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A NOTE ON TWO WEIGHT INEQUALITIES FOR THE DYADIC PARAPRODUCT

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Abstract. In this paper, we provide detailed proof of the Sawyer type characterization of the two weight estimate for the dyadic paraproduct. Although the dyadic paraproduct is known to be a well localized operators and the testing conditions obtained from checking boundedness of the given localized operator on a collection of test functions are provided by many authors. The main purpose of this paper is to present the necessary and sufficient conditions on the weights to ensure boundedness of the dyadic paraproduct directly.

1. Introduction

Weighted norm estimates for singular integral operators are widely encountered and studied in many areas of analysis. The one-weight case are now well understood. Precisely, one looks for a function $\phi(x)$ sharp in terms of its growth, such that

$$\|Tf\|_{L^2(w)} \leq C\phi([w]_{A_2})\|f\|_{L^2(w)}$$

where $[w]_{A_2}$ is the A_2-characteristic of the weight w . The answer of this question for the general Caldéron-Zygmund operator is given, in 2012, [5] and for their commutators [4]. The two-weight case are now only known for the maximal function [7] which is the first result obtained in two-weight setting, fractional and Poisson integrals, square functions and the Hilbert transform, the martingale transform, and positive and well localized dyadic operators. Recently, the author in [1] also provides the proof for essentially well localized operators. Although the dyadic paraproduct is a well localized operator, therefore, it follows the conditions for well localized operators, direct proof of the necessary and sufficient conditions on the weights for the boundedness of the dyadic paraproduct are not published yet except paraproduct type operators which appeared in the

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two weight estimate for the Hilbert transform. In this paper, we provide direct proof for the dyadic paraproduct. As we mentioned, we consider the boundedness of the dyadic paraproduct π_b acting from $L^2(\mathbb{R}, u)$ to $L^2(\mathbb{R}, v)$, i.e. we will characterize the following inequality

$$\|\pi_b f\|_{L^2(v)} \leq C\|f\|_{L^2(u)} \tag{1.1}$$

for all $f \in L^2(u)$. In what follows $\mathcal{D}$ denotes the dyadic intervals, h_I denotes the Haar function associated with I, and $\langle f, g\rangle := \int f\bar{g}$ denotes the inner product on $L^2(\mathbb{R})$. The dyadic paraproduct is defined by

$$\pi_b f = \sum_{I\in\mathcal{D}} \langle b, h_I\rangle \langle f\rangle_I h_I \,,$$

and the linear bound for the one weight case are obtained in [2] and the quantitative two weight estimates are obtained in [3], which are more geometric since the condition only involves the weights and not the operators, such as Carleson conditions or bilinear embedding conditions, and A_2-type conditions. It is common when one consider the two-weight problem to make the change of variables $f = wf \; u = w^{-1}$. Then it allows us to characterize the boundedness of $\pi_b(w\cdot)$ from $L^2(w)$ to $L^2(v)$, i.e. we need to characterize the inequality

$$\|\pi_b(wf)\|_{L^2(v)} \leq C\|f\|_{L^2(w)}$$

instead of the inequality (1.1). We now state the main theorem.

Theorem 1.1. *Let v and w be two Radon measures on $\mathbb{R}$. Then*

$$\|\pi_b(wf)\|_{L^2(v)} \leq C\|f\|_{L^2(w)}, \quad \|\pi_b^*(vf)\|_{L^2(w)} \leq C\|f\|_{L^2(v)} \tag{1.2}$$

if and only if for all $I, J \in \mathcal{D}$ and $I \cap J \neq \varnothing$, the following testing conditions hold:

$$\|\mathbb{1}_I \pi_b(w\mathbb{1}_I)\|_{L^2(v)} \leq C_1\|\mathbb{1}_I\|_{L^2(w)} \tag{1.3}$$

$$\|\mathbb{1}_I \pi_b^*(v\mathbb{1}_I)\|_{L^2(w)} \leq C_2\|\mathbb{1}_I\|_{L^2(v)} \tag{1.4}$$

$$|\langle \pi_b(w\mathbb{1}_I), v\mathbb{1}_J\rangle \leq C_3 w(I)^{1/2} v(J)^{1/2} \tag{1.5}$$

In Theorem 1.1, π_b^* stands for the adjoint operator of the dyadic paraproduct which is defined by

$$\pi_b^* f = \sum_{I\in\mathcal{D}} \langle b, h_I\rangle \langle f, h_I\rangle \frac{\mathbb{1}_I}{|I|} \,,$$

where $\mathbb{1}_I$ denotes the characteristic function of the interval I. We also use the notation $w(I)$ for the w-measure of an interval I, i.e. $w(I) = \int_I w dx$. Definitions and some useful lemmas are collected in Section 2. We give the proof of the main theorem and concluding remarks in Section 3 and in Section 4, respectively.

