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NEW APPROACHES OF INVERSE SOFT ROUGH SETS AND THEIR APPLICATIONS IN A DECISION MAKING PROBLEM

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Abstract. We present inverse soft rough sets by using inverse soft sets and soft rough sets. We study different approaches for inverse soft rough set and examine the relationships between them. We also discuss and explore the basic properties for these approaches. Moreover we develop an algorithm following these concepts and apply it to a decision-making problem to demonstrate the applicability of the proposed methods.

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1. Introduction

In recent days, mathematical modelling for an uncertain data has become an increasingly important topic in various research fields. Therefore, many researchers have worked on mathematical modelling to describe uncertainty. Rough set theory [13], based on the equivalence relations to describe uncertainty, was proposed by Pawlak in 1982. This theory was extended to covering based rough sets [15, 16]. Theories such as probability theory, fuzzy set theory [14], rough set theory [13] and interval mathematics theory [2, 3] are considered as a very successful tools to describe uncertainty. But each of them has its own inherent difficulties. In 1999, Molodtsov [11] introduced the new concept of soft sets to deal with the challenges of existing methods of uncertainty and established the fundamental results of this theory in solving many practical problems in economics, social science, medical science, etc. Studies on soft sets are progressing rapidly in recent years. In 2010, Feng et al. [5] described the concept of soft rough set using soft set and rough set. In [5, 7], basic properties of soft rough approximations were presented and supported by some illustrative

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examples. Moreover, Feng [6] gave an application of soft rough approximations in multicriteria group decision making problems.

In 2016, Cetkin, Aygunoglu and Aygun [4] initiated a new approach of inverse soft sets to find the optimal solution comparatively an easier and faster way than the existed algorithms.

In this paper, we initiate the inverse soft rough sets, which is an extension of inverse soft sets and soft rough sets, and present different approaches to this sets. We study different approaches for inverse soft rough set and examine the relationships between them. We also develop an algorithm to better determine the relationships between observed different approaches, and finally apply this algorithm to a decision making problem.

2. Preliminaries

First we recall some definitions and results to make this paper self contained.

Consider U be an initial universe, E be a set of parameters and P(U) denote the power set of U. Also let A and B are non-empty subsets of E.

Definition 2.1. [12] An information system is a pair $\wp = (U, A)$ of non-empty finite sets U and A, where U is a set of objects and A is a set of attributes; each attribute $a \in A$ is a function $a : U \to V_a$, where V_a is the set of values of attribute a.

Let U be a non-empty finite universe and R be an equivalence relation on U. The pair (U, R) is called a Pawlak approximation space. The equivalence relation R is often called an indiscernibility relation and related to an information system. Specifically, if $\wp = (U, A)$ is an information system and $B \subseteq A$, then an indiscernibility relation $R = I(B)$ can be defined by

$$(x, y) \in I(B) \iff a(x) = a(y), \quad \forall a \in B,$$

where $x, y \in U$, and $a(x)$ denotes the value of attribute a for object x. For any $X \subseteq U$ and using the indiscernibility relation R, one can define the following two operations

$$R_*X = \{x \in U : [x]_R \subseteq X\}, \quad R^*X = \{x \in U : [x]_R \cap X \neq 0\},$$

assigning to every subset $X \subseteq U$ two sets R^*X and R^*X called the $R-$lower and the $R-$upper approximation of X, respectively. Moreover, the sets

$$Pos_RX = R_*X, \quad Neg_RX = U - R^*X, \quad Bnd_RX = R^*X - R_*X$$

are referred to as the $R-$positive, the $R-$negative and the $R-$boundary region of X, respectively. If the $R-$boundary region of X is empty, i.e., $R_*X = R^*X$, then X is crisp (or exact) with respect to R. If $Bnd_RX \neq \emptyset$, then X is said to be rough (or inexact) with respect to R [12].

Note that sometimes the pair (R^*X, R_*X) is also referred to as the rough set of X with respect to R [9, 8].

Definition 2.2. [1, 10] A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \to P(U)$.
In other words, a soft set over U is a parameterized family of subsets of the universe U. For $e \in A$, $F(e)$ may be considered as the set of e-approximate elements of the soft set (F, A). It is worth noting that $F(e)$ may be an arbitrary: some of them may be empty, and some may have nonempty intersection [11]. The absence of any restrictions on the approximate description in soft set theory makes it very convenient and easily applicable in practice.

