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REVISIT TO ALEXANDER MODULES OF 2-GENERATOR KNOTS IN THE 3-SPHERE

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Abstract. It is known that a 2-generator knot K has a cyclic Alexander module $\mathbb{Z}[t,t^{-1}]/(\Delta(t))$ where $\Delta(t)$ is the Alexander polynomial of K. In this paper we explicitly show how to reduce 2-generator Alexander modules to cyclic ones by using Chiswell, Glass and Wilsons presentations of 2-generator knot groups

$$< x, y \mid (x^{\alpha_1})^{y^{\gamma_1}}, \cdots, (x^{\alpha_k})^{y^{\gamma_k}} >$$

where $a^b = bab^{-1}$.

1. Introduction

A knot K in the 3-sphere S^3 whose fundamental group is defined by a presentation with two generators (and hence one relator) is called *a 2-generator knot*.
An arc τ embedded in S^3 so that $K \cap \tau = \partial\tau$ is called *an unknotting tunnel* of K if the complement of a regular neighbourhood of $K \cup \tau$ in S^3 is H_2, a handlebody of genus 2. A knot with an unknotting is called *a tunnel 1-knot*. By attaching to H_2 a 2-handle corresponding to τ, one would get the exterior of K, the complement of a regular neighbourhood of K in S^3. Thus we see that a tunnel 1-knot is a 2- generator knot. The converse statement is one of intriguing conjectures in knot theory. Berge knots admitting lens space Dehn surgeries are well known examples of tunnel 1-knots. In particular, characterization of the Alexander polynomials of Berge knots seems somewhat intriguing subject. Recently Chiswell, Glass and Wilson [2] introduced a handy method of computing the Alexander polynomial of a 2-generator knot via its group presentation

$$< x, y \mid (x^{\alpha_1})^{y^{\gamma_1}}, \cdots, (x^{\alpha_k})^{y^{\gamma_k}} > .$$

It is induced by a presentation admitting a generator with zero exponent sum [5, Chapter V, Lemma 11.8].

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Indeed via Nielsen transformations [6, Chapter 3] corresponding to mutual subtractions in the Euclidean algorithm any 2-generator presentation of a knot group can be brought into $< x, y \mid w >$ so that w_y and w_x, the sum of exponents of y and x in w, are 0 and 1 respectively. Then it is easy to see that the relator w is cyclically conjugate to that introduced by Chiswell, Glass and Wilson.

Using such special presentations of 2-generator knots, we have:

Theorem 1.1. *Any 2-generator knot in the 3-sphere has a cyclic Alexander module* $\mathbb{Z}[t, t^{-1}]/(\Delta(t))$ *where* $\Delta(t)$ *denotes the Alexander polynomial of a knot.*

Milnor [7, Footnote, p. 120] asserted that a 2-generator knot has a cyclic Alexander module. This follows easily from the fact that the Alexander module has deficiency 0. See [4, p. 14] for more details. Hence the Alexander module arising from a 2-generator 1-relator knot group via Fox differential calculus can be always reduced to a cyclic one. The method shown in this paper may be thought of as explicit reducing steps for the desired cyclic Alexander modules. We have in mind a practical application of explicit knowledge of the Alexander polynomial to homology of the cyclic branched covering [9].

A knot K in S^3 is said to be *a (1,1)-knot* if K is split into a pair of trivial arcs in solid tori determined by a Heegaard torus of S^3. All torus knots, and all 2-bridge knots are (1,1)-knots. The author [8] showed that any (1,1)-knot in S^3 admits a cyclic Alexander module by explicitly constructing the infinite cyclic covering space of its exterior.

Finally it is pointed out that in a Chiswell, Glass and Wilson's presentation, *tidiness* of a relator word (for the definition see [2, p.2]) would not be necessary to get the desired Alexander polynomial because it is assumed to be in $\mathbb{Z}[t, t^{-1}]$ instead of $\mathbb{Z}[t]$.