2. Definitions and Useful lemmatas

In this section, we will review definitions, notations and some useful lemmas. Throughout the paper a constant C will be a numerical constant that may change from line to line. The symbol $A \lesssim B$ means there is a constant $c > 0$ such that $A \leq cB$. Given a weight w and an interval I we define the weighed Haar function associated to I as

$$h_I^w(x) = \frac{1}{w(I)^{1/2}} \left[\frac{w(I_-)^{1/2}}{w(I_+)^{1/2}} \mathbb{1}_{I_+} - \frac{w(I_+)^{1/2}}{w(I_-)^{1/2}} \mathbb{1}_{I_-} \right],$$

where I_+ and I_- denote the right and left half of I, respectively. The Haar systems $\{h_I\}_{I\in\mathcal{D}}$ and $\{h_I^w\}_{I\in\mathcal{D}}$ are orthonormal systems in L^2 and $L^2(w)$ respectively, where $L^2(w)$ is the collection of square integrable functions with respect to the measure wdx and it is a Hilbert space with the weighted inner product defined by $\langle f, g\rangle_w = \int fgwdx$. Then, every function $f \in L^2(w)$ can be written as

$$f = \sum_{I\in\mathcal{D}} \langle f, h_I^w\rangle_w h_I^w$$

where the sum converges almost everywhere in $L^2(w)$. Moreover,

$$\|f\|_{L^2(w)}^2 = \sum_{I\in\mathcal{D}} |\langle f, h_I^w\rangle_w|^2 .$$

For convenience, we will observe basic properties of the weighted Haar system. First observe that $\langle h_K, h_I^w\rangle_w$ could be non-zero only if $I \supseteq K$; moreover, for any $I \supseteq K$,

$$|\langle h_K, h_I^w\rangle_w| \leq \langle w\rangle_K^{1/2} .$$

For $I \supsetneq J$, h_I^w is constant on J. If we denote this constant by $h_I^w(J)$, the weighted average $\langle f\rangle_{J,w} := w(J)^{-1} \int_J gw$ can be written as follows

$$\langle f\rangle_{J,w} = \sum_{I\supsetneq J} \langle f, h_I^w\rangle_w h_I^w(J) .$$

Throughout the paper the following Weighted Caleson Embedding Theorem will be used frequently. This Theorem was first stated in [6].

Theorem 2.1 (Weighted Carleson Embedding Theorem). *Let $\{\alpha_J\}$ be a non-negative sequence such that for all dyadic intervals I,*

$$\sum_{J\in\mathcal{D}(I)} \alpha_J \leq Cw(I) .$$

Then for all $f \in L^2(w)$,

$$\sum_{J\in\mathcal{D}} \alpha_J \langle f\rangle_{J,w}^2 \leq 4C\|f\|_{L^2(w)}^2 .$$

We also use the following lemmas in the proof of Theorem 1.1.

Lemma 2.2. *For any weights w and v satisfying the testing condition (1.5) and weighted Haar functions h_I^w and h_I^v, the following estimate holds:*

$$|\langle \pi_b(wh_I^w), vh_I^v\rangle| \le 4C_3 \,,$$

Proof. Using the definition of the weighted Haar function and the well localized property of π_b, we have that

