), that $\phi U = -\mu W$. Differentiating the last equation covariantly and using (2.4), we find

$$g(AU, X)\xi - \phi\nabla_X U = (X\mu)W + \mu\nabla_X W.$$

Putting $X = \xi$ in this and using (4.12), we get

$$\mu\nabla_\xi W = 3AU - \alpha U + \nabla\alpha - (\xi\alpha)\xi - (\xi\mu)W, \tag{4.14}$$

which tells us that

$$W\alpha = \xi\mu. \tag{4.15}$$

In the next step suppose, throughout this paper, that $R_\xi\phi = \phi R_\xi$. Then from (4.1) we have

$$\alpha(\phi A - A\phi)X = g(A\xi, X)U + g(U, X)A\xi + 2kLX, \tag{4.16}$$

where we have used (3.11), (3.12) and (3.14). Applying this by L and using (2.19), (3.17) and (3.22), we find

$$\begin{aligned}\alpha\{AKX - k\eta(X)A\xi - \phi ALX\}& \\ + g(LU, X)A\xi + g(KU, X)U + 2kL^2X = 0,&\end{aligned} \tag{4.17}$$

which together with (2.16), (3.21) and (3.24) yields

$$\begin{aligned}k\alpha\{t(X)\xi - \eta(X)t + \mu(w(X)\xi - \eta(X)W\}& \\ + g(LU, X)A\xi - g(A\xi, X)LU - u(X)KU + g(KU, X)U = 0.&\end{aligned}$$

If we take the inner product with ξ to this and make use of (3.12), then we get

$$k\alpha\{t(X) - t(\xi)\eta(X) + g(A\xi, X) - \alpha\eta(X)\} + \alpha g(LU, X)U = 0. \tag{4.18}$$

Combining the last two relationships and making use of (2.18), we get

$$\mu\{w(X)LU - g(LU, X)W\} + u(X)KU - g(KU, X)U = 0. \tag{4.19}$$

We notice here that the following (cf. see [10]) :

Remark 4.1. *$\alpha \neq 0$ on Ω.*

Now, putting $X = U$ in (4.16) and using (2.16) and (2.19), we find

$$\alpha(\phi AU + \mu AW) = \mu^2 A\xi + 2kLU. \tag{4.21}$$

Remark 4.2. *$\Omega = \emptyset$ if $\theta = c$.*

In fact, since $\theta - c = 0$ was assumed, (3.16) implies that $L = 0$ and hence $KX = k\eta(X)\xi$ by virtue of (3.17). Thus, (3.20) becomes

$$k\{\eta(X)AY - \eta(Y)AX + \xi(\eta(X)t(Y) - \eta(Y)t(X))\} = 0,$$

which enables us to obtain $k\{t(X) + g(A\xi, X) - \sigma\eta(X)\} = 0$, where we have put $\sigma = \alpha + t(\xi)$. Therefore, combining the last two equations, it follows that

$$AX = \eta(X)A\xi + g(A\xi, X)\xi - \alpha\eta(X)\xi.$$

From this we have $AU = 0$ and $AW = \mu\xi$. Consequently we see from (4.21) $\mu = 0$, a contradiction because of Proposition 3.3.

Lemma 4.2. *Let M be a semi-invariant submanifold of codimension 3 in $M_{n+1}(c)$, $c \neq 0$ satisfying $dt = 2\theta\omega$ for a scalar $\theta(\neq 2c)$. If it satisfies $R_\xi\phi = \phi R_\xi$, then $R_\xi K = KR_\xi$ holds on Ω.*

Proof. Applying (4.16) by A and taking the trace obtained, we get $g(A^2\xi, U) = 0$ because of (3.29), which together with (2.16) and Lemma 3.1 gives $g(AW, U) = 0$.

If we take the inner product with U to (4.21) and make use of (2.19) and the last assertion, then we have $g(LU, U) = 0$.