2. Proof of the main theorem

Lemma 2.1. *A 2-generator knot in* S^3 *admits a presentation* $< x, y \mid w >$ *such that* $w_y = 0$*, and* $w_x = 1$*.*

Proof. If necessary replacing a generator to its inverse, we assume that for a knot group presentation $< a, b \mid r >$ both r_a and r_b are relative prime positive integers since the abelianized presentation of a knot group is isomorphic to $\mathbb{Z}$. Define $[r]$ to be the largest integer not greater than a real number r. If $r_a < r_b$, then replacing a by $ab^{-[\frac{r_b}{r_a}]}$ (and hence a^{-1} by $b^{[\frac{r_b}{r_a}]}a^{-1}$) in r, we end up with a presentation with a new pair of sums of exponents $(r_a, r_b - r_a[\frac{r_b}{r_a}])$. Otherwise exchanging roles of a and b, we end up with a presentation with a new pair of sums of exponents $(r_a - r_b[\frac{r_a}{r_b}], r_b)$. Inductively executing Nielsen transformations corresponding to mutual subtractions, we eventually end up with $< x, y \mid w >$ such that $w_y = 0$, and $w_x = 1$. □

A presentation $< x, y \mid w >$ of a knot group with $w_y = 0$ and $w_x = 1$ is said to be *normalized.*

Example 2.2. The fundamental group of a torus knot $t(5,7)$ has a presentation $< x, y \mid x^5y^{-7} >$. Put $w_0 = x^5y^7$. Replacing x by $xy^{-[\frac{7}{5}]} = xy^{-1}$ (and hence x^{-1} by yx^{-1}) in w_0, we have

$$w_1 = xy^{-1}xy^{-1}xy^{-1}xy^{-1}xy^6$$

where $(w_1)_x = 5$, and $(w_1)_y = 2$. Replacing y by $yx^{-[\frac{5}{2}]} = yx^{-2}$ in w_1, we have

$$w_2 = xy^{-1}x^3y^{-1}x^3y^{-1}x^3y^{-1}xyx^{-2}yx^{-2}yx^{-2}yx^{-2}yx^{-2}y$$

where $(w_2)_x = 1$, and $(w_2)_y = 2$. Finally replacing x by xy^{-2} in w_2, we have the desired normalized relator.

$$\begin{aligned} w(x,y) \quad = \quad & yxy^{-3}xy^{-2}xy^{-2}xy^{-3}xy^{-2}xy^{-2}xy^{-3}x \\ & y^{-2}xy^{-2}xy^{-3}xyx^{-1}y^2x^{-1}y^3x^{-1}y^2x^{-1}y^3x^{-1} \\ & y^2x^{-1}y^3x^{-1}y^2x^{-1}y^3x^{-1}y^2x^{-1} \end{aligned}$$

Remark 2.3. A normal presentation of a 2-generator knot group is not unique. For a normalized presentation $< x, y \mid w(x,y) >$, we may get another normalized presentation $< x, y \mid w(x, yx^k) >$ for any integer $k \in \mathbb{Z}$.

Lemma 2.4. *Assume that a presentation* $< x, y|w = y^{\beta_1}x^{\alpha_1}, \cdots, y^{\beta_k}x^{\alpha_k} >$ *is normalized so that* $w_y = 0$. *Then* w *is cyclically conjugate to a word*

$$(x_1^{\alpha})^{y^{\gamma_1}}, \cdots, (x_k^{\alpha})^{y^{\gamma_k}}.$$

Proof. For each $1 \le j \le k$, take $\gamma_j = \sum_{i=1}^{j} \beta_i$. Then the last term $(x_k^{\alpha_k})^{y^{\gamma_k}}$ is always equal to $x_k^{\alpha_k}$ since $w_y = 0$. □

Example 2.5.

