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I-SEMIREGULAR RINGS

Juncheol Han and Hyo-Seob Sim*

Abstract. Let R be a ring with unity, and let I be an ideal of R. Then R is called *I-semiregular* if for every $a \in R$ there exists $b \in R$ such that ab is an idempotent of R and $a - aba \in I$. In this paper, basic properties of I-semiregularity are investigated, and some equivalent conditions to the primitivity of e are observed for an idempotent e of an I-semiregular ring R such that $I \cap eR = (0)$. For an abelian regular ring R with the ascending chain condition on annihilators of idempotents of R, it is shown that R is isomorphic to a direct product of a finite number of division rings, as a consequence of the observations.

1. Introduction

Throughout this paper all rings are associative with unity unless otherwise specified. Let R be a ring and let $E(R)$ be the set of all idempotents of R, let $J(R)$ denote the Jacobson radical of R, and let $|S|$ denote the cardinality of a subset S of R. Denote the ring of integers (modulo n) by $\mathbb{Z}$ ($\mathbb{Z}_n$). Use $\mathbb{Q}$ to denote the field of rational numbers. A ring R is called *reduced* if R has no nonzero nilpotent. A ring is called *abelian* if every idempotent is central. It is easily checked that every reduced ring is abelian. An element a of a ring R is called *regular* if there exists $b \in R$ such that $aba = a$. A ring R is called *von Neumann regular* (simply, *regular*) if every element of R is regular. In [9], an element a of a ring R is called *semiregular* if it satisfies the following conditions of Proposition 1.1.

Proposition 1.1. *The following are equivalent for an element a of a ring R:*
(1) *There exists $e^2 = e \in aR$ such that $(1-e)a \in J(R)$.*
(2) *There exists $e^2 = e \in Ra$ such that $a(1-e) \in J(R)$.*
(3) *There exists a regular element $b \in R$ such that $a - b \in J(R)$.*
(4) *There exists $b \in R$ with $bab = b$ and $a - aba \in J(R)$.*

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*Corresponding author.

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A ring R is *semiregular* if every element of R is semiregular. Note that R is regular if and only if R is semiregular for a semisimple ring R (i.e., a ring R with $J(R) = (0)$). We will generalize the concept of semiregularity of a ring. An element a of a ring R is called I-*semiregular* for some ideal I of R if there exists $b \in R$ such that $ab \in E(R)$ and $a - aba \in I$. Obviously, a regular ring is (0)-semiregular and a semiregular ring is $J(R)$-semiregular. A ring is a I-*semiregular ring* if each of its elements is I-semiregular. A ring R is called *right* (resp., *left*) *attaching-idempotent* if for $a \in R$ there exists $0 \neq b \in R$ (resp., $0 \neq c \in R$) such that ab (resp., ca) is an idempotent. R is called *attaching-idempotent* if it is both left and right attaching idempotent. It is shown in [6] that the attaching idempotent property is not left-right symmetry, and finite ring rings are attaching idempotent. For any proper ideal I of a ring R, we have the following implications:

R is regular $\Rightarrow$ R is semiregular $\Rightarrow$ R is I-semiregular $\Rightarrow$ R is attaching-idempotent and R/I is regular.

The above implications are strict by the following examples:

Example 1.2. (1) $\mathbb{Z}_{12}$ is not regular but is semiregular.

(2) Let $R = \mathbb{Q} \times \mathbb{Z}$ and $I = \{0\} \times \mathbb{Z}$ be an ideal of R. Then $J(R) = (0)$ and R is not regular (equivalently, R is not semiregular). To show that R is I-semiregular, let $a = (x, y) \in R$ be an arbitrary element. if $a \in I$, then clearly, $(0, 0) \in aR$ and $a - a(0, 0)a = a \in I$. If $a \notin I$, then $x \neq 0$. Take an idempotent $e = (1, 0) \in R$. Then $e = (1.0) = (x, y)(x^{-1}, 0) \in aR$ and $(1 - e)a = a - ea = (0, y) \in I$, yielding that a is I-semiregular, and so R is I-semiregular.

(3) Consider an ideal $K = \{0\} \times 2\mathbb{Z}$ of $R = \mathbb{Q} \times \mathbb{Z}$. Then $R/K \simeq \mathbb{Q} \times \mathbb{Z}_2$ is regular. Clearly, R is attaching-idempotent. Observe that R is not K-semiregular. Indeed, assume that R is K-semiregular. Take $a = (1, 3) \in R$. Then there exists $b = (x, y) \in R$ such that $(ab)^2 = ab$ and $a - aba \in K$. By $(ab)^2 = ab$, we have that $b = (0, 0)$ or $b = (1, 0)$. If $b = (0, 0)$, then $a - aba = a \notin K$, a contradiction. If $b = (1, 0)$, then $a - aba = (0, 3) \notin K$, a contradiction. Hence R is not K-semiregular.

