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REMARKS ON CURVES OF MAXIMAL REGULARITY IN $\mathbb{P}^3$

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Abstract. For a nondegenerate projective curve $C \subset \mathbb{P}^r$ of degree d, it was shown that the Castelnuovo-Mumford regularity $\mathrm{reg}(C)$ of C is at most $d - r + 2$. And the curves of maximal regularity which attain the maximally possible value $d - r + 2$ are completely classified.
In this short note, we first collect several known results about curves of maximal regularity. We provide a new proof and some partial results. Finally we suggest some interesting questions.

1. Introduction

Through out this paper, we work over an algebraically closed field $\mathbb{K}$ of arbitrary characteristic. Let $C \subset \mathbb{P}^r$ $(r \geq 3)$ be a nondegenerate projective curve of degree d. Then we say C is m-regular if the following vanishing condition

$$H^i(\mathbb{P}^r, \mathcal{I}_C(m-i)) = 0 \quad \text{for all } i \geq 1$$

holds where $\mathcal{I}_C$ is the ideal sheaf of C. The (Castelnuovo-Mumford) regularity $\mathrm{reg}(C)$ of C is defined as the smallest integer m such that C is m-regular. L. Gruson, R. Lazarsfeld and C. Peskin in [GLP] proved that $\mathrm{reg}(C) \leq d - r + 2$. They also provided a complete classification theory about curves of maximal regularity $\mathrm{reg}(C) = d - r + 2$. It is interesting that C is a curve of maximal regularity if and only if C is a smooth rational curve having a $(d - r + 2)$-secant line $\mathbb{L}$ when $d \geq r + 2$. Now it is well known that C is contained in a rational normal threefold scroll $S(0, 0, r - 2)$ which is just the join C with $\mathbb{L}$. In particular, some curves of maximal regularity can be contained in a rational normal surface scroll $S(1, r - 2)$ which is a divisor of $S(0, 0, r - 2)$. Later, this geometric information has been playing a crucial role to understand the several invariants of the curves of maximal regularity. In [BS1] and [BS2], the authors investigated algebraic and geometric properties of C by analyzing the

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union $C \cup \mathbb{L}$. It was shown that the $(d-r+2)$-secant line $\mathbb{L}$ to C is unique if $r \geq 4$ and $C \cup \mathbb{L}$ is arithmetically Cohen-Macaulay if $d \leq 2r-2$ (see Theorem 2.2). They also described the Hartshorn-Rao module and some parts of the minimal free resolution of the homogeneous ideal I_C of C. The author of this paper has been studying the curves of maximal regularity according to the philosophy of M. Brodmann and P. Schenzel. In [LP1], the authors provided a complete description of all graded Betti numbers of C when $d \leq 2r-2$ (see Theorem 2.4). We also studied the syzygetic behaviors of C which is a divisor of $S(1, r-2)$. In [LY], [LJ] and [LP2], the shape of minimal generators of $C \subset \mathbb{P}^r$ for $r = 3, 4$ and the minimal free resolution of the defining ideal I_C of C lies on $S(1, r-2)$ for all $d \geq 5$ are precisely described, respectively (see Theorem 2.7, Theorem 2.8, Example 2.5 and Example 2.6). However the problem to determine explicitly the minimal generators and the graded Betti numbers of C is still widely open. In the sense of moduli space of curves of maximal regularity, the authors in [CLP] showed that the space $\Gamma_{r,d}$ of smooth rational curves of degree d in $\mathbb{P}^r$ of maximal regularity is an irreducible variety of dimension $3d+r^2-r-1$. Also they classified those rational curves up to projective motion.

In the present paper, we revisit to the results about the curves of maximal regularity in section 2. In section 3, we concentrate our interests on those curves C on $\mathbb{P}^3$ and provide some partial results and questions. In particular, we investigate the extremal secant line $\mathbb{L}$ to C and give a new proof about the existence of a one dimensional family of $(d-1)$-secant line when C is contained in a smooth quadric surface and the uniqueness of such a secant line otherwise (see Proposition 3.1).

2. Preliminaries

This section is devoted to arrange and ruminate several known results about the curves of maximal regularity. Let $C \subset \mathbb{P}^r$ be a nondegenerate projective curve of degree d.