$$\begin{aligned}
&|\langle \pi_b(wh_I^w), vh_I^v\rangle| \\
&\le \sqrt{\frac{w(I_-)v(I_-)}{w(I)w(I_+)v(I)v(I_+)}}\left|\left\langle \pi_b\left(w\mathbb{1}_{I_+}\right), v\mathbb{1}_{I_+}\right\rangle\right| \\
&\quad + \sqrt{\frac{w(I_+)v(I_+)}{w(I)w(I_-)v(I)v(I_-)}}\left|\left\langle \pi_b\left(w\mathbb{1}_{I_-}\right), v\mathbb{1}_{I_-}\right\rangle\right| \\
&\qquad + \sqrt{\frac{w(I_+)v(I_-)}{w(I)w(I_-)v(I)v(I_+)}}\left|\left\langle \pi_b\left(w\mathbb{1}_{I_-}\right), v\mathbb{1}_{I_+}\right\rangle\right| \\
&\qquad\quad + \sqrt{\frac{w(I_-)v(I_+)}{w(I)w(I_+)v(I)v(I_-)}}\left|\left\langle \pi_b\left(w\mathbb{1}_{I_+}\right), v\mathbb{1}_{I_-}\right\rangle\right| \\
&\le \sqrt{\frac{1}{w(I_+)v(I_+)}}\left|\left\langle \pi_b\left(w\mathbb{1}_{I_+}\right), v\mathbb{1}_{I_+}\right\rangle\right| + \sqrt{\frac{1}{w(I_-)v(I_-)}}\left|\left\langle \pi_b\left(w\mathbb{1}_{I_-}\right), v\mathbb{1}_{I_-}\right\rangle\right| \\
&\quad + \sqrt{\frac{1}{w(I_-)v(I_+)}}\left|\left\langle \pi_b\left(w\mathbb{1}_{I_-}\right), v\mathbb{1}_{I_+}\right\rangle\right| + \sqrt{\frac{1}{w(I_+)v(I_-)}}\left|\left\langle \pi_b\left(w\mathbb{1}_{I_+}\right), v\mathbb{1}_{I_-}\right\rangle\right| \\
&\le 4C_3 \,.
\end{aligned}$$

□

Lemma 2.3. *For any weights w and v satisfying the testing conditions (1.3) and (1.4), the following estimates hold with a fixed dyadic interval I_0:*

(1) $\left|\sum_{I\in\mathcal{D}(I_0)} \langle \pi_b(w\langle f, h_I^w\rangle_w h_I^w), v\langle g\rangle_{I_0,v}\mathbb{1}_{I_0}\rangle\right| \lesssim C_2\|f\|_{L^2(w)}\|g\|_{L^2(v)}$,

(2) $\left|\sum_{J\in\mathcal{D}(I_0)} \langle \langle f\rangle_{I_0,w}\pi_b(w\mathbb{1}_{I_0}), v\langle g, h_J^v\rangle_v h_J^v\rangle\right| \lesssim C_1\|f\|_{L^2(w)}\|g\|_{L^2(v)}$,

(3) $|\langle \langle f\rangle_{I_0,w}\pi_b(w\mathbb{1}_{I_0}), v\langle g\rangle_{I_0,v}\mathbb{1}_{I_0}\rangle| \lesssim C_1\|f\|_{L^2(w)}\|g\|_{L^2(v)}$.

Proof. For the estimate (1), we use the linearity of π_b, the Cauchy-Schwarz inequality, and the testing condition (1.4). Then we have

$$\begin{aligned}
&\left|\sum_{I\in\mathcal{D}(I_0)} \langle \pi_b(w\langle f, h_I^w\rangle_w h_I^w), v\langle g\rangle_{I_0,v}\mathbb{1}_{I_0}\rangle\right| \\
&\qquad \le |\langle g\rangle_{I_0,v}| \sum_{I\in\mathcal{D}(I_0)} |\langle \pi_b(w\langle f, h_I^w\rangle_w h_I^w), v\mathbb{1}_{I_0}\rangle|
\end{aligned}$$

$$\leq |\langle g\rangle_{I_0,v}| \left| \left\langle \pi_b \left(w \sum_{I\in\mathcal{D}(I_0)} \langle f, h_I^w\rangle_w h_I^w \right), v\mathbb{1}_{I_0} \right\rangle \right|$$

$$= |\langle g\rangle_{I_0,v}| \left| \left\langle w^{1/2} \sum_{I\in\mathcal{D}(I_0)} \langle f, h_I^w\rangle_w h_I^w, w^{1/2}\mathbb{1}_{I_0}\pi_b^*(v\mathbb{1}_{I_0}) \right\rangle \right|$$

$$\leq |\langle g\rangle_{I_0,v}| \left\| \sum_{I\in\mathcal{D}(I_0)} \langle f, h_I^w\rangle_w h_I^w \right\|_{L^2(w)} \|\mathbb{1}_{I_0}\pi_b^*(v\mathbb{1}_{I_0})\|_{L^2(w)}$$

$$\leq C_2 \frac{1}{v(I_0)} \left(\int_{I_0} |g|^2 v \right)^{1/2} \left(\int_{I_0} v \right)^{1/2} \|g\|_{L^2(v)}\|f\|_{L^2(w)}\|\mathbb{1}_{I_0}\|_{L^2(v)}$$

$$= C_2\|g\|_{L^2(v)}\|f\|_{L^2(w)}$$

Similarly to the estimate (1) one can immediately get the estimates (2) and (3). □

3. Proof of Theorem 1.1

First we will assume that f and g are finite linear combinations of characteristic functions $\mathbb{1}_I$ with $2^{-n}|I_0| \leq |I| \leq |I_0|$ for some dyadic interval I_0 and $n > 0$. Since we will get the estimates independent of I_0 and d, by the density of the simple function in $L^2(w)$, we can get the result for general f and g. Then our considering function f and g are compactly supported on a dyadic interval $I_0 \in \mathcal{D}$.