In section 2, the equivalent conditions of I-semiregularity are investigated with some basic propverties, and it is shown that if e is an idempotent of I-semiregular ring R such that $I \cap eR = (0)$, then e is primitive if and only if e is right (left)irreducible if and only if eRe is a division ring if and only if e is local; e is primitive if and only if $\overline{e}$ is primitive in $\overline{R} = R/I$.

In Section 3, it is shown that for an abelian I-semiregular ring R if e is an idempotent of R such that $I \cap eR = (0)$, then e is primitive if and only if $ann(e)$ is a maximal ideal of R if and only if $ann(e)$ is a prime ideal of R. It is also shown that for an abelian I-semiregular ring having a nonempty set $E_1 = \{e^2 = e \in R \mid I \cap eR = (0)\}$, if E_1 satisfies the ascending chain condition on annihilators of idempotents in E_1, then there exists at least one primitive idempotent of R, any distinct primitive idempotents in E_1 are orthogonal and the number of primitive idempotents in E_1 is finite. In particular, for an abelian ring R satisfying the ascending chain condition on annihilators of idempotents of R, as a consequence of the above observations, R is isomorphic to a direct product of a finite number of division rings.

2. I-semiregular rings

In this section, we will begin with the following proposition.

Proposition 2.1. *Let I be an ideal of a ring R. Then the following are equivalent for an I-semiregular element a of R:*
(1) *There exists $e^2 = e \in aR$ such that $(1-e)a \in I$.*
(2) *There exists $e^2 = e \in Ra$ such that $a(1-e) \in I$.*
(3) *There exists $b \in R$ such that $bab = b$ and $a - aba \in I$.*
(4) *There exists $b \in R$ such that $abab = ab$ and $a - aba \in I$.*
(5) *There exists $c \in R$ such that $caca = ca$ and $a - aca \in I$.*

Proof. $(1) \Rightarrow (3)$: Suppose that there exists $e^2 = e \in aR$ such that $(1-e)a \in I$. Let $e = ax$ for some $x \in R$. Then $axax = ax$ and $(1-e)a = a - axa \in I$. Let $b = xax$. Then $bab = (xax)a(xax) = x(ax) = b$. Since $(a - axa)xa = axa - aba \in I$, $a - aba = (a - axa) + (axa - aba) \in I$.

$(3) \Rightarrow (4)$: It is clear.

$(4) \Rightarrow (1)$: Suppose that there exists $b \in R$ such that $abab = ab$ and $a - aba \in I$. Let $e = ab$. Then $e^2 = e \in aR$ and $a - aba = a - ea = (1-e)a \in I$.

$(4) \Rightarrow (5)$: Suppose that there exists $b \in R$ such that $abab = ab$ and $a - aba \in I$. Let $c = bab$. Then $caca = (baba)(baba) = b(ab)a = ca$. Since $a - aba \in I$, $(a - aba)ba = aba - aca \in I$, and so $a - aca = (a - aba) + (aba - aca) \in I$.

$(5) \Rightarrow (4)$: It follows from the similar argument given in the proof of $(4) \Rightarrow (5)$.

$(2) \Leftrightarrow (3) \Leftrightarrow (5)$: It follows from the similar argument given in the proof of $(1) \Leftrightarrow (3) \Leftrightarrow (4)$. □

A ring is a *I-semiregular ring* if each of its element is I-semiregular.

Corollary 2.2. *Let I be an ideal of a ring R. Then R is I-semiregular if and only if for any $a \in R \setminus I$, there exists an element $b \in R$ such that $(ab)^2 = ab \neq 0$ and $a - aba \in I$.*

Proof. Suppose that R is I-semiregular. Since R is I-semiregular, for $a \in R \setminus I$, there exists $b \in R$ such that $(ab)^2 = ab$ and $a - aba \in I$. If $ab = 0$, then $a \in I$, a contradiction, and so $ab \neq 0$. The converse is clear. □

Corollary 2.3. *Let R be a ring. Then we have the following:*
(1) *R is regular if and only if R is (0)-semiregular.*
(2) *For any ideals I, K of R such that $I \subseteq K$, I-semiregular is K-semiregular.*
(3) *If R is I-semiregular for an ideal I of R, then $I \supseteq J(R)$.*
(4) *If R is I-semiregular for an ideal I of R lying in $J(R)$, then $I = J(R)$.*
(5) *If R is semiregular, then R is I-semiregular for any ideal $I \supseteq J(R)$ of R.*

Proof. (1). It follows from Proposition 2.1.

(2). It is clear.

(3) Since R is I-semiregular, R/I is regular by Proposition 2.1. Hence $0 = J(R/I) = J(R)/I$, and then $I \supseteq J(R)$.