Theorem 2.1 (GLP, 1983). *Suppose that $r \geq 3$.*

(1) *It holds that* $\operatorname{reg}(C) \leq d-r+2$.
(2) *Suppose that C is of maximal regularity and of degree $d \geq r+2$. Then C is the image of an isomorphic projection of a rational normal curve $\widetilde{C} \subset \mathbb{P}^d$ and C attains a $(d-r+2)$-secant line $\mathbb{L}$ to C.*

Remaining of this paper, we assume that C is of maximal regularity. M. Brodmann and P. Schenzel got a very useful idea to study algebraic and geometric properties of C by analyzing the union of C and $\mathbb{L}$.

Theorem 2.2 (BS1, 2001 and BLPS, 2017). *Suppose that $r \geq 4$.*

(1) *The $(d-r+2)$-secant line $\mathbb{L}$ to C is unique.*
(2) *If $d \leq 2r-2$, then $C \cup \mathbb{L}$ is arithmetically Cohen-Macaulay.*
(3) *If $d \geq 3r-3$, then $C \cup \mathbb{L}$ is not arithmetically Cohen-Macaulay.*

Recently, the authors in [LP1] and [LP2] described the explicit values of all graded Betti numbers for C with $d \leq 2r-2$ and for C which is contained in a rational normal surface scroll $S(1, r-2)$ as a divisor. To state the results, we need some terminology.

Notation and Remarks 2.3. For a projective curve $X \subset \mathbb{P}^r$, let $R := \mathbb{K}[x_0, x_1, \ldots, x_r]$ and I_X be respectively the homogeneous coordinate ring of $\mathbb{P}^r$ and the defining ideal of X. And let R_X be the homogeneous coordinate ring of X and $HF_X(k)$ be the Hilbert function of R_X.

(1) The graded Betti numbers $\beta_{i,j}(X)$ of X are defined as

$$\beta_{i,j}(X) = \dim_{\mathbb{K}} \mathrm{Tor}_i^R(I_X, \mathbb{K})_{i+j}.$$

In particular, X is m-regular if and only if $\beta_{i,j}(X) = 0$ for all $j \geq m+1$.

(2) The Hilbert series $\Phi_{R_X}(t)$ of R_X is given by

$$\Phi_{R_X}(t) = \sum_{k \geq 0} HF_X(k) \cdot t^k = \sum_{i \geq 0} (-1)^i \sum_{j \in \mathbb{Z}} \beta_{i,j} \cdot \frac{t^{i+j}}{(1-t)^{r+1}} \tag{1}$$

Theorem 2.4 (LP1, 2016)**.** *For $r+2 \leq d \leq 2r-2$, the graded Betti numbers $\beta_{i,j}(C)$ of C are completely determined as followings:*

$$\begin{cases} \beta_{i,2}(C) = \begin{cases} (i+1)\binom{r-1}{i+2} + (2r-2-d-i)\binom{r-1}{i} & \text{for } 0 \leq i \leq 2r-3-d, \\ (i+1)\binom{r-1}{i+2} & \text{for } 2r-2-d \leq i \leq r-1, \end{cases} \\ \beta_{i,3}(C) = \begin{cases} 0 & \text{for } 0 \leq i \leq 2r-3-d, \\ (i+d+3-2r)\binom{r-1}{i+1} & \text{for } 2r-2-d \leq i \leq r-1, \text{ and} \end{cases} \\ \beta_{i,d-r+2}(C) = \binom{r-1}{i}. \end{cases}$$

We here would like to mention about the main idea of Theorem 2.4 which has been a key tool to study the graded Betti numbers of effective divisors on a rational normal surface scroll $S(a,b) \subset \mathbb{P}^{a+b+1}$ in [LP2]. Consider the exact sequence

$$0 \to \mathcal{I}_{C \cup \mathbb{L}} \to \mathcal{I}_C \to \mathcal{O}_{\mathbb{P}^1}(-d+1) \to 0$$

Then we get a decomposition $\beta_{i,j}(C) = \beta_{i,j}(C \cup \mathbb{L}) \oplus \beta_{i,j}(E)$ of graded Betti numbers of C as those of $C \cup \mathbb{L}$ and E where E is the graded R-module associated to $\mathcal{O}_{\mathbb{P}^1}(-d+1)$. By applying this decomposition method to the curves $C \cup \mathbb{L}$, $C \cup 2\mathbb{L}$, $C \cup 3\mathbb{L}, \ldots$ the Betti table of C can be decomposed with other Betti tables as simple as possible. For details, see [LP2]. The authors in [LP2, Theorem 5.1] provided a nice decomposition of the Betti table of a smooth rational curve $C \subset S(1, r-2)$ as a sum of much simpler Betti tables. One can see that those curves are indeed of maximal regularity. We exhibit examples for $r = 3$ and $r = 4$ as the most simplest cases.