Since

$$\pi_b(h_I) = \sum_{J\subsetneq I} \langle b, h_J\rangle\langle h_I\rangle_J h_J \text{ and } \pi_b^*(h_I) = \langle b, h_I\rangle \frac{\mathbb{1}_I}{|I|},$$

the dyadic paraproduct and its adjoint operator are both well localized. Therefore we will consider functions f and g compactly supported on a dyadic interval $I_0 \in \mathcal{D}$. Then we can write

$$f = \sum_{I\in\mathcal{D}(I_0)} \langle f, h_I^w\rangle_w h_I^w + \langle f\rangle_{I_0,w}\mathbb{1}_{I_0}, \; g = \sum_{I\in\mathcal{D}(I_0)} \langle g, h_I^v\rangle_v h_I^v + \langle g\rangle_{I_0,v}\mathbb{1}_{I_0}.$$

For the necessary condition for boundedness of π_b and π_b^* can be easily obtained by replacing $f = g = \mathbb{1}_I$ in (1.2) and $f = \mathbb{1}_I$ and $g = \mathbb{1}_J$ in the following duality argument (3.1) for $I, J \in \mathcal{D}$. For the sufficient condition, by duality, it is enough to prove:

$$\langle \pi_b(fw), gv\rangle \leq C\|g\|_{L^2(v)}\|f\|_{L^2(w)}. \tag{3.1}$$

We start the proof of (3.1) by splitting the left-hand side of the inequality (3.1) into several sums.

$$\langle \pi_b(fw), gv\rangle = \sum_{I,J\in\mathcal{D}(I_0)} \langle \pi_b(w\langle f, h_I^w\rangle_w h_I^w), v\langle g, h_J^v\rangle_v h_J^v\rangle$$

$$+\sum_{I\in\mathcal{D}(I_0)}\langle \pi_b(w\langle f,h_I^w\rangle_w h_I^w), v\langle g\rangle_{I_0,v}\mathbb{1}_{I_0}\rangle$$
$$+\sum_{J\in\mathcal{D}(I_0)}\langle \langle f\rangle_{I_0,w}\pi_b(w\mathbb{1}_{I_0}), v\langle g,h_J^v\rangle_v h_J^v\rangle$$
$$+\langle \langle f\rangle_{I_0,w}\pi_b(w\mathbb{1}_{I_0}), v\langle g\rangle_{I_0,v}\mathbb{1}_{I_0}\rangle$$
$$:=\Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4\,.$$

By Lemma 2.3, we get the following for the last three sums:

$$|\Sigma_2|+|\Sigma_3|+|\Sigma_4|\lesssim (C_1+C_2)\|f\|_{L^2(w)}\|g\|_{L^2(v)}\,.$$

Thus it is enough to consider the first sum Σ_1. We will also split Σ_1 into several sums as follows.

$$\begin{aligned}|\Sigma_1| &= \left|\sum_{I,J\in\mathcal{D}(I_0)}\langle \pi_b(w\langle f,h_I^w\rangle_w h_I^w), v\langle g,h_J^v\rangle_v h_J^v\rangle\right|\\
&\le \left|\sum_{\substack{I,J\in\mathcal{D}(I_0)\\ J\supsetneq I}}\langle \pi_b(w\langle f,h_I^w\rangle_w h_I^w), v\langle g,h_J^v\rangle_v h_J^v\rangle\right|\\
&\quad+\left|\sum_{\substack{I,J\in\mathcal{D}(I_0)\\ I\supsetneq J}}\langle \pi_b(w\langle f,h_I^w\rangle_w h_I^w), v\langle g,h_J^v\rangle_v h_J^v\rangle\right|\\
&\quad+\left|\sum_{I\in\mathcal{D}(I_0)}\langle \pi_b(w\langle f,h_I^w\rangle_w h_I^w), v\langle g,h_I^v\rangle_v h_I^v\rangle\right|\\
&:=|\Sigma_5|+|\Sigma_6|+|\Sigma_7|\,.\end{aligned}$$