Putting $X = U$ in (4.19) and using this fact we have $KU = \tau U$, where τ is given by $\tau\mu^2 = g(KU, U)$ because of Proposition 3.3, which together with (3.14) implies that $LU = \tau\mu W$. Thus, if we combine this and (2.16) to (4.18), then we obtain

$$k\alpha\{t - t(\xi)\xi + \mu(1 + \frac{\tau}{k})W\} = 0.$$

Because of Remark 4.1, we have (4.9). Hence $t \in f(\xi, W)$. By Lemma 4.1, we conclude that $R_\xi K = KR_\xi$. □

In the next place, differentiating (4.1) covariantly along Ω, we find

$$\begin{aligned} g((\nabla_X R_\xi)Y, Z) = &-(k^2+c)\{\eta(Z)g(\nabla_X, \xi Y) + \eta(Y)g(\nabla_X \xi, Z)\} + (X\alpha)g(AY, Z) \\ &+ \alpha g((\nabla_X A)Y, Z) - g(A\xi, Z)\{g((\nabla_X A)\xi, Y) - g(A\phi AY, X)\} \\ &- g(A\xi, Y)\{g((\nabla_X A)\xi, Z) - g(A\phi AZ, X)\} + (Xk)g(KY, Z) \\ &+ kg((\nabla_X K)Y, Z) - 2k(Xk)\eta(Y)\eta(Z). \end{aligned}$$

Replacing X by ξ in this and using (2.5) and (4.11), we find

$$\begin{aligned} (\nabla_\xi R_\xi)X = &-(k^2+c)(u(X)\xi + \eta(X)U) + (\xi\alpha)AX + \alpha(\nabla_\xi A)X \\ &+ (\xi k)KX + k(\nabla_\xi K)X - 2k(\xi k)\eta(X)\xi \\ &- (3AU + \nabla\alpha)g(A\xi, X) - (3g(AU, X) + X\alpha)A\xi. \end{aligned} \tag{4.22}$$

For each point $p \in M$ and each unit tangent vector $X \in T_pM$, we defined R'_X by $R'_X = (\nabla_X R)(\cdot, X)X$. Then, in particular supposing that the structure vector field ξ of M is a geodesic vector field, it is easily seen that $R'_\xi = 0$ on M if and only if the Jacobi operator R_ξ is diagonalizable by a parallel orthonormal frame field along each trajectory of ξ and at the same time their eigenvalues are constant along each trajectory of ξ (cf. [3]).

Now, suppose that $R'_\xi = (\nabla_\xi R_\xi)\xi = 0$ on M. Then we have from (4.22)

$$\alpha(\nabla_\xi A)\xi + k(\nabla_\xi K)\xi = (k^2+c)U + \alpha(3AU + \nabla\alpha) + k(\xi k)\xi,$$

where we have used the first equation of (3.12). From $K\xi = k\xi$, we have $(\nabla_X K)\xi + K\nabla_X\xi = (X\xi)\xi + k\nabla_X\xi$, which shows $(\nabla_\xi K)\xi + KU = (\xi k)\xi + kU$. If we combine this to the last equation, then we find

$$\alpha AU + kKU + cU = 0, \tag{4.23}$$

where we have used (4.11).

In the rest of this paper, we shall suppose that M satisfies $R_\xi\phi = \phi R_\xi$ and $R'_\xi = 0$. Then from (4.3) and (4.23) we have

$$\alpha AU + (k\tau + c)U = 0,$$

which implies that

$$AU = \lambda U, \quad \alpha\lambda + k\tau + c = 0. \tag{4.24}$$

From this and (4.21) we have

$$\alpha AW = \alpha\mu\xi + (\mu^2 + 2k\tau + \alpha\lambda)W,$$

where we have used (2.16), (2.19) and (4.4), which implies that

$$AW = \mu\xi + (\rho - \alpha)W, \tag{4.25}$$

where we have put

$$\alpha(\rho-\alpha)=\mu^2+2k\tau+\alpha\lambda. \tag{4.26}$$

Differentiating (4.25) covariantly along Ω, we find

$$(\nabla_X A)W+A\nabla_X W=(X\mu)\xi+\mu\nabla_X\xi+X(\rho-\alpha)W+(\rho-\alpha)\nabla_X W. \tag{4.27}$$