Example 2.5 (GM, 1992 and LP2, 2019)**.** Suppose that C is linearly equivalent to a divisor $aH + bF$ of $S(1,1)$ in $\mathbb{P}^3$ where $a \geq 0$ and $b \geq 2$. Then C has the following Betti table:

i	0	1	2
$\beta_{i,a+b}$	$b+1$	$2b$	$b-1$
$\beta_{i,a+b-1}$	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\beta_{i,3}$	0	0	0
$\beta_{i,2}$	1	0	0

.

Example 2.6 (LP2, 2019)**.** Suppose that C is linearly equivalent to a divisor $aH+bF$ of $S(1,2)$ in $\mathbb{P}^4$ where either $a=0$ and $b\geq 3$ or else $a\geq 1$ and $b\geq 2$. Then C has following Betti tables according to the values of b and $\delta=\lceil\frac{b-1}{2}\rceil$:

i	0	1	2	3
$\beta_{i,a+b}$	1	3	3	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\beta_{i,a+\delta+2}$	1	3	3	1
$\beta_{i,a+\delta+1}$	1	6	5	1
$\beta_{i,a+\delta}$	1	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\beta_{i,3}$	0	0	0	0
$\beta_{i,2}$	3	2	0	0

i	0	1	2	3
$\beta_{i,a+b}$	1	3	3	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\beta_{i,a+\delta+2}$	1	3	3	1
$\beta_{i,a+\delta+1}$	3	6	3	0
$\beta_{i,a+\delta}$	0	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\beta_{i,3}$	0	0	0	0
$\beta_{i,2}$	3	2	0	0

$b=2\delta$ and $b=2\delta+1$

For a given projective variety $X\subset\mathbb{P}^r$ and its defining ideal I_X of X, it might be one of the most fundamental but difficult problem to determine explicit equations generate I_X. To the author's knowledge, there is no general theory about this problem. In this line, the following two Theorems are steps of beginning to describe a minimal generating set of the defining ideal I_C of $C\subset S(1,r-2)$ of maximal regularity.

Theorem 2.7 (LY, 2018)**.** *Let $C\subset\mathbb{P}^3$ be a rational curve of degree d defined as the parametrization*

$$C=\{[s^d(P):s^{d-1}t(P):st^{d-1}(P):t^d(P)]\mid P\in\mathbb{P}^1\}$$

where $d\geq 3$. Then C is contained in a rational normal surface scroll $S(1,1)$ as a divisor and the defining ideal I_C of C is minimally generated by

$$\{X_0X_3-X_1X_2,F_{d,1},F_{d,2},\ldots,F_{d,d-1}\}$$

where $F_{d,i}=X_0^{d-i-1}X_2^i-X_1^{d-i}X_3^{i-1}$ for $1\leq i\leq d-1$.

Theorem 2.8 (LJ, 2019). *Let $C \subset \mathbb{P}^4$ be a rational curve of degree d defined as the parametrization*

$$C = \{[s^d(P) : s^{d-1}t(P) : s^2t^{d-2}(P) : st^{d-1}(P) : t^d(P)] \mid P \in \mathbb{P}^1\}$$

where $d \geq 3$. Let

$$\begin{cases} G_{[n,i]} = X_1X_3^{i-1}X_4^{n-i} - X_2^{n+i-2}X_3^{2-i} & \text{for } i = 1, 2 \\ H_{n+j-1} = X_0^{2j-1}X_2^{n-j} - X_1^{2j}X_4^{n-j-1} & \text{for } 1 \leq j \leq n-1 \\ F_{n+j-1} = X_0^{2j-2}X_2^{n-j+1} - X_1^{2j-1}X_4^{n-j} & \text{for } 1 \leq j \leq n \end{cases}$$

be respectively homogenous polynomials of degree n and $n+j-1$ for $n \geq 2$. Then C is contained in a rational normal surface scroll $S(1,2)$ as a divisor and the defining ideal I_C of C is minimally generated as followings:

(1) *If $d = 2n$, then*

$$I_C = \langle X_0X_3 - X_1X_2, X_0X_4 - X_1X_3, X_2X_4 - X_3^2, G_{[n,1]}, G_{[n,2]}, H_n, H_{n+1}, \cdots, H_{2n-2}\rangle.$$