Although rough sets and soft sets are two different mathematical tools for modelling vagueness, there are some interesting connections between them. At first, we note that information systems and soft sets are closely related. Let $S = (F, A)$ be a soft set over U. If U and A are both non-empty finite sets, then S could induce an information system $\wp = (U, A)$ in a natural way. In fact, for any attribute $a \in A$, one can define a function $a : U \to V_a = \{0, 1\}$ by

$$a(x) = \begin{cases} 1, & \text{if } x \in F(a) \\ 0, & \text{otherwise} \end{cases}$$

Therefore every soft set may be considered as an information system.

Definition 2.3. [7] Let $S = (F, A)$ be a soft set over U. Then the pair $P = (U, S)$ is called a soft approximation space. For any $X \subseteq U$ and using the soft approximation space P, we define the following two operations

$$\underline{apr}_p(X) = \{u \in U : \exists a \in A, [u \in F(a) \subseteq X]\},$$

$$\overline{apr}_p(X) = \{u \in U : \exists a \in A, [u \in F(a) \cap X \neq \emptyset]\}$$

assigning to every subset $X \subseteq U$ two sets $\underline{apr}_p(X)$ and $\overline{apr}_p(X)$, which are called the soft $P-$lower approximation and the soft $P-$upper approximation of X, respectively. In general, we refer to $\underline{apr}_p(X)$ and $\overline{apr}_p(X)$ as soft rough approximations of X with respect to P. Moreover, the sets

$$Pos_p(X) = \underline{apr}_p(X),$$

$$Neg_p(X) = -\overline{apr}_p(X),$$

$$Bnd_p(X) = \overline{apr}_p(X) - \underline{apr}_p(X)$$

are called the soft $P-$positive region, the soft $P-$negative region and the soft $P-$boundary region of X, respectively. If $\underline{apr}_p(X) = \overline{apr}_p(X)$, X is said to be soft $P-$definable; otherwise X is called a soft $P-$rough set.

Proposition 2.4. [7] *Let $S = (F, A)$ be a soft set over U and $P = (U, S)$ be a soft approximation space. Then we have*

$$\underline{apr}_p(X) = \bigcup_{a \in A} \{F(a) : F(a) \subseteq X\},$$

and

$$\overline{apr}_p(X) = \bigcup_{a \in A} \{F(a) : F(a) \cap X \neq \emptyset\},$$

for all $X \subseteq U$.

Proposition 2.5. [7] *Let* $S = (F, A)$ *be a soft set over* U *and* $P = (U, S)$ *be a soft approximation space. Then for any* $X \subseteq U$, X *is soft* $P-$*definable if and only if* $\overline{apr}_p(X) \subseteq X$.

Definition 2.6. [7] Let $S = (F, A)$ be a soft set over U. If $\cup_{a \in A} F(a) = U$, then S is said to be a full soft set.

Definition 2.7. [4] A mapping $\Lambda : U \to P(E)$ is called an inverse soft set (for short, ISS) on U.
Note that an ISS can be seen as a collection of subsets of the parameter set E, i.e., $\Lambda = \{\Lambda(u)\}_{u \in U} \subseteq P(E)$.

For an ISS Λ on X, the subset $\Lambda(u)$ of E denotes the membership parameters of u to the ISS Λ and $E - \Lambda(u)$ denotes the non-membership parameters of u to the ISS Λ. The family of all ISSs on X is denoted by $ISS(X)$.

3. Inverse Soft Rough Approximations and Inverse soft Rough Sets

Throughout this paper, U and E refers to an initial universe and the set of all parameters for U, respectively. X is a parameterized family of subsets of the universe U. Also, A and B denote the subsets of E, otherwise specified.

Definition 3.1. Let $S = (F, X)$ be a inverse soft set over E. If $\cup_{x \in X} F(x) = E$, then S is said to be a full inverse soft set.

Definition 3.2. A full soft set $S = (F, X)$ over E is called a inverse covering soft set if $F(x) \neq 0$, $\forall x \in X$.

We indicate a inverse covering soft set with IC_S.

Definition 3.3. Let $S = (F, X)$ be a inverse covering soft set over E. Then the pair $IP = (E, IC_S)$ is called a inverse soft covering approximation space.