Taking the inner product W to this and using (2.21) and (4.25), we find

$$g((\nabla_X A)W,W)=-2g(AU,X)+X\rho-X\alpha \tag{4.28}$$

because W is orthogonal to ξ. If we apply (4.27) by ξ and take account of (2.21), we also find

$$\mu g((\nabla_X A)W,\xi)=(\rho-2\alpha)g(AU,X)+\mu(X\mu) \tag{4.29}$$

or, using (3.18)

$$\mu(\nabla_\xi A)W=(\rho-2\alpha)AU+\mu\nabla\mu-k\mu LW-cU. \tag{4.30}$$

From this we verify, using (3.12), (3.18), (4.6) and (4.28), that

$$\mu(\nabla_W A)\xi=(\rho-2\alpha)AU-2cU+\mu\nabla\mu. \tag{4.31}$$

Putting $X=\xi$ in (4.28) and using (4.29), we obtain

$$W\mu=\xi\rho-\xi\alpha. \tag{4.32}$$

Replacing X by ξ in (4.27) and using (4.6) and (4.30), we find

$$\begin{aligned}(\rho-2\alpha)AU-k\tau U-cU+\mu\nabla\mu+\mu(A\nabla_\xi W-(\rho-\alpha)\nabla_\xi W)\\ =\mu(\xi\mu)\xi+\mu^2U+\mu(\xi\rho-\xi\alpha)W,\end{aligned}$$

which together with (4.14) and (4.15) gives

$$\begin{aligned}3A^2U-2\rho AU+(\alpha\rho-\beta-k\tau-c)U+A\nabla\alpha+\frac{1}{2}\nabla\beta-\rho\nabla\alpha\\ =2\mu(W\alpha)\xi+(2\alpha-\rho)(\xi\alpha)\xi+\mu(\xi\rho)W.\end{aligned} \tag{4.33}$$

Differentiating the equation $AU=\lambda U$ covariantly, we find

$$(\nabla_X A)U+A\nabla_X U=(X\lambda)U+\lambda\nabla_X U,$$

from which, taking the skew-symmetric part,

$$\begin{aligned}\mu(k\tau+c)(\eta(X)w(Y)-\eta(Y)w(X))+g(A\nabla_X U,Y)-g(A\nabla_Y U,X)\\ =(X\lambda)u(Y)-(Y\lambda)u(X)+\lambda(g(\nabla_X U,Y)-g(\nabla_Y U,X)),\end{aligned}$$

where we have used (2.16), (2.19), (3.18) and (4.4).

Replacing X by U in this and taking account of $AU=\lambda U$, we find

$$A\nabla_U U - \lambda\nabla_U U = (U\lambda)U - \mu^2\nabla\lambda. \qquad (4.34)$$

If we take the inner product this with ξ and remember (4.25), then we also find

$$\mu g(\nabla_U U, \xi) + \mu^2(W\lambda) + (\rho - \alpha - \lambda)g(\nabla_U U, W) = 0. \qquad (4.35)$$

On the other hand, differentiating (4.3) covariantly, we find

$$(\nabla_X K)U + K\nabla_X U = \tau\nabla_X U. \qquad (4.36)$$

If we take the inner product with U to this, then we have $g((\nabla_X K)U, U) = 0$. But, from (3.19), (4.2) and (4.4) we have $(\nabla_U K)U = 0$, which together with (4.7) and (4.36) yields $g(\nabla_U U, W) = 0$. Thus, (4.35) turns out to be

$$\mu g(\nabla_U U, \xi) + \mu^2(W\lambda) = 0.$$

However, the first term of this vanishes identically by virtue of (2.20) and (4.25). Thus, it follows that $\mu(W\lambda) = 0$ and hence $W\lambda = 0$ by virtue of Proposition 3.3.

In the same way, we can verify, using (2.20) and (4.25), that $\xi\lambda = 0$. Summing up, we have

$$\xi\lambda = 0, \quad W\lambda = 0. \qquad (4.37)$$

If we put $X = \mu W$ in (3.28) and take account of (2.16), (3.12), (4.2), (4.6) and (4.25), then we find

$$(\theta - c)\{AU - (\rho - \alpha)U\} + k\tau\{AU + (\rho - \alpha)U\} = 0,$$

which together with (4.5) and (4.24) yields

$$\lambda(k + \tau) + (\rho - \alpha)(k - \tau) = 0. \qquad (4.38)$$

Finally, differentiating (2.16) covariatly and using (2.5), we find

$$(\nabla_X A)\xi + A\phi AX = (X\alpha)\xi + \alpha\phi AX + (X\mu)W + \mu\nabla_X W.$$

If we put $X = \mu W$ in this and make use of (4.24), (4.25) and (4.31), then we find

$$\mu^2\nabla_W W - \mu\nabla\mu = (2\rho\lambda - 3\alpha\lambda + \alpha^2 - \alpha\rho - 2c)U - \mu(W\alpha)\xi - \mu(W\mu)W. \quad (4.39)$$

5. Semi-invariant submanifolds satisfying $R_\xi\phi = \phi R_\xi$ and $R_\xi^{'} = 0$

We will continue our arguments under the same hypotheses as that in section 3. Further, we assume that $R_\xi\phi = \phi R_\xi$ and $R_\xi^{'} = 0$ hold on M. Then all equations obtained in section 4 are valid.