(2) *If $d = 2n+1$, then*

$$I_C = \langle X_0X_3 - X_1X_2, X_0X_4 - X_1X_3, X_2X_4 - X_3^2, F_n, F_{n+1}, \cdots, F_{2n-1}\rangle.$$

3. Space curves of maximal regularity

In this section, we focus our attention to the curves of maximal regularity in $\mathbb{P}^3$. First we begin with the uniqueness of extremal secant line to the curve C of maximal regularity with an exceptional case in $\mathbb{P}^3$. Proposition 3.1 was proved as a geometric method in [CLP, proposition 2.1]. We here provide another proof of Proposition 2.1 in [CLP] by using the parametrization of C described in [CLP, section 2].

Proposition 3.1. *Let $C \subset \mathbb{P}^r$ be a curve of degree $d \geq r+2$ and of maximal regularity $d - r + 2$.*

(1) *If $r \geq 4$, then there exists a unique $(d-r+2)$-secant line to C.*

(2) *If $r = 3$, then one of the followings holds:*
   (i) *There exists a unique $(d-1)$-secant line to C.*

   (ii) *C is contained in a rational normal surface scroll $S(1,1)$. In paticular, there exists one dimensional family of $(d-1)$-secant lines to C.*

Proof. Remind that there exists such a $(d-r+2)$-secant line to C by [GLP]. Suppose that there are two different $(d-r+2)$-secant lines $\mathbb{L}_1$ and $\mathbb{L}_2$ to C. We may assume that

$$\mathbb{L}_1 = \mathbb{V}(x_2, \ldots, x_r) \quad \text{and} \quad \mathbb{L}_2 = \mathbb{V}(H_2, \ldots, H_r)$$

where $H_i = \sum_{j=0}^{r} a_{ij}x_j$ are linear forms in R. And T denotes the homogeneous coordinate ring $\mathbb{K}[s,t]$ of $\mathbb{P}^1$. Then by Proposition 2.5 in [CLP], we have a parametrization

$$\{[f_0(P), f_1(P), fs^{r-2}(P), \ldots, ft^{r-2}(P)] \mid P \in \mathbb{P}^1\}$$

of C for the line $\mathbb{L}_1$ where $f_0, f_1 \in T$ and $f \in T$ are respectively homogeneous forms of degree d and $d-r+2$. Since $\mathbb{L}_2$ is a $(d-r+2)$-secant line to C, we obtain the following system of polynomials

$$H_i = a_{i0}f_0 + a_{i1}f_1 + a_{i2}fs^{r-2} + \cdots + a_{ir-1}fst^{r-3} + a_{ir}ft^{r-2} \quad \text{for } 2 \leq i \leq r$$

of degree d with respect to s and t. Also every H_i has a common factor $g \in T$ of degree $d-r+2$. Note that if $a_{ij} \neq 0$ for $j = 0, 1$(resp. for $2 \leq j \leq r$) then f_i(resp. $fs^{r-j}t^{j-2}$) has the factor g. Now we claim that a_{ij} should be zero for all i and $2 \leq j \leq r$. That is, $r = 3$. For the claim, we assume that $r \geq 4$. Note that at least one of a_{i0} and a_{i1} is not zero for some i. Otherwise, $\{H_2, H_3, \ldots, H_r\}$ defines the line $\mathbb{L}_1$ because they are linearly independent linear forms. That is, $\mathbb{L}_1 = \mathbb{L}_2$ and it is a contradiction. Now consider a $(r-1) \times (r-1)$ matrix M which entries are a_{ij} for $2 \leq i, j \leq r$. Then it follows that the rank of M is at least $r-3$ by linearly independency of $\{H_2, H_3, \ldots, H_r\}$. More precisely, if both a_{i0} and $a_{i'1}$ are not zero for some $2 \leq i, i' \leq r$ (i.e., $\mathbb{L}_1 \cap \mathbb{L}_2 = \emptyset$) then at least $(r-3)$-polynomials among $\{fs^{r-2}, fs^{r-3}t \ldots, ft^{r-2}\}$ have the factor g. And if either $a_{i0} = 0$ or $a_{i1} = 0$ for all i (i.e., $\mathbb{L}_1 \cap \mathbb{L}_2 \neq \emptyset$) then at least $(r-2)$-polynomials among $\{fs^{r-2}, fs^{r-3}t \ldots, ft^{r-2}\}$ have the factor g. On the other hand, the degree of common factor of f and g is at most 1 since $\mathbb{L}_1$ and $\mathbb{L}_2$ are different to each other. Therefore we conclude that a system of $(r-3)$-forms of degree d among $\{fs^{r-2}, fs^{r-3}t \ldots, ft^{r-2}\}$ can have a factor g of degree at most 3. Thus it contradicts to $d-r+2 \geq 4$. Now suppose that $\mathbb{L}_1 \neq \mathbb{L}_2$. Then it should be $r = 3$. And it follows that $\mathbb{L}_1$ and $\mathbb{L}_2$ are skew. So it can be written that