(4) It follows form (3).

(5) It follows from (2) and (3). □

Note that the converse of (2) of Corollary 2.3 may not be true by considering $\mathbb{Z}_{12}$. We can easily check that $\mathbb{Z}_{12}$ is semiregular, and so $\mathbb{Z}_{12}$ is both I-semiregular and K-semiregular where $I = 2\mathbb{Z}_{12}, K = 3\mathbb{Z}_{12}$, even though $I \nsubseteq K, K \nsubseteq I$.

Proposition 2.4. *Let I be an ideal of a ring R. Then we have the following.*

(1) *If $a \in R$ is I-semiregular, then there exists a regular element $b \in R$ such that $a - b \in I$.*

(2) *If $b = b\alpha b \in R$ for some $\alpha \in R$ (i.e., b is regular) such that $a - b \in I$ and $I \cap eR = (0)$ where $e = b\alpha$, then $a \in R$ is I-semiregular.*

Proof. (1) Since $a \in R$ is I-semiregular, there exists $e^2 = e \in aR$ such that $(1 - e)a \in I$ by Proposition 2.1. Let $e = ax$ for some $x \in R$. Let $b = ea$. Then $a - ea = a - b \in I$ and $b = eeb = e(ax)b = bxb$, yielding that $b \in R$ is a regular element of R.

(2) Since $e^2 = e = b\alpha$ and $a - b \in I$, $a - ea = (1 - e)(a - b) \in I$. Let $x = e - a\alpha \in R$. Then $x = (b - a)\alpha \in I$. Since $ex \in I \cap eR = (0)$, $ex = 0$, and so $e = ea\alpha = ea\alpha e$. Let $\beta = a\alpha e$. Then $(\beta)^2 = (a\alpha e)(a\alpha e) = (a\alpha)(ea\alpha e) = a\alpha e = \beta \in aR$. Note that $e - \beta = e - a\alpha e = (1 - a\alpha)e = (1 + x - e)e = xe \in I$. Thus $a - \beta a = (a - ea) + (ea - \beta a) = (a - ea) + (e - \beta)a \in I$, which implies that a is I-semiregular by Proposition 2.1 as desired. □

We say that an idempotent e ($\neq 0$) is *right* (resp., *left*) *irreducible* if eR (resp., Re) is a minimal right (resp., left) ideal of R. Recall that an idempotent e of a ring R is local if eRe is a local ring. In [9], Nicholson has shown that in a semiregular ring every primitive idempotent is local.

Proposition 2.5. *Let R be an I-semiregular ring for some ideal I of R. If $e \in R$ is a nonzero idempotent such that $I \cap eR = (0)$, then the following are equivalent:*

(1) *e is primitive;*

(2) *e is right (resp. left) irreducible;*

(3) *eRe is a division ring;*

(4) *e is local;*

(5) *$\overline{e}$ is primitive in $\overline{R} = R/I$.*

Proof. (3) $\Rightarrow$ (4) $\Rightarrow$ (1) : It is clear.

(1) $\Rightarrow$ (2) : Suppose that e is primitive. Let K be any nonzero right ideal of eR. Take a nonzero element $a \in K$. Since R is I-semiregular, there exists an element $b \in R$ such that $(ab)^2 = ab$ and $a - aba \in I$ by Proposition 2.1. Let

$e_1 = ab$, which is a nonzero idempotent by assumption that $I \cap eR = (0)$. Thus $e_1 \in eR$, and so $e_1 = ee_1$. Then (e_1e) and $(e - e_1e)$ are idempotents which are orthogonal. Since $e = e_1e + (e - e_1e)$ is a sum of orthogonal idempotents and e is primitive, $e_1e = 0$ or $e - e_1e = 0$. Assume that $e_1e = 0$. Then $e_1 \in e_1K \subseteq e_1eR = 0$, yielding that $e_1 = 0$, a contradiction. Hence we have that $e - e_1e = 0$, i.e., $e = e_1e$. Since $e = e_1e \in e_1R \subseteq K$, $eR \subseteq K$, and so $eR = K$, which implies that eR is a minimal right ideal of R. Hence e is right irreducible idempotent of R. Similarly, e is a left irreducible idempotent of R.

(2) $\Rightarrow$ (3) : It follows from [8, Proposition 21.16].

(4) $\Rightarrow$ (5) : Suppose that e is local. Then $eRe/J(eRe)$, (which is isomorphic to $\overline{e}\overline{R}\overline{e}$) is a division ring. Thus $\overline{e}$ is primitive in R/I.