Lemma 5.1. *Let M be a semi-invariant submanifold of codimension 3 in $M_{n+1}(c)$, $c \neq 0$, $n > 2$ such that $dt = 2\theta\omega$ for a scalar $\theta(\neq 2c)$. If it satisfies $R_\xi \phi = \phi R_\xi$ and $R_\xi^{'} = 0$, then we have*

$$\nabla k = (k - \tau)U. \tag{5.1}$$

Proof. Differentiating the second relationship of (4.24) with respect to W and using (4.37), we find $\lambda(W\alpha) = 0$ with the aid of (4.10).

But, we notice here that $\lambda \neq 0$ if $\xi k \neq 0$. In fact, if not, then we have $\lambda = 0$ on this open subset. So, we have $k\tau + c = 0$ on the set because of (4.24), which shows $\tau \nabla k = 0$ on the set. Thus, we have $\nabla k = 0$ because of Remark 4.2, a contradiction. Hence $\xi k = 0$ on Ω is proved. □

Because of (2.17) we can write (4.9) as

$$t(Y) = t(\xi)\eta(Y) - (1 + \frac{\tau}{k})g(\phi U, Y)$$

for any vector field Y. Differentiating this covariantly along Ω, we find

$$\begin{aligned} X(t(Y)) =& X(t(\xi))\eta(Y) + t(\xi)g(\phi AX, Y) + \frac{\tau}{k^2}(k - \tau)\mu u(X)w(Y) \\ &- (1 + \frac{\tau}{k})\{\lambda u(X)\eta(Y) + g(\phi \nabla_X U, Y)\}, \end{aligned}$$

from which, taking the skew-symmetric part and making use of (2.19), (2.22), (3.1), (4.24) and (5.1) implies that

(5.2)

$$\begin{aligned} &2\theta g(\phi X, Y) + \frac{\tau}{k^2}(k - \tau)\mu\{u(Y)w(X) - u(X)w(Y)\} \\ &+ t(\xi)\{g(\phi AX, Y) - g(\phi AY, X)\} = X(t(\xi))\eta(Y) - Y(t(\xi)\eta(X) \\ &+ (1 + \frac{\tau}{k})\{2cg(\phi X, Y) + \lambda(u(X)\eta(Y) - u(Y)\eta(X)) + (X\alpha)\eta(Y) - (Y\alpha)\eta(X) \\ &+ 2g(A\phi AX, Y) + \alpha(g(\phi AX, Y) - g(\phi AY, X))\}. \end{aligned}$$

Putting $Y = \xi$ in this and using (2.5) and (4.24), we find

$$\begin{aligned} X(t(\xi)) =& \xi(t(\xi))\eta(X) + t(\xi)u(X) \\ &+ (1 + \frac{\tau}{k})\{(\alpha - 2\lambda)u(X) - (\xi\alpha)\eta(X) + X\alpha\}. \end{aligned} \tag{5.3}$$

Lemma 5.2. *Under the same hypotheses as those in Lemma 5.1, we have $k - \tau \neq 0$ on Ω.*

Proof. If not, then we have $k-\tau=0$ on an open subset of Ω. We discuss our arguments on such a place. Then we have $\lambda=0$ because of (4.38). Thus, (4.24) tells us that $AU=0$ and $\tau^2+c=0$, which together with (4.5) yields $\theta=0$. We also have from (2.18) and (4.26)

$$\beta-\rho\alpha+2\tau^2=0. \tag{5.4}$$

In the next step, differentiating (4.7) covariantly and taking the skew-symmetric part, and using (3.19) and (4.6), we find

$$\begin{aligned}\frac{\tau}{\mu}\{t(Y)u(X)-t(X)u(Y)\}+g(K\nabla_XW,Y)-g(K\nabla_YW,X)\\ =\tau\{(\nabla_YW)X-(\nabla_XW)Y\}.\end{aligned} \tag{5.5}$$

