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CONJUGATING AUTOMORPHISMS OF CERTAIN HNN EXTENSIONS

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Abstract. We consider HNN extensions $\langle B, t : t^{-1}Ht = K\rangle$, where H, K are in the center of B. We show that conjugating automorphisms of those HNN extensions are inner if B satisfies certain conditions.

1. Introduction

An automorphism α of a group G is called a *conjugating* (or class-preserving, or point-wise inner) *automorphism* if, for each $g \in G$, $\alpha(g)$ and g are conjugate in G. Clearly inner automorphisms are conjugating automorphisms. But there exist some groups admitting conjugating automorphisms which are not inner (see [3, 10, 11]).

Definition 1 (Grossman [5]). A group G has *Property A* if, for each conjugating automorphism α of G, there exists a single element $k \in G$ such that $\alpha(g) = k^{-1}gk$ for all $g \in G$.

For example, Grossman [5] proved that free groups and fundamental groups of compact orientable surfaces have Property A. She also proved that outer automorphism groups of finitely generated conjugacy separable groups with Property A are residually finite. Hence outer automorphism groups of free or surface groups are residually finite. In this direction, it was shown that outer automorphism groups of some Fuchsian groups [1, 9], most of Seifert 3-manifold groups [2] and certain 1-relator groups [6, 7] are residually finite.

In this paper we show that certain HNN extension

$$G = \langle B, t : t^{-1}Ht = K\rangle$$

has Property A (Theorem 2.4), when H, K are in the center of B.

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Throughout this paper we use standard notation and terminology.

If $g \in G$, Inn_g denotes the inner automorphism of G induced by g.

$x \sim_G y$ means that x and y are conjugate in G, otherwise $x \not\sim_G y$.

We use $Z(G)$ to denote the center of G.

We shall make extensive use of the following result by D. J. Collins.

Theorem 1.1 (Collins [4]). *Let x and y be cyclically reduced elements of the HNN-extension $G = \langle B, t : t^{-1}Ht = K\rangle$. Suppose that $x \sim_G y$. Then $\|x\| = \|y\|$, and one of the following holds.*
(1) $\|x\| = \|y\| = 0$ *and there is a finite sequence $z_1, z_2, \ldots, z_m$ of elements in $H \cup K$ such that $x \sim_B z_1 \sim_{B,t^*} z_2 \sim_{B,t^*} \cdots \sim_{B,t^*} z_m \sim_B y$, where $u \sim_{B,t^*} v$ means one of: (i) $u \sim_B v$, or (ii) $u \in H$ and $v = t^{-1}ut (\in K)$, or (iii) $u \in K$ and $v = tut^{-1} (\in H)$.*
(2) $\|x\| = \|y\| \geq 1$ *and $y \sim_{H\cup K} x^*$ where x^* is a cyclic permutation of x.*

2. Main results

Remark 1. [12] *Let $G = \langle B, t : t^{-1}Ht = K\rangle$ be an HNN extension of a base group B. Then every element $g \in G$ can be expressed as a normal form (*[8]*, p.181),*

$$g = a_0 t^{\epsilon_1} a_1 \cdots t^{\epsilon_n} a_n, \text{ where } \epsilon_i = \pm 1 \text{ and } a_i \in B,$$

satisfying the following:
(i) a_n is an arbitrary element of the base group B,
(ii) if $\epsilon_i = -1$, then a_{i-1} is a represntative of a coset of K in B,
(iii) if $\epsilon_i = 1$, then a_{i-1} is a represnetative of a coset of H in B, and
(iv) there is no consecutive subsequence $t^{\epsilon}\ 1\ t^{-\epsilon}$.

In this note, we mean that a cyclically reduced element $g \in G$ is either $g \in B$ or $g = t^{\epsilon_1} a_1 \cdots t^{\epsilon_n} a_n$, where $t^{\epsilon_i} a_i \cdots t^{\epsilon_n} a_n t^{\epsilon_1} a_1 \cdots t^{\epsilon_{i-1}} a_{i-1}$ is reduced for each i.