3.1. First Type of Inverse Soft Covering Based Rough Sets.

Definition 3.4. Let $S = (F, X)$ be a inverse soft covering set over E. Then the pair $IP = (E, S)$ is called a inverse soft covering approximation space. Based on the inverse soft covering approximation space IP, we define the following two operations

$$\underline{IP}_1(A) = \{e \in E : \exists x \in X, [e \in F(x) \subseteq A]\},$$
$$\overline{IP}_1(A) = \{e \in E : \exists x \in X, [e \in F(x) \cap A \neq \emptyset]\}$$

assigning to every subset $A \subseteq E$ two sets $\underline{IP}_1(A)$ and $\overline{IP}_1(A)$, which are called the inverse soft covering $IP-$lower approximation and the inverse soft covering

$IP-$upper approximation of A, respectively. In general, we refer to $\underline{IP}_1(A)$ and $\overline{IP}_1(A)$ as inverse soft covering based rough approximations of A with respect to IP. Moreover, the sets

$$IPos_{IP}(A) = \underline{IP}_1(A),$$
$$INeg_{IP}(A) = -\overline{IP}_1(A),$$
$$IBnd_{IP}(A) = \overline{IP}_1(A) - \underline{IP}_1(A)$$

are called the inverse soft covering $IP-$positive region, the inverse soft covering $IP-$negative region and the inverse soft covering $IP-$boundary region of A, respectively. If $\underline{IP}_1(A) = \overline{IP}_1(A)$, A is said to be inverse soft covering based $IP-$definable; otherwise A is called a inverse soft covering based $IP-$rough set.

Example 3.5. Let $U = \{u_1, u_2, u_3, u_4\}$, $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ and $X = \{u_1, u_3, u_4\} \subseteq U$. Let $S = (F, X)$ be a inverse covering soft set over E given by Table 1 and the inverse soft covering approximation space $IP = (E, S)$.

Table 1. Tabular representation of the inverse covering soft set S.

S	e_1	e_2	e_3	e_4	e_5	e_6	e_7
u_1	1	0	1	0	0	0	0
u_3	0	0	1	0	0	0	1
u_4	0	1	0	0	1	0	1

For $A = \{e_1, e_3, e_4, e_6\} \subseteq E$, $\underline{IP}_1(A) = \{e_1, e_3\}$, and $\overline{IP}_1(A) = \{e_1, e_3, e_7\}$. Thus $\underline{IP}_1(A) \neq \overline{IP}_1(A)$ and A is a inverse soft covering based $IP-$rough set. Moreover, it is easy to see that $IPos_{IP}(X) = \{e_1, e_3\}$, $INeg_{IP}(X) = \{e_2, e_4, e_5, e_6\}$ and $IBnd_{IP}(X) = \{e_7\}$. On the other hand, one can consider $A_1 = \{e_6\} \subseteq E$. Since $\underline{IP}_1(A_1) = \emptyset = \overline{IP}_1(A_1)$, by definition, A_1 is a inverse soft covering based $IP-$definable set.

Proposition 3.6. *Let $S = (F, X)$ be a inverse covering soft set over E and $IP = (E, S)$ a inverse soft covering approximation space. Then we have*

$$\underline{IP}_1(A) = \bigcup_{x \in X} \{F(x) : F(x) \subseteq A\}$$

and

$$\overline{IP}_1(A) = \bigcup_{x \in X} \{F(x) : F(x) \cap A \neq \emptyset\},$$

for all $A \subseteq E$.

Proof. This is easily obtained from the definition of inverse soft covering based rough approximations. □

Proposition 3.7. *Let $S = (F, X)$ be a inverse soft covering set over E and $IP = (E, IC_S)$ a inverse soft covering approximation space. Then for any $A \subseteq E$, A is inverse soft covering based definable if and only if $\overline{IP}_1(A) \subseteq A$.*