If we put $X=\xi$ in this and take account of (2.21), (4.3), (4.7), (4.14) and the fact that $AU=0$, then we find

$$K\nabla\alpha+\tau\nabla\alpha=2\tau(\xi\alpha)\xi+\tau(2\alpha+t(\xi))U. \tag{5.6}$$

Replacing X by W in (5.5) and using (4.39), we also obtain

$$\mu(K\nabla\mu+\tau\nabla\mu)=2\tau(\mu^2-\alpha^2+\rho\alpha+2c)U+2\mu\tau(W\alpha)\xi.$$

If we take the inner product with U to this and make use of (4.3), then we find $\mu(U\mu)=(\mu^2-\alpha^2+\rho\alpha+2c)\mu^2$, which connected to (2.18) and (5.4) gives $\mu(U\mu)=2(\mu^2+\tau^2+c)\mu^2$ by virtue of (4.5). Hence, it follows that

$$\mu(U\mu)=2\mu^4. \tag{5.7}$$

However, if we take the inner product with U to (4.30), and use (5.4) and the fact that $\tau^2+c=0$ and $AU=0$, then we get $\mu(U\mu)=(\rho-\alpha)U\alpha+2c\mu^2$, which together with (5.7) yields

$$(\rho-\alpha)U\alpha=2(\mu^2-c)\mu^2. \tag{5.8}$$

Since we know that $k=\tau$, $\theta=0$ and $\lambda=0$, we can write (5.2) as

$$\begin{aligned}&-4cg(\phi X,Y)-t(\xi)\{g(\phi AX,Y)-g(\phi AY,X)\}\\ &\quad=X(t(\xi))\eta(Y)-Y(t(\xi))\eta(X)+2\{2g(A\phi AX,Y)\\ &\quad+\alpha(g(\phi AX,Y)-g(\phi AY,X))+(X\alpha)\eta(Y)-(Y\alpha)\eta(X)\}.\end{aligned}$$

Putting $Y=\xi$ in this and remembering $AU=0$, we get

$$X(t(\xi))+2(X\alpha)=\{\xi(t(\xi))+2\xi\alpha\}\eta(X)+(2\alpha+t(\xi))u(X).$$

Substituting this into the last equation, we obtain

$$\begin{aligned}(t(\xi)+2\alpha)\{u(X)\eta(Y)-u(Y)\eta(X)+g(\phi AX,Y)-g(\phi AY,X)\}\\ +4g(A\phi AX,Y)+4cg(\phi X,Y)=0,\end{aligned}$$

where we have used (4.5), and the fact that $\tau^2 + c = 0$ and $AU = 0$.

If we put $X = \mu W$ in this and take account of (2.17), (4.25) and the fact that $AU = 0$ and $\theta = 0$, then we get

$$(2\alpha + t(\xi))(\rho - \alpha) + 4c = 0.$$

However, applying (5.6) by U and using (4.3), we find

$$2U\alpha = \{2\alpha + t(\xi)\}\mu^2.$$

From the last two relationships it follows that $(\rho - \alpha)U\alpha = -2\mu^2$, which together with (5.8) will produce a contradiction. Therefore, $k - \tau \neq 0$ on Ω is proved. □

Now, differentiating (5.1) covariantly, and taking the skew-symmetric part obtained, $du = 0$ because of $k - \tau \neq 0$. Hence, we have $du(\xi, X) = 0$ for any vector X. If we take account of (2.5), (2.20), (4.12) and (4.24), then we see from this

$$3\lambda\phi U + A\xi - \beta\xi + \phi\nabla\alpha + \mu AW = 0,$$

or, using (2.16), (2.17) and (4.38),

$$\nabla\alpha = (\xi\alpha)\xi + (\rho - 3\lambda)U. \tag{5.9}$$

We are now going to prove that $\xi\alpha = 0$. Differentiating the second equation of (4.24) with respect to ξ and taking account of (4.37) and Lemma 5.1, we obtain $\lambda\xi\alpha = 0$. But, the function λ does not vanish on Ω because of (4.24), (5.1) and Lemma 5.2. Thus, (5.9) is reformed as

$$\nabla\alpha = (\rho - 3\lambda)U. \tag{5.10}$$

Lemma 5.3. *Under the same hypotheses as those stated in Lemma 5.1, we have $\Omega = \emptyset$.*

Proof. We already know that $du = 0$. So, from (4.36) we have

$$g(K\nabla_X U, Y) - g(K\nabla_Y U, X) + \mu\tau\{t(X)w(Y) - t(Y)w(X)\} = 0,$$

where we have used (3.19) and (4.4).