If $x = t^{\epsilon_1} a_1 \cdots t^{\epsilon_n} a_n$ is cyclically reduced, then a cyclic permutation x^* of x is either

$$x^* = t^{\epsilon_i} a_i \cdots t^{\epsilon_n} a_n t^{\epsilon_1} a_1 \cdots t^{\epsilon_{i-1}} a_{i-1} \text{ or}$$
$$x^* = a_{i-1} t^{\epsilon_i} a_i \cdots t^{\epsilon_n} a_n t^{\epsilon_1} a_1 \cdots t^{\epsilon_{i-1}}$$

for some i. As mentioned in [12, Remark 2.2], if $y = t^{\delta_1} b_1 \cdots t^{\delta_n} b_n$ is cyclically reduced and $y = c^{-1}x^*c$ for $c \in H \cup K$ in Theorem 1.1 (2), then it is enough to consider only the cyclic permutations $x^* = t^{\epsilon_i} a_i \cdots t^{\epsilon_n} a_n t^{\epsilon_1} a_1 \cdots t^{\epsilon_{i-1}} a_{i-1}$ of x.

We shall consider the HNN extension

$$G = \langle B, t : t^{-1}Ht = K\rangle,$$

satisfying the following conditions:
(C1) If $h_1 \sim_B h_2$ for $h_1, h_2 \in H$ then $h_1 = h_2$ and if $k_1 \sim_B k_2$ for $k_1, k_2 \in K$ then $k_1 = k_2$.
(C2) There exists an element $z \in B$ such that $z \not\sim_B x$ for any $x \in H \cup K$.

For example, if $H, K \subset Z(B)$, then (C1) holds. Also if $H, K \subset Z(B)$ and if there is a element $z \in B \backslash H \cup K$, then (C2) holds.

The proof of the following is quite similar to that of Lemma 2.3 in [12].

Lemma 2.1. *Let $G = \langle B, t : t^{-1}Ht = K\rangle$ and let B satisfy (C2) above. Suppose α is a conjugating automorphism of G such that $\alpha(z) = z$, where $z \in B$ in (C2). Then there exists an element $a \in B$ such that $\alpha(t) = a^{-1}ta$.*

Lemma 2.2. *Let $G = \langle B, t : t^{-1}Ht = K\rangle$ and let B satisfy (C1) and (C2) above. Suppose α is a conjugating automorphism of G such that $\alpha(t) = t$ and $\alpha(z) = a^{-1}za$, where $z \in B$ in (C2) and $a \in B$. If $H, K \subset Z(B)$, then $\alpha(z) = a^{-1}za = zc$ and $t^{-1}ct = c$ for some $c \in H \cap K$.*

Proof. Let $a \in B$ and $\alpha(z) = a^{-1}za = a_0$, where $z \in B$ in (C2). Then $a_0 \in B\backslash(H \cup K)$. Note $t\,z\,t^{-1}\,z^{-1}$ is cyclically reduced of length 2 and $t\,z\,t^{-1}\,z^{-1} \sim_G \alpha(t\,z\,t^{-1}\,z^{-1}) = t\,a_0\,t^{-1}\,a_0^{-1}$. By Theorem 1.1 (2) we have

$$t\,a_0\,t^{-1}\,a_0^{-1} = x^{-1}(t\,z\,t^{-1}\,z^{-1})^*x$$

for some $x \in H \cup K$, where $(t\,z\,t^{-1}\,z^{-1})^*$ is a cyclic permutation of $t\,z\,t^{-1}\,z^{-1}$. Hence we have $t\,a_0\,t^{-1}\,a_0^{-1} = x^{-1}(t\,z\,t^{-1}\,z^{-1})x$ for some $x \in H \cup K$. Thus $tz^{-1}t^{-1}xt\,a_0\,t^{-1} = z^{-1}xa_0$. We have

$$x = h_1 \in H,\ t^{-1}h_1t = k_1,\ z^{-1}k_1a_0 \in K, \text{ and } z^{-1}xa_0 \in H.$$