We use Lemma 2.2 for Σ_7. Then we have

$$\begin{aligned}|\Sigma_7| &= \left|\sum_{I\in\mathcal{D}(I_0)}\langle \pi_b(w\langle f,h_I^w\rangle_w h_I^w), v\langle g,h_I^v\rangle_v h_I^v\rangle\right|\\
&= \left|\sum_{I\in\mathcal{D}(I_0)}\langle f,h_I^w\rangle_w\langle g,h_I^v\rangle_v\,\langle \pi_b(wh_I^w), vh_I^v\rangle\right|\\
&\le \sup_{I\in\mathcal{D}(I_0)}|\langle \pi_b(wh_I^w), vh_I^v\rangle|\left(\sum_{I\in\mathcal{D}(I_0)}\langle f,h_I^w\rangle_w^2\right)^{1/2}\left(\sum_{I\in\mathcal{D}(I_0)}\langle g,h_I^v\rangle_v^2\right)^{1/2}\\
&\le 4C_3\|f\|_{L^2(w)}\|g\|_{L^2(v)}\end{aligned}$$

Since Σ_5 and Σ_6 are symmetric, one can get the estimate of Σ_6 similar to Σ_5. Now we only have Σ_5 to finish the proof.

Stopping intervals: In order to get the estimate for Σ_5 we will use the stopping time argument. We follow the standard construction of stopping intervals appeared in [8] to construct collection $\mathcal{S} \subset \mathcal{D}$ of stopping intervals. For a interval J, let $\widetilde{\mathcal{S}}(J)$ be the collection of maximal intervals $I \in \mathcal{D}(J)$ such that

$$\langle g \rangle_{J,v} > 2\langle g \rangle_{J,v} \,.$$

Let $S := \bigcup_{I \in \widetilde{\mathcal{S}}(J)} I$, and let $\mathcal{R}(J) := \mathcal{D}(J) \setminus \bigcup_{K \in \widetilde{\mathcal{S}}} \mathcal{D}(K)$. Then we see immediately that the collection of stopping intervals $\widetilde{\mathcal{S}}(J)$ satisfies the following properties:

(1) For any $I \in \mathcal{R}(J)$ we have $\langle g \rangle_{I,v} \le 2\langle g \rangle_{J,v}$.
(2) $v(G(J)) \le \frac{1}{2} v(J)$.

In order to construct the collection $\mathcal{S}$ of stopping intervals, consider all maximal $I \in \mathcal{D}(J)$ with $|I| \le 2^K$ where K is a fixed large integer. Then collection of these maximal intervals will be the first generation $\widetilde{\mathcal{S}}_1$ of stopping intervals. We get the second generation of stopping intervals for each $I \in \widetilde{\mathcal{S}}_1$, construct the collection $\widetilde{\mathcal{S}}(I)$ of stopping intervals and define the second generation $\widetilde{\mathcal{S}}_2 = \bigcup_{I \in \widetilde{\mathcal{S}}_1} \widetilde{\mathcal{S}}(I)$. We define recursively a sequence of next generations

$$\widetilde{\mathcal{S}}_{n+1} = \bigcup_{I \in \widetilde{\mathcal{S}}_n} \widetilde{\mathcal{S}}(I)$$

and the collection of stopping intervals $\mathcal{S} := \bigcup_{n \ge 1} \widetilde{\mathcal{S}}_n$. Property (2) gives us the following Carleson embedding condition, for all $J \in \mathcal{D}$.

$$\sum_{I \in \mathcal{S} \cap \mathcal{D}(J)} v(I) \le 2v(J) \,.$$

For every interval $I \in \mathcal{D}(I_0)$, we define the stopping parent and the projection as follows

$$\mathcal{P}(I) := \min\{J \in \mathcal{S} \,|\, J \supseteq I\} \,,$$

$$\mathbb{P}^w_J f = \sum_{I : \mathcal{P}(I) = J} \langle f, h^w_I \rangle h^w_I \,, \ \mathbb{P}^v_J g = \sum_{I : \mathcal{P}(I) = J} \langle g, h^v_I \rangle h^v_I \,.$$

Then we can write $f = \sum_{I \in \mathcal{S}} \mathbb{P}^w_I f$ and $g = \sum_{J \in \mathcal{S}} \mathbb{P}^v_J g$. Furthermore, we have

$$\begin{aligned}
\Sigma_5 &= \sum_{I,J \in \mathcal{S}} \langle \pi_b(\mathbb{P}^v_I(wf), v\mathbb{P}^v_J g \rangle \\
&= \sum_{I \in \mathcal{S}} \langle \pi_b(\mathbb{P}^w_I(wf), v\mathbb{P}^v_I g \rangle + \sum_{\substack{I,J \in \mathcal{S} \\ J \supsetneq I}} \langle \pi_b(\mathbb{P}^w_I(wf), v\mathbb{P}^v_J g \rangle \\
&:= \Sigma_8 + \Sigma_9 \,.
\end{aligned}$$