(5) $\Rightarrow$ (1) : Suppose that $\overline{e}$ is primitive in R/I. Let $e = e_1 + e_2$ be a sum of two orthogonal idempotents $e_1, e_2 \in R$. Assume that $e_1, e_2 \neq 0$. Then $\overline{e} = \overline{e_1} + \overline{e_2}$ is a sum of two orthogonal idempotents $\overline{e_1}, \overline{e_2}$ of $\overline{R}$. Since $\overline{e}$ is primitive in $\overline{R}$, $\overline{e_1} = \overline{0}$ or $\overline{e_2} = \overline{0}$, i.e., $e_1 \in I$ or $e_2 \in I$. Clearly, $e_1R, e_2R \subseteq eR$, and so $I \cap e_1R, I \cap e_2 \subseteq I \cap eR = (0)$, which yields that $e_1, e_2 \notin I$, a contradiction. Hence $e_1 = 0$ or $e_2 = 0$, and so e is primitive in R. □

Proposition 2.6. *Let I be an ideal of a ring R. Then we have the following:*

(1) *If R is I-semiregular, then R/I is regular and idempotents can be lifted modulo I.*

(2) *Let $\overline{a} = a + I \in R/I$ be regular such that $a - aba \in I$ for some $b \in R$. If idempotent $\overline{ab} \in R/I$ can be lifted to an idempotent $e \in R$ modulo I such that $I \cap eR = (0)$, then $a \in R$ is I-semiregular.*

Proof. (1) If R is I-semiregular, then R/I is regular by Proposition 2.1. Let $a \in R$ with $a^2 - a \in I$. Since a is I-semiregular, there exists $b \in R$ such that $(ab)^2 = ab$ and $a - aba \in I$. Let $e = ab$. Then $e^2 = e \in aR$ and $a - ea \in I$. Let $f = e + ea(1 - e) \in R$. Then $f^2 = f$. Since $e - ae = (a - a^2)b \in I$, $f - a = e(e - ae) + (ea - a) \in I$.

(2) Let $x = ab - e \in I$. Since $ex \in I \cap eR = (0)$, $ex = 0$, and so $e = eab = eabe$. Let $\beta = abe$. Then $\beta^2 = (abe)(abe) = ab(eabe) = abe = \beta \in aR$. On the other hand, we have that $aba - \beta a = ab(1 - e)a = (x + e)(1 - e)a = x(1 - e)a \in I$, and then $a - \beta a = (a - aba) + (aba - \beta a) \in I$, and so $a \in R$ is I-semiregular by Proposition 2.1. □

Let I, K be ideals of a ring R such that $I \subseteq K$. An element $a \in K$ is called *I-semiregular* if there exists $e^2 = e \in aK$ such that $(1 - e)a \in I$.

Proposition 2.7. *Let I, K be ideals of a ring R such that $I \subseteq K$. Then the following are equivalent for an I-semiregular element a of K:*

(1) *There exists $e^2 = e \in aK$ such that $(1 - e)a \in I$.*
(2) *There exists $e^2 = e \in Ka$ such that $a(1 - e) \in I$.*
(3) *There exists $b \in K$ such that $bab = b$ and $a - aba \in I$.*
(4) *There exists $b \in K$ such that $abab = ab$ and $a - aba \in I$.*
(5) *There exists $c \in K$ such that $caca = ca$ and $a - aca \in I$.*

Proof. It follows from the similar argument given in the proof of Proposition 2.1. □

Proposition 2.8. *Let I, J, K be ideals of a ring R such that $I \subseteq J \subseteq K$. If K is I-semiregular, then K/J is regular and J is I-semiregular.*

Proof. Let $\overline{a} = a + J \in K/J$ be arbitrary. Since K is I-semiregular, there exists $b \in K$ such that $(ab)^2 = ab$ and $a - aba \in I$. Since $a - aba \in I \subseteq J$, $\overline{a} = \overline{a}\overline{b}\overline{a}$ in K/J, and so K/J is regular. Next, let $c \in J$ be arbitrary. By Proposition2.1, there exists $e^2 = e \in cR$ such that $c - ec \in I$. Then $e = cr$ for some $r \in R$, and so $e = ee = c(rcr) \in cJ$, which implies that J is I-semiregular. □

Note that the converse of Proposition 2.8 need not be true, as shown in Example 1.2 (3): Let $K = \{0\} \times 2\mathbb{Z}$ be an ideal of $R = \mathbb{Q} \times \mathbb{Z}$. Then $R/K \simeq \mathbb{Q} \times \mathbb{Z}_2$ is regular and K is clearly K-semiregular. But R is not K-semiregular.