For w in Example 2.2, we have the following product of conjugates ;

$$\begin{aligned} & x^y x^{y^{-2}} x^{y^{-4}} x^{y^{-6}} x^{y^{-9}} x^{y^{-11}} x^{y^{-13}} \\ & x^{y^{-16}} x^{y^{-18}} x^{y^{-20}} x^{y^{-23}} \\ & (x^{-1})^{y^{-22}} (x^{-1})^{y^{-20}} (x^{-1})^{y^{-17}} (x^{-1})^{y^{-15}} (x^{-1})^{y^{-12}} \\ & (x^{-1})^{y^{-10}} (x^{-1})^{y^{-7}} (x^{-1})^{y^{-5}} (x^{-1})^{y^{-2}} x^{-1} \end{aligned}$$

Let X be a standard 2-complex associated with a presentation of $< x, y \mid w >$ of a knot group G with a single 0-cell v, two 1-cells x, y and one 2-cell w such that $\pi_1(X, v) = G$. And let $\tilde{X}$ be a infinite cyclic covering space of X such that $\pi_1(\tilde{X}, \tilde{v}) = G$, the commutator subgroup of G where $\tilde{v}$ is 0-cell chosen in the 0-skelecton $\tilde{X}^0$ of $\tilde{X}$. Under action of the covering transformation group $G/G =< t^n | n \in \mathbb{Z} >$, $H_1(\tilde{X}) = G/G$ admits a $\mathbb{Z}[t, t^{-1}]$ module structure so called the Alexander module of a knot. For the canonical homomorphism $\phi : G =< x, y|w > \to G_{ab} \cong < t^n | n \in \mathbb{Z} > \cong G/G$. The linear extension to the group ring is also denoted by $\phi : \mathbb{Z}G \to \mathbb{Z}[t, t^{-1}]$, and $\phi(w) = w^{\phi}$ is denoted by

$[w]$ for $w \in \mathbb{Z}G$. Fox derivatives of $w \in \mathbb{Z}G$ with respect to x, y are denoted by $\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}$ respectively.
Lemma 2.6 follows immediately from Fox differential calculus, lemma 2.4 and the fact that the canonical homomorphism ϕ carries x, y to $1, t$ respectively.

Lemma 2.6. *For a presentation* $< x, y \mid w = (x^{\alpha_1})^{y^{\gamma_1}}, \cdots, (x^{\alpha_k})^{y^{\gamma_k}} >$ *with* $w_x = 1$, *we have:*
(1) $[\frac{\partial w}{\partial x}] = \sum_{i=1}^{k} \alpha_i t^{\gamma_i}$, *and*
(2) $[\frac{\partial w}{\partial y}] = 0$

For any positive integer n, a tamed embedding of the $n-$ sphere S^n in the $n+2-$ sphere S^{n+2} is said to be $n-$ *knot.* From lemma 2.6, we have:

Corollary 2.7. *If a* $n-$ *knot has a presentation*

$$< x, y \mid w = (x^{\alpha_1})^{y^{\gamma_1}}, \cdots, (x^{\alpha_k})^{y^{\gamma_k}} >,$$

then it has the Alexander polynomial $\Delta(t) = \sum_{i=1}^{k} \alpha_i t^{\gamma_i}$.

Example 2.8. The Alexander polynomial corresponding to the normal presentation in Example 2.5 is

$$\begin{aligned} & t + t^{-2} + t^{-4} + t^{-6} + t^{-9} + t^{-11} + t^{-13} + t^{-16} + t^{-18} + t^{-20} + t^{-23} \\ & -t^{-22} - t^{-20} - t^{-17} - t^{-15} - t^{-12} - t^{-10} - t^{-7} - t^{-5} - t^{-2} - 1 \\ = \; & t + t^{-4} + t^{-6} + t^{-9} + t^{-11} + t^{-13} + t^{-16} + t^{-18} + t^{-23} \\ & -t^{-22} - t^{-17} - t^{-15} - t^{-12} - t^{-10} - t^{-7} - t^{-5} - 1 \end{aligned}$$

We recover the Alexander polynomial in $\mathbb{Z}[t]$ by multiplying a unit t^{23} of $\mathbb{Z}[t, t^{-1}]$ to the above Laurant polynomial.

$$t^{24} + t^{19} + t^{17} + t^{14} + t^{12} + t^{10} + t^7 + t^5 + 1 - t - t^6 - t^8 - t^{11} - t^{13} - t^{16} - t^{18} - t^{23}$$

The following example is prepared to show that we may get the desired Alexander polynomial from a normalized presentation $< x, y \mid w >$ without the tidy condition of w in [2].