3.1. Estimating Σ_8

For the sum Σ_8, we have

$$\begin{aligned}
\Sigma_8 &= \sum_{I\in\mathcal{S}} \langle \pi_b(\mathbb{P}_I^w(wf)), v\mathbb{P}_I^v g\rangle \\
&= \sum_{I\in\mathcal{S}} \sum_{\substack{K\subsetneq L\in\mathcal{D}(I)\\ \mathcal{P}(K)=\mathcal{P}(L)=I}} \langle \pi_b(w\langle f, h_K^w\rangle_w h_K^w), v\langle g, h_L^v\rangle_v h_L^v\rangle \\
&= \sum_{I\in\mathcal{S}} \sum_{\substack{K\in\mathcal{D}(I)\\ \mathcal{P}(K)=I}} \left\langle \pi_b(w\langle f, h_K^w\rangle_w h_K^w), v \sum_{\substack{L:K\subsetneq L\in\mathcal{D}(I)\\ \mathcal{P}(L)=I}} \langle g, h_L^v\rangle_v h_L^v \right\rangle \\
&= \sum_{I\in\mathcal{S}} \sum_{\substack{K\in\mathcal{D}(I)\\ \mathcal{P}(K)=I}} \langle \pi_b(w\langle f, h_K^w\rangle_w h_K^w), v\left(\langle g\rangle_{K,v}\mathbb{1}_K - \langle g\rangle_{I,v}\mathbb{1}_I\right)\rangle \\
&= \sum_{I\in\mathcal{S}} \sum_{\substack{K\in\mathcal{D}(I)\\ \mathcal{P}(K)=I}} \langle \pi_b(w\langle f, h_K^w\rangle_w h_K^w), v\langle g\rangle_{K,v}\mathbb{1}_K\rangle \\
&\quad - \sum_{I\in\mathcal{S}} \sum_{\substack{K\in\mathcal{D}(I)\\ \mathcal{P}(K)=I}} \langle \pi_b(w\langle f, h_K^w\rangle_w h_K^w), v\langle g\rangle_{I,v}\mathbb{1}_I\rangle \; .
\end{aligned}$$

Then we get

$$\begin{aligned}
|\Sigma_8| &\le \left| \sum_{I\in\mathcal{S}} \sum_{\substack{K\in\mathcal{D}(I)\\ \mathcal{P}(K)=I}} \langle \pi_b(w\langle f, h_K^w\rangle_w h_K^w), v\langle g\rangle_{K,v}\mathbb{1}_K\rangle \right| \\
&\quad + \left| \sum_{I\in\mathcal{S}} \sum_{\substack{K\in\mathcal{D}(I)\\ \mathcal{P}(K)=I}} \langle \pi_b(w\langle f, h_K^w\rangle_w h_K^w), v\langle g\rangle_{I,v}\mathbb{1}_I\rangle \right| \\
&:= |\Sigma_{10}| + |\Sigma_{11}| \, .
\end{aligned}$$

It is good to remind the stopping condition which is $\langle g\rangle_{K,v} \le \langle g\rangle_{I,v}$. Using this fact, we have the followings

$$\begin{aligned}
|\Sigma_{10}| &= \left| \sum_{I\in\mathcal{S}} \sum_{\substack{K\in\mathcal{D}(I)\\ \mathcal{P}(K)=I}} \langle \pi_b(w\langle f, h_K^w\rangle_w h_K^w), v\langle g\rangle_{K,v}\mathbb{1}_K\rangle \right| \\
&\le \sum_{I\in\mathcal{S}} \sum_{\substack{K\in\mathcal{D}(I)\\ \mathcal{P}(K)=I}} |\langle g\rangle_{I,v}| \, |\langle \pi_b(w\langle f, h_K^w\rangle_w h_K^w), v\mathbb{1}_K\rangle|
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{I\in\mathcal{S}} |\langle g\rangle_{I,v}| \sum_{\substack{K\in\mathcal{D}(I)\\ \mathcal{P}(K)=I}} |\langle w\langle f,h_K^w\rangle_w h_K^w, \pi_b^*(v\mathbb{1}_K)\rangle| \\
&\leq \sum_{I\in\mathcal{S}} |\langle g\rangle_{I,v}| \sum_{\substack{K\in\mathcal{D}(I)\\ \mathcal{P}(K)=I}} \|\langle f,h_K^w\rangle_w h_K^w\|_{L^2(w)} \|\pi_b^*(v\mathbb{1}_K)\|_{L^2(w)} \\
&\leq C_2 \sum_{I\in\mathcal{S}} |\langle g\rangle_{I,v}| \sum_{\substack{K\in\mathcal{D}(I)\\ \mathcal{P}(K)=I}} \|\langle f,h_K^w\rangle_w h_K^w\|_{L^2(w)} \|v\mathbb{1}_K\|_{L^2(w)} \\
&\leq C_2 \sum_{I\in\mathcal{S}} |\langle g\rangle_{I,v}| \left(\sum_{\substack{K\in\mathcal{D}(I)\\ \mathcal{P}(K)=I}} \|\langle f,h_K^w\rangle_w h_K^w\|_{L^2(w)}^2\right)^{1/2} \left(\sum_{\substack{K\in\mathcal{D}(I)\\ \mathcal{P}(K)=I}} w(K)\right)^{1/2} \\
&\leq C_2 \sum_{I\in\mathcal{S}} |\langle g\rangle_{I,v}| v(I)^{1/2} \|\mathbb{P}_I^w f\|_{L^2(w)} .
\end{aligned}
$$