Proposition 2.9. *Let R be an I-semiregular ring for some proper ideal I of R. Then the following are equivalent:*

(1) *R is abelian;*

(2) *For each $0 \neq x \in R \setminus I$, there exists $y \in R \setminus I$ such that $(xy)^2 = xy = yx \neq 0$;*

(3) *$xy = yx$ whenever $(xy)^2 = xy$ for $x, y \in R \setminus I$.*

Proof. (1) $\Rightarrow$ (2) : Since R is I-semiregular, for each $0 \neq x \in R \setminus I$ there exists $z \in R$ such that $(xz)^2 = xz \neq 0$ and $x - xzx \in I$ by Corollary 2.2. If $z \in I$, then $x = (x - xzx) + xzx \in I$, a contradiction, and so $z \in R \setminus I$. Let $y = zxz$. Then $xy = x(zxz) = xz$ and $yxyx = (zxz)x(zxz)x = zxzx = yx$ are idempotents. Since R is abelian, xy, yx are central idempotents, and so $xy = (xy)(xy) = x(yx)y = (yx)xy = y(xy)x = yx$, as desired. Observe that $y \notin I$. Indeed, if $y \in I$, then $xy = xz \in I$, and so $xzx \in I$, and then $x = (x - xzx) + xzx \in I$, a contradiction,

(2) $\Rightarrow$ (3) : Suppose that for each $0 \neq x \in R \setminus I$, there exists $y \in R \setminus I$ such that $(xy)^2 = xy = yx \neq 0$. We observe that R is reduced (i.e., R has no nonzero nilpotent). Indeed, assume that there exists $0 \neq x$ such that $x^2 = 0$. Then there exists $y \in R \setminus I$ such that $(xy)^2 = xy = yx \neq 0$, and then $xy = x^2y^2 = 0$, a contradiction. Let $(xy)^2 = xy$ for $x, y \in R \setminus I$. Since R is reduced, $xy(1 - xy) = 0$ implies that $x(1 - xy)y = yx(1 - xy) = y(1 - xy)x = 0$ by [[1], Theorem 1.3], entailing $xy = xxyy$, $yx = yxxy$, and $yx = (yx)^2$. Since reduced ring is clearly abelian, xy and yx are central idempotents of R. Thus $xy = (xy)(xy) = x(yx)y = (yx)(xy) = yx$.

(3) $\Rightarrow$ (1) : Suppose that $xy = yx$ whenever $(xy)^2 = xy$ for $x, y \in R \setminus I$. Let $e \in R$ be any nonzero idempotent of R. If $e \notin I$, then $eu, u^{-1} \in R \setminus I$ for any unit $u \in R$. By assumption, $e^2 = e = (eu)u^{-1} = u^{-1}(eu)$, and so $ue = eu$, which yields that e is central by [[5], Corollary 2.2]. If $e \in I$, then $1 - e \notin I$. By the above argument, $(1 - e)u = u(1 - e)$, yields that $eu = ue$, and so e is central. □

Corollary 2.10. *The following are equivalent for a regular ring R:*
(1) *R is abelian;*
(2) *For each $0 \neq x \in R$, there exists $y \in R$ such that $(xy)^2 = xy = yx \neq 0$;*
(3) *$xy = yx$ whenever $(xy)^2 = xy$ for $x, y \in R$.*

Proof. It follows from Proposition 2.9. □

Theorem 2.11. *Let I be an ideal of a ring R. If R is abelian I-semiregular, then the center of R is I-semiregular.*

Proof. Let S be the center of R, and let $x \in S$. Since R is abelian I-semiregular, there exists $y \in R$ such that $xyxy = xy = yx$ and $x - xyx \in I$ by Proposition 2.1 and Proposition 2.9. Let $z = yxy$. Then $xz = x(yxy) = xy = yx = yxyx = zx$, and so $xzxz = xz = zx$. To show $z \in S$, let $r \in R$ be arbitrary. Then $rz = r(yxy) = yxry = yrxy = yxyr = zr$ because $xy = yx$ are central and $x \in S$. Clearly, $x - xzx = x - xyx \in I$, yielding that S is I-semiregular. □

We say that a ring R ($\neq 0$) is *indecomposable* if R is not a direct sum of two nonzero ideals. This is the case if and only if R has no nontrivial central idempotents. The following is similar to [3, Corollary 1.15].

Corollary 2.12. *Let R be an abelian I-semiregular ring for some proper ideal I of R. Then R is indecomposable (as a ring) if and only if its center is a field.*

Proof. Suppose that R is indecomposable. Let S be the center of R and let $0 \neq x \in S$ be arbitrary. By Theorem 2.11, $xzxz = xz$ and $x - xzx \in I$ for some $z \in S$. Note that we can take $z \neq 0$. Indeed, assuming that $xz = 0$ for all $z \in S$, then $x \in I$, and so $1 \in S \subseteq I$, a contradiction. Thus xz is a nonzero central idempotent of R. Since R is abelian I-semiregular, $xz = zx$ by Proposition 2.9. Since R is indecomposable, $xz = 1$. Therefore, S is a field. The converse is clear. □

3. Idempotents of abelian I-semiregular rings

Recall that a prime ideal P of a ring R is an associated prime if $P = ann(y)$ for some $y \in R$. It was well known that for a Noetherian ring, the set of associated primes is finite. In addition, if R is commutative, then any maximal element of the family of ideals $\Lambda = \{ann(x) \mid 0 \neq x \in R\}$ is an associated prime (see [2], IV §1.1 Proposition 2). Since $ann(0) = R$, an element $x \in R$ whose annihilator is a prime ideal is necessarily $\neq 0$.