$$C := \{[g(a_0s + b_0t)(P), g(a_1s + b_1t)(P), fs(P), ft(P)] \mid P \in \mathbb{P}^1\}$$

where $f, g \in T$ are relatively prime forms of degree $d-1$ since two lines $\mathbb{L}_1$ and $\mathbb{L}_2$ are skew. As the last step of the proof, it can be shown that C is projectively equivalent to the curve

$$\overline{C} := \{[gs(P), gt(P), fs(P), ft(P)] \mid P \in \mathbb{P}^1\}$$

by a projective transformation

$$M^{-1} \begin{bmatrix} g(a_0s + b_0t) \\ g(a_1s + b_1t) \\ fs \\ ft \end{bmatrix} = \begin{bmatrix} gs \\ gt \\ fs \\ ft \end{bmatrix} \quad \text{where} \quad M = \begin{bmatrix} a_0 & b_0 & 0 & 0 \\ a_1 & b_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathrm{Aut}(\mathbb{P}^3).$$

Moreover, one can see that $\overline{C}$ is contained in the rational normal surface scroll $S(1,1)$ as a divisor linearly equivalent to $H + (d-2)F$ where H and F are respectively the hyperplane section and a ruling line. That is, every line section on $S(1,1)$ is a $(d-1)$-secant line to C. $\square$

Now let $C \subset \mathbb{P}^3$ be a curve of degree $d \geq 5$ and of maximal regularity. Suppose that C has a unique $(d-1)$- secant line $\mathbb{L}$. And Z denotes the union $C \cup \mathbb{L}$.

Lemma 3.2. *C is not contained in any quadric surface.*

Proof. Suppose that the defining ideal I_C of C has a quadratic generator Q. Then the possible candidate of Q is either a cone of smooth rational curve on $\mathbb{P}^2$ or the rational normal surface scroll $S(1,1)$. For the later case, the curve C has a one dimensional family of $(d-1)$-secant lines to C by Proposition 3.1.(2) and hence it contradict to the uniqueness of extremal secant line. For the former case, it is well known that any effective divisor on a cone of smooth rational curve is arithmetically Cohen-Macaulay (See [Fe]). On the other hand, C is of depth 1 by Theorem 2.1.(2). This completes the proof. $\square$

Proposition 3.3. *Let C and Z be as stated above. Suppose that Z is arithmetically Cohen-Macaulay. Then Z is 3-regular and C has the following Hilbert function $HF_C(k)$ and the Betti table:*

$$HF_C(k) = \begin{cases} 1 & \text{if } k = 0 \\ 4 & \text{if } k = 1 \\ 10 & \text{if } k = 2 \\ (d+1)k - d + 3 & \text{if } 3 \leq k \leq d-2 \\ dk + 1 & \text{if } k \geq d-1 \end{cases}$$

and

i	0	1	2
$\beta_{i,d-1}$	1	2	1
$\beta_{i,d-2}$	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\beta_{i,4}$	0	0	0
$\beta_{i,3}$	$14-2d$	$28-5d$	0
$\beta_{i,2}$	0	0	0

Proof. Consider the two exact sequences

$$0 \to \mathcal{I}_C \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_C \to 0 \tag{2}$$

$$0 \to \mathcal{I}_Z \to \mathcal{I}_C \to \mathcal{O}_{\mathbb{P}^1}(-d+1) \to 0 \tag{3}$$