Then for the sum Σ_{11} we have

$$
\begin{aligned}
|\Sigma_{11}| &= \left| \sum_{I\in\mathcal{S}} \sum_{\substack{K\in\mathcal{D}(I)\\ \mathcal{P}(K)=I}} \langle \pi_b(w\langle f,h_K^w\rangle_w h_K^w), v\langle g\rangle_{I,v}\mathbb{1}_I\rangle \right| \\
&\leq \sum_{I\in\mathcal{S}} |\langle g\rangle_{I,v}| \left| \sum_{\substack{K\in\mathcal{D}(I)\\ \mathcal{P}(K)=I}} \langle \pi_b(w\langle f,h_K^w\rangle_w h_K^w), v\mathbb{1}_I\rangle \right| \\
&\leq \sum_{I\in\mathcal{S}} |\langle g\rangle_{I,v}| \left| \left\langle \pi_b\left(w \sum_{\substack{K\in\mathcal{D}(I)\\ \mathcal{P}(K)=I}} \langle f,h_K^w\rangle_w h_K^w \right), v\mathbb{1}_I \right\rangle \right| \\
&\leq \sum_{I\in\mathcal{S}} |\langle g\rangle_{I,v}| \left| \left\langle w \sum_{\substack{K\in\mathcal{D}(I)\\ \mathcal{P}(K)=I}} \langle f,h_K^w\rangle_w h_K^w, \mathbb{1}_I \pi_b^*\left(v\mathbb{1}_I\right) \right\rangle \right| \\
&\leq \sum_{I\in\mathcal{S}} |\langle g\rangle_{I,v}| \|\mathbb{P}_I^w f\|_{L^2(w)} \|\mathbb{1}_I \pi_b^*(v\mathbb{1}_I)\|_{L^2(w)} \\
&\leq C_2 \sum_{I\in\mathcal{S}} |\langle g\rangle_{I,v}|\ v(I)^{1/2} \|\mathbb{P}_I^w f\|_{L^2(w)} .
\end{aligned}
$$

Combining the estimates for Σ_{10} and Σ_{11} and using the Carleson Embedding Theorem, we have

$$
\begin{aligned}
|\Sigma_8| &\le C_2 \sum_{I\in\mathcal{S}} |\langle g\rangle_{I,v}| v(I)^{1/2} \|\mathbb{P}_I^w f\|_{L^2(w)} \\
&\le C_2 \|f\|_{L^2(w)} \left(\sum_{I\in\mathcal{S}} |\langle g\rangle_{I,v}|^2 v(I) \right)^{1/2} \le C_2 \|f\|_{L^2(w)} \|g\|_{L^2(v)} .
\end{aligned}
$$