Proof. Note first that if A is inverse soft covering based definable, then $\underline{IP}_1(A) = \overline{IP}_1(A)$, and so $\underline{IP}_1(A) = \overline{IP}_1(A) \subseteq A$.
Conversely, suppose that $\overline{IP}_1(A) \subseteq A$, for $A \subseteq E$. To show that A is inverse soft covering based definable, we only need to prove that $\overline{IP}_1(A) \subseteq \underline{IP}_1(A)$, since the reverse inequality is trivial. Let $e \in \overline{IP}_1(A)$. Then $e \in F(x)$ and $F(x) \cap A \neq \emptyset$, for some $x \in X$. It follows that $e \in F(x) \subseteq \overline{IP}_1(A) \subseteq A$. Hence $e \in \underline{IP}_1(A)$, and so $\overline{IP}_1(A) \subseteq \underline{IP}_1(A)$ as required. □

Proposition 3.8. *Let $S = (F, X)$ be a inverse soft covering set over E, $IP = (E, IC_S)$ a inverse soft covering approximation space and $A, B \subseteq E$. One can verify that first type of inverse soft covering lower and upper approximations satisfy the following properties:*
(1) $\underline{IP}_1(\emptyset) = \overline{IP}_1(\emptyset) = \emptyset$;
(2) $\underline{IP}_1(E) = \overline{IP}_1(E) = \bigcup_{x\in X} f(x) = E$;
(3) $\overline{IP}_1(A \cup B) = \overline{IP}_1(A) \cup \overline{IP}_1(B)$;
(4) $\overline{IP}_1(A \cap B) \subseteq \overline{IP}_1(A) \cap \overline{IP}_1(B)$;
(5) $\underline{IP}_1(A \cap B) \subseteq \underline{IP}_1(A) \cap \underline{IP}_1(B)$;
(6) $\underline{IP}_1(A \cup B) \supseteq \underline{IP}_1(A) \cup \underline{IP}_1(B)$;
(7) $A \subseteq B \Rightarrow \overline{IP}_1(A) \subseteq \overline{IP}_1(B)$;
(8) $A \subseteq B \Rightarrow \underline{IP}_1(A) \subseteq \underline{IP}_1(B)$.

Proof. From Definition 3.4, we can easily prove (1) and (2).
(3) From Definition 3.4, we have $\overline{IP}_1(A) = \{e \in E : \exists x \in X, [e \in F(x) \cap A \neq \emptyset]\}$ and $\overline{IP}_1(B) = \{e \in E : \exists x \in X, [e \in F(x) \cap B \neq \emptyset]\}$. So $\overline{IP}_1(A) \cup \overline{IP}_1(B) = \big(\{e \in E : \exists x \in X, [e \in F(x) \cap A \neq \emptyset]\}\big) \cup \big(\{e \in E : \exists x \in X, [e \in F(x) \cap B \neq \emptyset]\}\big) = \{e \in E : \exists x \in X, [e \in F(x) \cap (A \cup B) \neq \emptyset]\} = \overline{IP}_1(A \cup B)$.
(4) From Definition 3.4, we have $\underline{IP}_1(A) = \{e \in E : \exists x \in X, [e \in F(x) \subseteq A]\}$ and $\underline{IP}_1(B) = \{e \in E : \exists x \in X, [e \in F(x) \subseteq B]\}$. So $\underline{IP}_1(A) \cap \underline{IP}_1(B) = \big(\{e \in E : \exists x \in X, [e \in F(x) \subseteq A]\}\big) \cap \big(\{e \in E : \exists x \in X, [e \in F(x) \subseteq B]\}\big) \supseteq \{e \in E : \exists x \in X, [e \in F(x) \subseteq (A \cap B)]\} = \overline{IP}_1(A \cap B)$.
The proofs of (5) and (6) are obtained similar to (4).
(7) By the definition of $\overline{IP}_1(A) = \bigcup_{x\in X}\{F(x) : F(x) \cap A \neq \emptyset\}$, $a_1, a_2, ...a_m \in A$, and $F(x_1), F(x_2), ..., F(x_m) \in S = (F, X)$ such that $a_i \in F(x) \cap A$, $1 \leq i \leq m$, $\overline{IP}_1(A)$ is expressed as $\overline{IP}_1(A) = F(x_1) \cup F(x_2) \cup ... \cup F(x_m)$. It is obvious that $F(x_i) \subseteq \overline{IP}_1(A)$. Since $A \subseteq B$ and $a_1, a_2, ...a_m \in A$, we obtain $a_i \in B$. For $1 \leq i \leq m$, $F(x_i) \subseteq \overline{IP}_1(B)$. Therefore, $\overline{IP}_1(A) \subseteq \overline{IP}_1(B)$.
The proof of (8) is obtained similar to (7). □

Following example show that reverse inclusions of (4), (5) and (6) in Proposition 3.8 do not hold.