Then it follows that $H^0(\mathbb{P}^3, \mathcal{I}_C(2)) = 0$ from Lemma 3.2. Thus we have

$$h^1(\mathbb{P}^3, \mathcal{I}_C(2)) = h^0(\mathbb{P}^3, \mathcal{O}_C(2)) - h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) = 2d - 9$$

from the exact sequence (2). We claim that $h^2(\mathbb{P}^3, \mathcal{I}_Z(j)) = 0$ for all $j \geq 1$. That is, Z is 3-regular. For the claim, it suffices to show that $h^2(\mathbb{P}^3, \mathcal{I}_Z(1)) = 0$. Since Z is arithmetically Cohen-Macaulay, we have

$$0 \to H^1(\mathbb{P}^3, \mathcal{I}_C(1)) \to H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-d+2)) \to H^2(\mathbb{P}^3, \mathcal{I}_Z(1)) \to H^2(\mathbb{P}^3, \mathcal{I}_C(1)) \to 0$$

from the exact sequence (3). It is easy to see that $h^2(\mathbb{P}^3, \mathcal{I}_C(1)) = h^1(\mathbb{P}^3, \mathcal{O}_C(1)) = 0$ and $h^1(\mathbb{P}^3, \mathcal{I}_C(1)) = d-3$ by Theorem 2.1.(2). Thus we have the desired vanishing since $h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-d+2)) = d-3$. Now we turn to describe the Betti table of C. Let $E = \bigoplus_{j\in\mathbb{Z}} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-d+1+j))$ be the graded R-module associated to $\mathcal{O}_{\mathbb{P}^1}(-d+1)$. Then the exact sequence (3) yields that

$$\beta_{i,j}(C) = \begin{cases} \beta_{i,j}(Z) & \text{for } j = 2,3 \\ \beta_{i,j}(E) & \text{for } j \geq 4 \end{cases}$$

since Z is 3-regular. Also it is well known that

$$\dim_{\mathbb{K}} \mathrm{Tor}^R_{i-1}(E,K)_{i+j-1} = \begin{cases} \binom{2}{i-1} & \text{for } j = d-1 \\ 0 & \text{otherwise} \end{cases}$$

(cf. see [LP2, Proposition 4.1]). Since Z is arithmetically Cohen-Macaulay and $H^2(\mathbb{P}^3, \mathcal{I}_Z(j)) = 0$ for $j \geq 1$, the Hashorne-Rao module of C is obtained as

$$\dim_{\mathbb{K}} H^1(\mathbb{P}^3, \mathcal{I}_C(j)) = \dim_{\mathbb{K}} H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-d+1+j)) = d-j-2$$

for $j \geq 1$ from the exact sequence (3). Then we obtain the desired Hilbert function $HF_C(k)$ of C from the exact sequence (2). Therefore the Betti numbers of C follow by applying Hilbert funtion $HF_C(k)$ to the Hilbert series (1) in Notation and Remarks 2.3. □

Remark and Questions 3.4. (1) In the Betti table in Proposition 3.3, if $d = 5$ then C has the following Betti diagram:

i	1	2	3	4
$\beta_{i,3}$	1	2	1	0
$\beta_{i,2}$	4	3	0	0
$\beta_{i,1}$	0	0	0	0

On the other hand, if $d \geq 6$ then the values of the Betti table in Proposition 3.3 can be negative. It is a clue to believe that Z is not arithmetically Cohen-Macaulay for $r = 3$ and $d \geq 6$ (cf. Theorem 2.2.(3)). However we unfortunately did not find any crucial reason. From our observation, we shall propose the following questions: Let $C \subset \mathbb{P}^3$ be a curve of maximal regularity which is not contained a quadric surface. Suppose that $d \geq 5$.

Q1. Prove that Z is arithmetically Cohen-Macaulay if $d = 5$.
Q2. Prove that Z is not arithmetically Cohen-Macaulay if $d \geq 6$
Q3. Describe the Hartshorne-Rao module and Hilbert function of C.
Q4. Describe the graded Betti numbers of C when Z is not arithmetically Cohen-Macaulay.

(2) In the case where the curve $C \subset \mathbb{P}^3$ is contained in the rational normal surface scroll $S(1,1)$, the minimal generators and the graded Betti numbers are completely determined (see Theorem 2.4 and Theorem 2.7). Also the space $\Gamma_{3,d}$ of smooth rational curves in $\mathbb{P}^3$ of maximal regularity is an irreducible variety of dimension $3d+5$ (see [CLP, Theorem 3.5 and Propositon 3.6]. Finally, we

pose one more question:

Q5. Study to describe the space $\Gamma_{3,d}$ and the subspace of the smooth rational curves $C \subset S(1,1)$ in the sense of geometric invariant theory.

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