Example 2.9. Kanenobu and Sumi [3, Example 2.1] showed that a ribbon 2-knot $K2 = R(1, 2, -3, 1)$ admits a knot group presentation

$$< x, y \mid x^{-1}y^{-1}x^3y^{-2}x^{-1}yxy^2x^{-3}y >,$$

which is normalized to a presentation

$$< x, y \mid y^{-1}x^2y^{-1}x^{-1}yx^{-1}yx^{-1}y^{-2}xyx^{-1}y^2xy^{-1}x > .$$

The relator word can be brought into the product of conjugates;

$$(x^2)^{y^{-1}}(x^{-1})^{y^{-2}}(x^{-1})^{y^{-1}}(x^{-1})(x)^{y^{-2}}(x^{-1})^{y^{-1}}(x)^y x.$$

Finally we end up with the desired Alexander polynomial

$$\begin{aligned}\Delta(t) &= 2t^{-1} - t^{-2} - t^{-1} + t^{-2} - t^{-1} + t + 1 \\ &= t\end{aligned}$$

From the homology long exact sequence of of a pair $(\tilde{X}, \tilde{X^0})$, we have a short exact sequence

$$0 \to H_1(\tilde{X}) \to H_1(\tilde{X}, \tilde{X_0}) \xrightarrow{\partial} keri_* \to 0$$

where the boundary homomorphism ∂ has the right inverse σ, and hence the short exact sequence is split in such a way that $H_1(\tilde{X}, \tilde{X^0}) \cong H_1(\tilde{X}) \bigoplus \mathbb{Z}[t, t^{-1}]$ where $\mathbb{Z}[t, t^{-1}]$ stands for a free $\mathbb{Z}[t, t^{-1}]$ -module of rank 1 generated by $(t-1)\tilde{v}$ From [1, Proposition 9.2] we have:

Lemma 2.10. *Let $< x, y \mid w >$ be a knot group presentation, $\tilde{x}, \tilde{y}$ lifted 1-cells of x, y respectively, and $\tilde{w}$ a lifted 2-cell of w. Then $H_1(\tilde{X}, \tilde{X^0})$ admits a $\mathbb{Z}[t, t^{-1}]$ -module presentation*

$$< \tilde{x}, \tilde{y} \mid \tilde{w} = [\frac{\partial w}{\partial x}]\tilde{x} + [\frac{\partial w}{\partial y}]\tilde{y} >$$

where $\partial\tilde{x} = ([x] - 1)\tilde{v}$, and $\partial\tilde{y} = ([y] - 1)\tilde{v}$ for the connecting homomorphism $\partial : H_1(\tilde{X}, \tilde{X^0}) \to keri_*$

From lemma 2.10, we have:

Proposition 2.11. *If a knot group presentation $< x, y \mid w >$ is normalized, then $H_1(\tilde{X})$ admits a $\mathbb{Z}[t, t^{-1}]$ -module presentation*

$$< \tilde{x} \mid \tilde{w} = [\frac{\partial w}{\partial x}]\tilde{x} > \cong \mathbb{Z}[t, t^{-1}]/(\Delta(t))$$

Proof. Since $[\frac{\partial w}{\partial y}] = 0$, $H_1(\tilde{X}, \tilde{X^0})$ admits a $\mathbb{Z}[t, t^{-1}]$ -module presentation

$$< \tilde{x}, \tilde{y} | \tilde{w} = [\frac{\partial w}{\partial x}]\tilde{x} > .$$

Furthermore since $\partial\tilde{y} = (t - 1)\tilde{v}$, removing $\tilde{y}$ corresponding to the free $\mathbb{Z}[t, t^{-1}]$ -module generator from the presentation of $H_1(\tilde{X}, \tilde{X^0})$ we get the desired cyclic module presentation of $H_1(\tilde{X})$.

□

Theorem 1.1 follows from Lemma 2.1 and Proposition 2.11.

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