Since $H, K \subset Z(B)$, we have $z^{-1}a_0 \in K$ and $z^{-1}a_0 \in H$. Hence $a_0 = zc$ where $c \in H \cap K$. Therefore, $tz^{-1}k_1zct^{-1} = z^{-1}h_1zc$. Since $H, K \subset Z(B)$, we get $tk_1ct^{-1} = h_1c$, and $tct^{-1} = c$. □

Lemma 2.3. *Let $G = \langle B, t : t^{-1}Ht = K\rangle$ and let B satisfy (C1) and (C2) above. Suppose α is a conjugating automorphism of G such that $\alpha(t) = t$ and $\alpha(z) = a^{-1}za$, where $z \in B$ in (C2) and $a \in B$. Let $H, K \leq Z(B)$.*
(1) For each $b \in B\backslash H$, $\alpha(b) = bc$, where $c \in H \cap K$ and $t^{-1}ct = c$, and
(2) for each $b \in B\backslash K$, $\alpha(b) = bc$, where $c \in H \cap K$ and $t^{-1}ct = c$.

Proof. By Lemma 2.2, $\alpha(z) = a^{-1}za = zc$ and $t^{-1}ct = c$ for some $c \in H \cap K$. Let $b \in B$ and $\alpha(b) = k_b^{-1}b\,k_b$ and let $k_b = a_0t^{\epsilon_1}a_1 \cdots t^{\epsilon_n}a_n$, where $\epsilon_i = \pm 1$ and $a_i \in B$, be in its normal form, as Remark 1. Since

$$b\,z \sim_G \alpha(b\,z) = \alpha(b)\alpha(z) = a_n^{-1}t^{-\epsilon_n} \cdots a_1^{-1}t^{-\epsilon_1}a_0^{-1}\,b\,a_0t^{\epsilon_1}a_1 \cdots t^{\epsilon_n}a_n\,zc,$$

we have

$$b\,z \sim_G t^{-\epsilon_n} \cdots a_1^{-1}t^{-\epsilon_1}a_0^{-1}ba_0t^{\epsilon_1}a_1 \cdots t^{\epsilon_n}a_nzc\,a_n^{-1}. \tag{1}$$

By (C2), we have $a_nzc\,a_n^{-1} \notin H \cup K$. In (1) above, if

$$t^{-\epsilon_n} \cdots a_1^{-1}t^{-\epsilon_1}a_0^{-1}ba_0t^{\epsilon_1}a_1 \cdots t^{\epsilon_n} \notin B,$$

then the R.H.S. of (1) is cyclically reduced of even length ≥ 1, contrary to Theorem 1.1. Hence $\alpha(b) = a_n^{-1}(t^{-\epsilon_n} \cdots a_1^{-1}t^{-\epsilon_1}a_0^{-1}ba_0t^{\epsilon_1}a_1 \cdots t^{\epsilon_n})\,a_n \in B$. Let $\alpha(b) = u \in B$.

(1) Let $b \in B\backslash H$. Consider $t\,z^{-1}\,t^{-1}\,b \sim_G \alpha(t\,z^{-1}\,t^{-1}\,b) = t\,zc\,t^{-1}\,u$. By Theorem 1.1 (2), we have $t\,c^{-1}z^{-1}\,t^{-1}\,u = x^{-1}(t\,z^{-1}\,t^{-1}\,b)^*x$ for some $x \in H \cup K$, where $(t\,z^{-1}\,t^{-1}\,b)^*$ is a cyclic permutation of $t\,z^{-1}\,t^{-1}\,b$. Hence we have

$$t\,c^{-1}z^{-1}\,t^{-1}\,u = x^{-1}(t\,z^{-1}\,t^{-1}\,b)x$$

for some $x \in H \cup K$. Thus $tzt^{-1}xt\,c^{-1}z^{-1}\,t^{-1} = bxu^{-1}$. From this, we have $x = h_1 \in H$, $t^{-1}h_1t = k_1$ and $t(zk_1c^{-1}z^{-1})t^{-1} = bxu^{-1}$. Since $H, K \leq Z(B)$, we have $zk_1c^{-1}z^{-1} = k_1c^{-1}$. Hence $tk_1c^{-1}t^{-1} = bxu^{-1}$. Since $k_1 = t^{-1}h_1t = t^{-1}xt$, we have $xtc^{-1}t^{-1} = bxu^{-1} = xbu^{-1}$. Therefore $tc^{-1}t^{-1} = bu^{-1}$. Since $t^{-1}ct = c$ from the beginning of this proof, we have $\alpha(b) = u = bc$.