Example 3.9. Consider Example 3.5. For $B = \{e_2, e_3, e_7\}$, we have

$$\underline{IP}_1(B) = \bigcup_{x\in X} \{F(x) : F(x) \subseteq B\} = \{e_3, e_7\}.$$

For $A \cap B = \{e_3\}$, we have

$$\underline{IP}_1(A \cap B) = \bigcup_{x \in X} \{F(x) : F(x) \subseteq A \cap B\} = \emptyset.$$

Here, let us check (5)

$$\emptyset = \underline{IP}_1(A \cap B) \not\supseteq \underline{IP}_1(A) \cap \underline{IP}_1(B) = \{e_3\}.$$

For $C = \{e_1, e_2, e_5, e_7\}$, we have

$$\overline{IP}_1(C) = \bigcup_{x \in X} \{F(x) : F(x) \cap (A \cap C) \neq \emptyset\} = \{e_1, e_2, e_3, e_5, e_7\}.$$

For $A \cap C = \{e_1\}$, we have

$$\overline{IP}_1(A \cap C) = \bigcup_{x \in X} \{F(x) : F(x) \cap (A \cap C) \neq \emptyset\} = \{e_1, e_3\}.$$

Here, let us check (4)

$$\{e_1, e_3\} = \overline{IP}_1(A \cap C) \not\supseteq \overline{IP}_1(A) \cap \overline{IP}_1(C) = \{e_1, e_3, e_7\},$$

For $D = \{e_3, e_7\}$, we have

$$\underline{IP}_1(D) = \bigcup_{x \in X} \{F(x) : F(x) \subseteq D\} = \{e_3, e_7\}.$$

For $A \cup D = \{e_3\}$, we have

$$\underline{IP}_1(A \cup D) = \bigcup_{x \in X} \{F(x) : F(x) \subseteq A \cup D\} = \{e_1, e_2, e_3, e_5, e_7\}.$$

Here, let us check (6)

$$\{e_1, e_2, e_3, e_5, e_7\} = \underline{IP}_1(A \cup B) \not\subseteq \underline{IP}_1(A) \cup \underline{IP}_1(B) = \{e_1, e_3, e_7\}.$$

Theorem 3.10. *Let $S = (F, X)$ be a inverse soft covering set over E and $IP = (E, S)$ a inverse soft covering approximation space. Then the following conditions are equivalent:*
(1) *S is a full inverse soft covering set.*
(2) *$\underline{IP}_1(E) = E$;*
(3) *$\overline{IP}_1(E) = E$;*
(4) *$A \subseteq \overline{IP}_1(A)$ for all $A \subseteq E$;*
(5) *$\overline{IP}_1(\{e\}) \neq \emptyset$ for all $e \in E$.*

Proof. (1) $\Rightarrow$ (2) From Definition 3.4, we have $\underline{IP}_1(E) = \bigcup_{x \in X} \{F(x) : F(x) \subseteq E\}$. Since S is a full inverse soft covering set, $F(x) \neq \emptyset$ for all $x \in X$ implemented. So $\underline{IP}_1(E) = E$.
(2) $\Rightarrow$ (3) It is easily seen from Definition 3.4.
(3) $\Rightarrow$ (4) $\overline{IP}_1(A) \subseteq \overline{IP}_1(E)$ is implemented because of Proposition 3.8(7). Therefore, $A \subseteq \bigcup_{x \in X} \{F(x) : F(x) \cap A \neq \emptyset\} \subseteq \overline{IP}_1(E) = E$ is obtained from Definition 3.4.
(4) $\Rightarrow$ (5) Take $A = \{e\}$. From Definition 3.4, we have $\{e\} \subseteq \overline{IP}_1(\{e\}) =$

$\bigcup_{x\in X}\{F(x) : F(x)\cap\{e\}\neq\emptyset\}$. So $\overline{IP}_1(\{e\})\neq\emptyset$, for all $e\in E$.
(5) $\Rightarrow$ (1) The proof is obvious. $\square$