If we put $X = \xi$ in this and make use of (2.19), (2.20), (4.12) and (4.24), then we find

$$K(3\lambda\phi U + \alpha A\xi - \beta\xi + \phi\nabla\alpha) + k\mu AW + \mu\tau t(\xi)W = 0,$$

which connected to (2.16), (3.12), (3.14), (4.7), (4.25) and (5.10) gives

$$\tau t(\xi) + (\rho - \alpha)(k + \tau) = 0, \tag{5.11}$$

or, using (4.38),

$$\tau(k-\tau)t(\xi)=\lambda(k+\tau)^2. \tag{5.12}$$

Using (5.10), we can write (5.3) as

$$X(t(\xi))=\xi(t(\xi))\eta(X)+\{(1+\frac{\tau}{k})(\lambda+\alpha-\rho)+t(\xi)\}u(X). \tag{5.13}$$

Differentiating (5.11) covariantly and using (5.1), we find

$$\tau X(t(\xi))=(\alpha-\rho)(k-\tau)u(X)+(k+\tau)(X\alpha-X\rho),$$

which connected to (4.38) gives

$$\tau X(t(\xi))=(k+\tau)(X\alpha-X\rho+\lambda u(X)). \tag{5.14}$$

By the way, if we differentiate (4.38) with respect to ξ and using (4.37) and (5.1), we get $(k-\tau)(\xi\rho-\xi\alpha)=0$, which together with Lemma 5.2 gives $\xi\rho-\xi\alpha=0$. Thus, (5.14) tells us that $\xi(t(\xi))=0$ because of $\xi\rho-\xi\alpha=0$.

Hence, (5.13) can be written as

$$\tau X(t(\xi))=\{(k+\frac{\tau^2}{k}+2\tau)(\alpha-\rho)+\tau\lambda(1+\frac{\tau}{k})\}u(X),$$

where we have used (5.11).

Combining this to (5.14), we obtain

$$(k+\tau)(\nabla\alpha-\nabla\rho+\lambda U)=(1+\frac{\tau}{k})\{(k+\tau)(\alpha-\rho)+\tau\lambda\}U,$$

which connected to (4.38) and Lemma 5.2 yields

$$k(\nabla\alpha-\nabla\rho)=2\tau(\lambda+\alpha-\rho)U. \tag{5.15}$$

On the other hand, if we differentiate (5.13) and use (5.1) and itself, then we find

$$\lambda(k+\tau)^2U+\tau(k-\tau)\nabla t(\xi)=(k+\tau)^2\nabla\lambda+2\lambda(k^2-\tau^2)U,$$

or, using (5.14) and (5.15),

$$(k+\tau)\nabla\lambda=6\tau\lambda U, \tag{5.16}$$

where we have used (4.38) and Lemma 5.2.

Now, if we put $X=U$ and $Y=W$ in (5.2) and make use of (2.17), (4.24), (4.25) and (5.10), then we find

$$\begin{aligned}&\theta k(k-\tau)-\tau\alpha\lambda(k-\tau)-\tau^2(k-\tau)^2\\&\qquad=c(k^2-\tau^2)+\lambda^2(k+\tau)^2-\tau\lambda(k+\tau)(t(\xi)+\rho),\end{aligned}$$

or, using (4.5), (4.38) and (5.1), we obtain

$$\lambda^2(k+\tau)^2 + 2\lambda\alpha\tau(k-\tau) + (k-\tau)^2(\tau^2-c) = 0. \tag{5.17}$$

If we use (4.38), (5.1) and (5.10) and (5.16), then we can write this as $\lambda = 0$. Thus, (5.17) implies $\tau^2 = c$, a contradiction because of Proposition 3.3. Therefore, we conclude that $k = 0$ on M, that is, $\Omega = \emptyset$. This completes the proof of Lemma 5.3. □

6. Main theorem

We will continue our arguments under the same hypotheses as those in section 5. Then, by Lemma 5.3 we have $k = 0$ on M and hence (3.6) can be written as

$$K^2X = (\theta - c)(X - \eta(X)\xi), \tag{6.1}$$

where we have used (3.12).