For the sum Σ_9, using the localized property of π_b i.e. $\pi_b(w\mathbb{P}_I^w(wf)) = \mathbb{1}_I \pi_b(w\mathbb{P}_I^w(wf)$ and $\langle g\rangle_{I,v}\mathbb{1}_I = \mathbb{1}_I \sum_{J\supsetneq I} \langle g, h_K^v\rangle_v h_K^v$ we have that

$$
\begin{aligned}
\Sigma_9 &= \sum_{\substack{I,J\in\mathcal{S}\\ J\supsetneq I}} \langle \pi_b(\mathbb{P}_I^w(wf)), v\mathbb{P}_J^v g\rangle = \sum_{I\in\mathcal{S}} \left\langle \pi_b(\mathbb{P}_I^w(wf)), v \sum_{J\in\mathcal{S}, J\supsetneq I} \mathbb{P}_J^v g \right\rangle \\
&= \sum_{I\in\mathcal{S}} \sum_{K\supsetneq I} \langle \pi_b(\mathbb{P}_I^w(wf)), v\mathbb{1}_I \langle g, h_K^v\rangle_v h_K^v\rangle \\
&= \sum_{I\in\mathcal{S}} \langle g\rangle_{I,v} \, \langle \pi_b(\mathbb{P}_I^w(wf)), v\mathbb{1}_I\rangle
\end{aligned}
$$

Thus, we get the last estimate

$$
\begin{aligned}
|\Sigma_9| &\le \sum_{I\in\mathcal{S}} |\langle g\rangle_{I,v} \, \langle \pi_b(\mathbb{P}_I^w(wf)), v\mathbb{1}_I\rangle| \le \sum_{I\in\mathcal{S}} |\langle g\rangle_{I,v} \, \langle \mathbb{P}_I^w(wf), \mathbb{1}_I \pi_b^*(v\mathbb{1}_I)\rangle| \\
&\le \sum_{I\in\mathcal{S}} |\langle g\rangle_{I,v}| \, \|\mathbb{P}_I^w(wf)\|_{L^2(w)} \|\mathbb{1}_I \pi_b^*(v\mathbb{1}_I)\|_{L^2(w)} \\
&\le C_2 \|f\|_{L^2(w)} \left(\sum_{I\in\mathcal{S}} |\langle g\rangle_{I,v}|^2 \, v(I) \right)^{1/2} \le C_2 \|f\|_{L^2(w)} \|g\|_{L^2(v)} .
\end{aligned}
$$

Once again we use the Carleson Embedding Theorem for the last inequality. Since the sum Σ_6 is symmetric to Σ_5, by the estimates for the sums Σ_7, Σ_8, Σ_9, Σ_{10}, and Σ_{11}, we get the estimate for the sum Σ_1 as follows

$$
|\Sigma_1| \le C \|f\|_{L^2(w)} \|g\|_{L^2(v)} ,
$$

where $C \lesssim C_1 + C_2 + C_3$ and this indeed finished the proof.

4. Concluding Remarks

As we mentioned in the beginning, the authors in [3] obtained the quantitative estimate for the dyadic paraproduct. They prove the following Theorem.

Theorem 4.1. *Let (u, v) be a pair of measurable functions on $\mathbb{R}$ such that v and u^{-1}, the reciprocal of u, are weights on $\mathbb{R}$ and such that*

(i) $(u, v) \in \mathcal{A}_2^d$, that is $[u, v]_{\mathcal{A}_2^d} := \sup_{I\in\mathcal{D}} m_I(u^{-1})\, m_I v < \infty$.

(ii) there is a constant $\mathcal{D}_{u,v} > 0$ such that

$$\sum_{I\in\mathcal{D}(J)} |\Delta_I v|^2 |I|\, m_I(u^{-1}) \le \mathcal{D}_{u,v} v(J) \qquad \text{for all } J \in \mathcal{D},$$

where $\Delta_I v := m_{I_+} v - m_{I_-} v$.

Assume that $b \in Carl_{u,v}$, that is $b \in L^1_{loc}(\mathbb{R})$ and there is a constant $\mathcal{B}_{u,v} > 0$ such that

$$\sum_{I\in\mathcal{D}(J)} \frac{|\langle b, h_I\rangle|^2}{m_I v} \le \mathcal{B}_{u,v} u^{-1}(J) \qquad \text{for all } J \in \mathcal{D}.$$

Then π_b, the dyadic paraproduct associated to b, is bounded from $L^2(u)$ into $L^2(v)$. Moreover, there exists a constant $C > 0$ such that for all $f \in L^2(u)$

$$\|\pi_b f\|_{L^2(v)} \le C\sqrt{[u,v]_{\mathcal{A}_2^d}\mathcal{B}_{u,v}}\left(\sqrt{[u,v]_{\mathcal{A}_2^d}} + \sqrt{\mathcal{D}_{u,v}}\right)\|f\|_{L^2(u)},$$

where $\pi_b f := \sum_{I\in\mathcal{D}} m_I f\, \langle b, h_I\rangle\, h_I$.

Also, they provide a necessary condition for boundedness of π_b. However, there are no results connecting Theorem 1.1 and Theorem 4.1 yet. This might be a very challenging and interesting problem.

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