(2) Let $b \in B\backslash K$. Consider $t^{-1}\,z^{-1}\,t\,b \sim_G \alpha(t^{-1}\,z^{-1}\,t\,b) = t^{-1}\,c^{-1}z^{-1}\,t\,u$. As above, we have $\alpha(b) = u = bc$. □

Now we can apply these above lemmas to get our criterion for certain HNN extensions having Property A.

Theorem 2.4. *Let $G = \langle B, t : t^{-1}Ht = K\rangle$ and let $H, K \subset Z(B)$. Suppose there exists $a_1 \in B\backslash H \cup K$ and suppose there exists $a_2 \in B\backslash H$ such that $a_1a_2 \notin H$ or there exists $a_2 \in B\backslash K$ such that $a_1a_2 \notin K$. Then G has Property A.*

Proof. Clearly (C1) and (C2) hold. Here we consider $z = a_1$ in (C2). Let α be a conjugating automorphism of G. Without loss of generality, we assume $\alpha(z) = z$, where $z \in B$ in (C2). By Lemma 2.1, there exists $a \in B$ such that $\alpha(t) = a^{-1}ta$. Considering $Inn_a \circ \alpha$, we may assume that $\alpha(t) = t$ and $\alpha(z) = aza^{-1} = a_0 \in B$. By Lemma 2.2, $a_0 = zc$ where $c \in H \cap K$. By Lemma 2.3, $\alpha(b) = bc$ for each $b \in B\backslash H$. Hence, for $h \in H$, we have $\alpha(zh) = zhc$ and $\alpha(zh) = \alpha(z)\alpha(h) = zc\,\alpha(h)$. Thus $\alpha(h) = h$ for $h \in H$. Similarly, $\alpha(k) = k$ for $k \in K$.

Let $a_2 \in B\backslash H$ such that $a_1a_2 \notin H$. By Lemma 2.3, $\alpha(a_1a_2) = a_1a_2c$. Also $\alpha(a_1a_2) = \alpha(a_1)\alpha(a_2) = a_1ca_2c = a_1a_2c^2$. Hence $c = 1$ and $\alpha(b) = b$ for each $b \in B\backslash(H \cup K)$. Therefore α is the identity map. This follows that G has Property A. □

Theorem 2.5. *Let $G = \langle B, t : t^{-1}Ht = K\rangle$ and let $H, K \subset Z(B)$. Suppose there exists $z \in B\backslash H \cup K$ and suppose there exists no element of order 2 in $H \cap K$. Then G has Property A.*

Proof. Clearly (C1) and (C2) hold. Let α be a conjugating automorphism of G. As before, we can assume $\alpha(t) = t$ and $\alpha(z) = aza^{-1}$ for some $a \in B$. By Lemma 2.2, $\alpha(z) = zc$ and $t^{-1}ct = c$ for some $c \in H \cap K$. Hence $c \in Z(G)$ and $\alpha(c) = c$. Note that $z^{-1}c \notin H \cup K$. By Lemma 2.3, $\alpha(z^{-1}c) = z^{-1}c^2$. Thus $c = \alpha(c) = \alpha(zz^{-1}c) = \alpha(z)\alpha(z^{-1}c) = zcz^{-1}c^2 = c^3$. Since $H \cap K$ has no element of order 2, we have $c = 1$. Then, as above, we know that α is the identity map, and G has Property A. □

Corollary 2.6. *Let B be a finitely generated nilpotent group and $h \in Z(B)$ of infinite order. Then $G = \langle B, t : t^{-1}h^{\alpha}t = h^{\beta}\rangle$ has Property A if $|\alpha| \neq 1 \neq |\beta|$.*

In particular, the Baumslag-Solitar group $G = \langle h, t : t^{-1}h^2t = h^3\rangle$ *has Property A.*

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