Lemma 3.1. *Let R be a ring and $e, f \in R$ be idempotents. Then $ann(e) = ann(f)$ if and only if $e = f$.*

Proof. Suppose that $ann(e) = ann(f)$. Since $1 - e \in ann(e) = ann(f)$ (resp., $1 - f \in ann(f) = ann(e)$), $f = ef$ (resp., $e = ef$), and so $e = ef = f$. The converse is clear. □

Theorem 3.2. *Let R be an abelian I-semiregular ring for some ideal I of R. If $e \in R$ is a nonzero idempotent such that $I \cap eR = (0)$, then the following are equivalent:*

(1) *e is primitive;*
(2) *$ann(e)$ is a maximal element in $\Lambda = \{ann(x) \mid 0 \neq x \in R\}$;*
(3) *$ann(e)$ is a maximal ideal of R;*
(4) *$ann(e)$ is a prime ideal of R.*

Proof. (1) $\Rightarrow$ (2) : Suppose that e is primitive. Let $ann(e) \subseteq ann(x)$ for any $ann(x) \in \Lambda$. We observe that $x \notin I$. Indeed, since $1 - e \in ann(e) \subseteq ann(x)$, $x = ex = xe$, and so if $x \in I$, then $x \in I \cap eR = (0)$ by assumption, i.e., $x = 0$, a contradiction. Since R is abelian I-semiregular and $x \notin I$, there exists $y \in R \setminus I$ such that $(xy)^2 = xy = yx \neq 0$ by Proposition 2.9. Since $xy = yx$, we have that $ann(x) \subseteq ann(xy)$. Let $f = xy$. Since $1 - e \in ann(e) \subseteq ann(f)$, $f = ef = fe$. Note that $e = ef + e(1 - f)$ is a sum of two orthogonal central idempotents ef and $e(1 - f)$ of R. Assume that $ef = 0$. Then $f \in ann(e) \subseteq ann(f) \in \Lambda$, and so $f^2 = f = 0$, a contradiction. Thus $ef \neq 0$. Since e is a primitive idempotent of R and $ef \neq 0$, $e(1-f) = 0$, i.e., $e = ef$, yielding that $e = f$, and so $ann(e) = ann(x)$, which implies that $ann(e)$ is a maximal element in Λ.

(2) $\Rightarrow$ (4) : Suppose that $ann(e)$ is a maximal element in $\Lambda = \{ann(x) \mid 0 \neq x \in R\}$. Since $e \neq 0$, $ann(e) \neq R$. Let b, c be elements of R such that $bc \in ann(e)$ and $c \notin ann(e)$. Then it is clear that $ec = ce \neq 0$, and so $ann(ec) \in \Lambda$. Since e is central, $ann(e) \subseteq ann(ec)$. Since $ann(e)$ is a maximal element in Λ, $b \in ann(ec) = ann(e)$, hence $ann(e)$ is a prime ideal of R.

(4) $\Rightarrow$ (1) : Suppose that $ann(e)$ is a prime ideal of R. Let $e = \alpha + \beta$ for some idempotents $\alpha, \beta \in R$ with $\alpha\beta = \beta\alpha = 0$. Since α and β are central, we have that $(\alpha)(\beta) = (\alpha\beta) = (0)$ where (x) is a principal ideal of R generated by $x \in R$. Since $ann(e)$ is prime and $(\alpha)(\beta) = (0) \subseteq ann(e)$, $(\alpha) \subseteq ann(e)$ or $(\beta) \subseteq ann(e)$, and so $\alpha \in ann(e)$ or $\beta \in ann(e)$. Thus $\alpha = \alpha e = 0$ or $\beta = \beta e = 0$, which implies that e is a primitive idempotent of R.

(3) $\Rightarrow$ (4) : It is clear.