By virtue of (3.24) we have $KU = 0$ and hence $\tau U = 0$ because of (4.3). Thus, (3.28) can be written as $\tau(\phi A - A\phi) = 0$. Therefore, it follows that

$$A\phi = \phi A, \tag{6.2}$$

which implies $A\xi = \alpha\xi$. From (6.2) and (3.18) with $k = 0$, we can verify that (cf. [6], [11])

$$A^2 = \alpha A + c(I - \eta \otimes \xi). \tag{6.3}$$

Further, (3.27) is reduced to

$$2(\nabla_X L)LY = \tau^2\{t(X)\phi Y + \eta(Y)A\phi X + \eta(X)\phi AY\}.$$

Applying this by L and making use of (2.5), (3.16) and (3.22) we get,

$$(\nabla_X L)Y = -t(X)KY + \eta(X)AKY + \eta(Y)AKX + g(AX, KY)\xi. \tag{6.4}$$

In the same, we have from (6.1)

$$(\nabla_X K)Y = t(X)LY - \eta(X)LAY - \eta(Y)ALX - g(AX, LY)\xi. \tag{6.5}$$

Since we have $T_rL = 0$, $K\xi = 0$ and $A\xi = \alpha\xi$, taking the trace of (6.4), we obtain

$$T_r(AK) = 0 \tag{6.6}$$

and hence

$$T_r(A^2K) = 0 \tag{6.7}$$

because of (6.3).

Since we have $AK = KA$ because of (3.21) with $k = 0$. it follows that A and K are diagonalizable at the same time. So, using (6.2) and (6.3) and the fact that $A\xi = \alpha\xi$, we can verify that A has two constant eigenvalues α and $(\alpha - \sqrt{D})/2$ with multiplicities 1, $2(n-1)$ respectively, and D denoted by $D = \alpha^2 + 4c$, where we have used (6.6) and (6.7). Consequently the trace h of A is given by (for detail, see (4.16) of [15])

$$h = n\alpha - (n-1)\sqrt{D}. \tag{6.8}$$

On the other hand, differentiating (6.5) covariantly along M and using the previously obtained formulas and the Ricci identity for K, we have (for detail, see (4.20) and (4.22) of [15])

$$(h + 3\alpha)(h - \alpha) = 4(n-1)\{(n+1)\theta - 2c(n+2)\}, \tag{6.9}$$

$$(\theta - 3c)(h - \alpha) = 2(n-1)(\theta - 2c)\alpha. \tag{6.10}$$

By the way, we have from (6.8) and (6.9)

$$\alpha(\alpha - \sqrt{D}) = 2(\theta - 3c).$$

Thus, if we combine (6.8) and (6.10) to the last relationship, we obtain

$$(\theta - 3c)^2 = (\theta - 2c)\alpha^2. \tag{6.11}$$

From (2.23) we see that the Ricci tensor S of M is given by

$$SX = c\{(2n+1)X - 3\eta(X)\xi\} + hAX - A^2X - K^2X - L^2X$$

because of Lemma 5.3, which together with (3.16) and (6.1) implies that

$$SX = \{c(2n+1) - 2\tau^2\}X + (2\tau^2 - 3c)\eta(X)\xi + hAX - A^2X.$$

Thus, the scalar curvature $\bar{r}$ of M is given by

$$\bar{r} = 2(n-1)(2n+1)c - 4(n-1)\tau^2 + h(h - \alpha). \tag{6.12}$$

where we have used (6.3).

By the way, it is clear, using (4.5), that $\theta - 3c \neq 0$ for $c < 0$. But, we also have $\theta - 3c \neq 0$ for $c > 0$ if $\bar{r} - 2(n-1)c \leq 0$.

In fact, if not, then we have $\theta = 3c$ on this subset of M. We discuss our arguments on this set. So we have $\alpha = 0$ because of (6.11). Hence, (6.3) and (6.9) imply respectively $h_{(2)} = 2(n-1)c$, $h^2 = 4(n-1)^2c$.

Using these facts and (4.5), we can write (6.12) as $\bar{r} - 2(n-1)c = 4(n-1)(2n-3)c$, a contradiction because $\bar{r} - 2(n-1)c \leq 0$. Thus, if we combine (6.10) to (6.11), then we obtain $\alpha(h - \alpha) = 2(n-1)(\theta - 3c)$, which together with (6.9) yields

$$h(h - \alpha) = 2(n-1)(2n-1)\tau^2 - 4n(n-1)c.$$

Using this fact, we can write (6.12) as $\bar{r} - 2(n-1)c = 2(n-1)(2n-3)\tau^2$.