(4) $\Rightarrow$ (3) : Suppose that $ann(e)$ is a prime ideal of R. Let K be any ideal of R such that $ann(e) \subsetneq K \subseteq R$. Then there exists some nonzero $x \in K \setminus ann(e)$, i.e., $0 \neq ex \in R$. Note that $ex \notin I$ by assumption $I \cap eR = (0)$. Since R is I-semiregular and $ex \notin I$, there exists some $y \in R$ such that $(ex)y$ is a nonzero idempotent of R. Clearly, $ann(e) \subseteq ann(exy) \in \Lambda$. Since $ann(e)$ is a prime ideal of R (equivalently, $ann(e)$ is a maximal element in Λ by the proof of (1) $\Leftrightarrow$ (4)), $ann(e) = ann(exy)$. By Lemma 3.1, $e = exy$, i.e., $e(1 - xy) = 0$. Since e is central, $(e)(1 - xy) \subseteq (e(1 - xy)) = (0) \subseteq ann(e)$. Since $ann(e)$ is a prime ideal of R, $(e) \subseteq ann(e)$ or $(1 - xy) \subseteq ann(e)$, and so $e \in ann(e)$ or $1 - xy \in ann(e)$. If $e \in ann(e)$, then $e^2 = e = 0$, a contradiction. Hence $1 - xy \in ann(e) \subsetneq K$, and then $1 = (1 - xy) + xy \in K$, which implies that $K = R$. Therefore, $ann(e)$ is a maximal ideal of R. □

Corollary 3.3. *Let R be an abelian I-semiregular ring for some ideal I of R. If $e \in R$ is a nonzero idempotent such that $I \cap eR = (0)$, then the following are equivalent:*

(1) *e is primitive;*
(2) *e is irreducible;*
(3) *eRe is a division ring;*
(4) *e is local;*
(5) *$\overline{e}$ is primitive in $\overline{R} = R/I$;*
(6) *$ann(e)$ is a maximal element in $\Lambda = \{ann(x) \mid 0 \neq x \in R\}$;*
(7) *$ann(e)$ is a prime ideal of R;*
(8) *$ann(e)$ is a maximal ideal of R.*

Proof. It follows from Proposition 2.5 and Theorem 3.2. □

Corollary 3.4. *Let R be an abelian regular ring. If $e \in R$ is a nonzero idempotent, then the following are equivalent:*

(1) *e is primitive;*
(2) *e is irreducible;*
(3) *eRe is a division ring;*
(4) *e is local;*
(5) *$ann(e)$ is a maximal element in $\Lambda = \{ann(x) \mid 0 \neq x \in R\}$;*
(6) *$ann(e)$ is a prime ideal of R;*
(7) *$ann(e)$ is a maximal ideal of R.*

Proof. It follows from Corollary 3.3. □

Theorem 3.5. *Let R be an abelian I-semiregular ring for some ideal I of R and $E_1 = \{e^2 = e \in R \mid I \cap eR = (0)\}$ be a nonempty set. Suppose that R satisfies the ascending chain condition on annihilators of idempotents in E_1. Then we have the following:*

(1) *There exists at least one primitive idempotent of R.*
(2) *Any distinct primitive idempotents in E_1 are orthogonal.*
(3) *The number of all primitive idempotents in E_1 is finite.*

Proof. (1) Let $\Lambda = \{ann(x) \mid 0 \neq x \in R\}$ and $\Lambda_1 = \{ann(e) \in \Lambda \mid e \in E_1\}$. Clearly, $(\Lambda_1, \subseteq)$ is a partially ordered set under the set inclusion $\subseteq$. Take $ann(e_1) \in \Lambda_1$. If $ann(e_1)$ is a maximal element in Λ_1, then we will show that $ann(e_1)$ is a maximal element in Λ. To show this, let $ann(e_1) \subseteq ann(x) \in \Lambda$ for any nonzero $x \in R$. We observe that $x \notin I$. Indeed, if $x \in I$, then $1 - e_1 \in ann(e_1) \subseteq ann(x)$, yielding that $x = e_1x = xe_1 \in I \cap e_1R = (0)$, i.e., $x = 0$, a contradiction. Since R is abelian I-semiregular and $x \notin I$, there exists $y \in R \setminus I$ such that $(xy)^2 = xy = yx \neq 0$ by Proposition 2.9. Since $xy = yx$, $ann(x) \subseteq ann(xy)$. Let $f = xy$. We will show that $f \in E_1$, (equivalently, $I \cap fR = (0)$). Since $1 - e_1 \in ann(e_1) \subseteq ann(f)$, $f = e_1f = fe_1$. Let $a \in I \cap fR$. Then $a = fr$ for some $r \in R$. Thus $a = fr = e_1fr = e_1a \in I \cap e_1R = (0)$, i.e., $a = 0$, and so $I \cap fR = (0)$. Since $ann(e_1)$ is a maximal in Λ_1 and

$ann(e_1) \subseteq ann(f) \in \Lambda_1$, $ann(e_1) = ann(f)$. Therefore $ann(e_1) = ann(x)$, which implies that $ann(e_1)$ is a maximal element in Λ, and so e_1 is primitive by Theorem 3.2 as desired. If $ann(e_1)$ is not a maximal element in Λ_1, we can take $ann(e_2) \in \Lambda_1$ such that $ann(e_1) \subsetneq ann(e_2)$. If $ann(e_2)$ is a maximal element in Λ_1, then e_2 is primitive by the similar argument. Continuing in this way, we have the following chain in Λ_1:

$$ann(e_1) \subsetneq ann(e_2) \subsetneq ann(e_3) \subsetneq \cdots$$

By assumption, there exists a positive integer k such that $ann(e_k) = ann(e_j)$ for all $j \geq k$, and so $ann(e_k)$ is the upper bound of the chain. By Zorn's Lemma, there exists a maximal element $ann(e)$ in Λ_1, yielding that e is primitive by the previous argument.