Therefore we have $\tau = 0$ if $\bar{r} - 2(n-1)c \leq 0$ and hence $K = L = 0$ on M by virtue of (3.16) and (6.1).

Let $N_0(p) = \{\nu \in T_p^{\perp}(M) : A_\nu = 0\}$ and $H_0(p)$ be the maximal J-invariant subspace of $N_0(p)$. Since $K = L = 0$, the orthogonal complement of $H_0(p)$ is invariant under parallel translation with respect to the normal connection because of $\nabla^{\perp} C = 0$. Thus, by the reduction theorem in [9] and by Lemma 3.2 and Proposition 3.3, we conclude that

Theorem 6.1. *Let M be a real $(2n-1)$-dimensional $(n > 2)$ semi-invariant submanifold of codimension 3 in a complex space form $M_{n+1}(c)$, $c \neq 0$ with constant holomorphic sectional curvature $4c$ such that the third fundamental form t satisfies $dt = 2\theta\omega$ for a scalar $\theta - 2c \neq 0$ and $\bar{r} - 2c(n-1) \leq 0$, where $\omega(X,Y) = g(\phi X, Y)$ for any vector fields X and Y on M. If M satisfies $R_\xi \phi = \phi R_\xi$ and at the same time $R'_\xi = 0$, then M is a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$.*

Since we have $\nabla^{\perp} C = 0$, we can write (3.18) and (4.16) as

$$(\nabla_X A)Y - (\nabla_Y A)X = c\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\},$$
$$\alpha(\phi AX - A\phi X) - g(A\xi, X)U - g(U, X)A\xi = 0$$

respectively. Making use of (2.4), (2.5) and the above equations, it is prove in [15] that $g(U,U) = 0$, that is, M is a Hopf real hypersuface. Hence, we conclude that $\alpha(A\phi - \phi A) = 0$ and hence $A\xi = 0$ or $A\phi = \phi A$. Here, we note that the case $\alpha = 0$ correspond to the case of tube of radius $\pi/4$ in $P_n\mathbb{C}$([5],[6]). But, in the case $H_n\mathbb{C}$ it is known that α never vanishes for Hopf hypersurfaces (cf.[19]) Thus, owing to Theorem 6.1 and main theorem in [18] and [20], we have

Theorem 6.2. *Let M be a real $(2n-1)$-dimensional $(n > 2)$ semi-invariant submanifold of codimension 3 in a complex space form $M_{n+1}(c)$, $c \neq 0$ with constant holomorphic sectional curvature $4c$ such that $R'_\xi = 0$ and the third fundmental form t satisfies $dt = 2\theta\omega$ for a scalar θ $-2c(\neq 0)$, where R'_X is defined by $R'_X = (\nabla_X R)(\cdot, X)X$ for any unit vector field X. Then $R_\xi\phi = \phi R_\xi$ holds on M if and only if $A\xi = 0$ or M is locally congruent to one of the following hypersurfaces provided that the scalar curvature $\bar{r}$ of M satisfies $\bar{r} - 2(n-1)c \leq 0$:*

- (I) *in case that $M_n(c) = P_n\mathbb{C}$ with $\eta(A\xi) \neq 0$,*
  - (A_1) *a geodesic hypersphere of radius r, where $0 < r < \pi/2$ and $r \neq \pi/4$,*
  - (A_2) *a tube of radius r over a totally geodesic $P_k\mathbb{C}$ for some $k \in \{1, ..., n-2\}$, where $0 < r < \pi/2$ and $r \neq \pi/4$;*
- (II) *in case that $M_n(c) = H_n\mathbb{C}$,*
  - (A_0) *a horosphere,*
  - (A_1) *a geodesic hypersphere or a tube over a complex hyperbolic hyperplane $H_{n-1}\mathbb{C}$,*
  - (A_2) *a tube over a totally geodesic $H_k\mathbb{C}$ for some $k \in \{1, ..., n-2\}$.*

From (4.22) and Theorem 6.2 we have

Corollary 6.3. *If we replace the condition* $R_\xi^{'} = 0$ *by* $\nabla_\xi R_\xi = 0$ *in Theorem 6.2, then we verify that* M *is the same type as those stated in Theorem 6.2.*

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