(2) Let $e, f \in E_1$ $(e \neq f)$ be arbitrary primitive idempotents. By Theorem 3.2, $ann(e)$ and $ann(f)$ are maximal ideals of R. Assume that $ef \neq 0$. Then $ann(e), ann(f) \subseteq ann(ef)$ and clearly $ef \in E_1$. Since $ann(e)$ and $ann(f)$ are maximal ideals of R and $ann(ef) \neq R$, $ann(e) = ann(ef) = ann(f)$, yielding that $e = ef = f$ by Lemma 3.1, a contradiction. Hence any distinct primitive idempotents in E_1 are orthogonal.

(3) Let M_1 be the set of all primitive idempotents of E_1. Then M_1 is a nonempty set by (1). Assume that M_1 is infinite. Let $f_k = \sum_{i=1}^{k} a_i$ $(k \geq 1)$ for $a_i \in M_1$. Since M_1 is orthogonal by (2), $f_k \in E_1$ $(k \geq 1)$. Define a relation $\leq$ on E_1 by $e \leq f$ if $e = ef = fe$ for all $e, f \in E_1$. Clearly, $(E_1, \leq)$ is a partially ordered set with a partial ordering $\leq$. Consider the following chain in E_1:

$$f_1 \leq f_2 \leq f_3 \leq \cdots .$$

Then we also have the following chain of annihilators of idempotents of E_1:

$$ann(1 - f_1) \subseteq ann(1 - f_2) \subseteq ann(1 - f_3) \subseteq \cdots .$$

Since the above chain has an ascending chain condition by assumption, there exists a positive integer n such that $ann(1 - f_n) = ann(1 - f_k)$ for all $k \geq n$, and so $1 - f_n = 1 - f_k$ (i.e., $f_n = f_k$) for all $k \geq n$ by Lemma 3.1, which yields that $a_k = 0$ for all $k \geq n$, a contradiction. Hence M_1 is finite. □

Corollary 3.6. *Let R be an abelian regular ring. Suppose that R satisfies the ascending chain condition on annihilators of idempotents in R. Then we have the following:*

(1) *There exists at least one primitive idempotent of R.*
(2) *Any distinct primitive idempotents in R are orthogonal.*
(3) *The number of all primitive idempotents in R is finite.*

Proof. It follows from Theorem 3.5. □

We finally have a consequence of our observation as follows:

Theorem 3.7. *Let R be an abelian regular ring with the ascending chain condition on annihilators of idempotents of R. Then we have the following:*

(1) *The unity 1 of R can be expressed a sum of a finite number primitive idempotents of R.*

(2) *Every idempotent can be expressed a sum of a finite number primitive idempotents of R.*

(3) *R is isomorphic to a direct product of a finite number of division rings.*

Proof. (1) Let $M(R)$ be the set of all primitive idempotents of R. By Corollary 3.6, $M(R)$ is nonempty, finite and orthogonal. If $1 \in M(R)$, then we are done. Suppose that $1 \notin M(R)$. Then $1 = e_1 + b_1$ for some nontrivial idempotents $e_1, b_1 \in R$. If $e_1, b_1 \in M(R)$, then we are done. Suppose that $e_1 \notin M(R)$ or $b_1 \notin M(R)$ (say, $b_1 \notin M(R)$). Then $b_1 = e_2 + b_2$ some nontrivial idempotents $e_2, b_2 \in R$. Since $M(R)$ is finite, continuing in this way, 1 can be expressed a sum of a finite number of primitive idempotents of R.

(2) By (1), we have that $1 = \sum_{i=1}^{n} e_i$ for some $e_i \in M(R)$. Let $a \in R$ be an arbitrary idempotent. Then $a = \sum_{i=1}^{n} ae_i$. Note that if $ae_i \neq 0$, then $ae_i \in M(R)$ by [4, Corollary 2.11]. Hence a can be expressed as a sum of finite number of primitive idempotents of R.

(3) It follows from (1) and Corollary 3.4. □

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Juncheol Han

Department of Mathematics Education, Pusan National University, Pusan 46277, Korea

E-mail address: jchan@pusan.ac.kr

Hyo-Seob Sim

Department of Applied Mathematics, Pukyong National University, Pusan 48513, Korea

E-mail address: hsim@pknu.ac.kr