3.2. Second Type of Inverse Soft Covering Based Rough Sets.

Definition 3.11. Let $IP=(E,IC_S)$ be a inverse soft covering approximation space and $e\in E$. Then the soft minimal description of e is defined as follows:

$$\begin{aligned} & IMd_{IP}(a) \\ = \quad & \{F(x) : x\in X\wedge a\in F(x)\wedge(\forall u\in X\wedge a\in F(u)\subseteq F(x)\Rightarrow F(x)=F(u))\}. \end{aligned}$$

Definition 3.12. Let $IP=(E,IC_S)$ be a soft covering approximation space. For a set $A\subseteq E$, soft covering lower and upper approximations are, respectively, defined as

$$\underline{IP}_2(A)=\bigcup_{x\in X}\{F(x) : F(x)\subseteq A\},$$
$$\overline{IP}_2(A)=\bigcup\{IMd_{IP}(a) : a\in A\}.$$

In addition,

$$IPos_{IP}(A)=\underline{IP}_2(A),$$
$$INeg_{IP}(A)=E-\overline{IP}_2(A),$$
$$IBnd_{IP}(A)=\overline{IP}_2(A)-\underline{IP}_2(A)$$

are called the soft covering positive, negative, and boundary regions of A, respectively.

Definition 3.13. Let $IP=(E,IC_S)$ be a soft covering approximation space. A subset $A\subseteq E$ is called inverse soft covering based definable if $\overline{IP}_2(A)=\underline{IP}_2(A)$; in the opposite case, that is, if $\overline{IP}_2(A)\neq\underline{IP}_2(A)$, A is said to be inverse soft covering based rough set.

Example 3.14. Let $IP=(E,IC_S)$ be a soft covering approximation space, where $U=\{u_1,u_2,u_3,u_4,u_5,u_6\}$, $A=\{e_1,e_2,e_3,e_4,e_5,e_6\}\subseteq E$, $F(u_1)=\{e_3,e_5\}$, $F(u_2)=\{e_2,e_3,e_5,e_6\}$, $F(u_3)=\{e_2,e_6\}$, $F(u_4)=\{e_1,e_2,e_4,e_6\}$, $F(u_5)=\{e_1,e_3,e_4,e_5\}$ and $F(u_6)=\{e_1,e_4,e_6\}$. For $A_1=\{e_2,e_3,e_5\}$, we have

$$\underline{IP}_2(A_1)=\{F(u) : u\in X\wedge F(u)\subseteq\ A_1\}=\{e_3,e_5\}.$$
$$\overline{IP}_2(A_1)=IP_*(A_1)\cup\{IMd_p(a) : a\in A_1-\underline{IP}_2(A_1)\}=\{e_2,e_3,e_5,e_6\}.$$

Thus, $\underline{IP}_2(A_1)\neq\overline{IP}_2(A_1)$ and A_1 is a inverse soft covering based rough set. For $A_2=\{e_3,e_5\}$, we have

$$\underline{IP}_2(A_2)=\{F(u) : u\in X\wedge F(u)\subseteq\ A_2\}=\{e_3,e_5\},$$
$$\overline{IP}_2(A_2)=IP_-(A_2)\cup\{IMd_p(a) : a\in A_2-\underline{IP}_2(A_2)\}=\{e_3,e_5\}.$$

Thus, $\underline{IP}_2(A_2)=\overline{IP}_2(A_2)$ and A_2 is a inverse soft covering based definable set.

Proposition 3.15. *Let $S = (F, X)$ be a inverse soft covering set over E, $IP = (E, IC_S)$ a inverse soft covering approximation space and $A, B \subseteq E$. Then the second type of inverse soft covering lower and upper approximations have the following properties:*
(1) $\underline{IP}_2(\emptyset) = \overline{IP}_2(\emptyset) = \emptyset$;
(2) $\underline{IP}_2(E) = \overline{IP}_2(E) = \bigcup_{x \in X} f(x) = E$;
(3) $\underline{IP}_2(A) \subseteq A \subseteq \overline{IP}_2(A)$ *for all* $A \subseteq E$;
(4) $\overline{IP}_2(A \cup B) = \overline{IP}_2(A) \cup \overline{IP}_2(B)$;
(5) $\overline{IP}_2(A \cap B) \subseteq \overline{IP}_2(A) \cap \overline{IP}_2(B)$;
(6) $\underline{IP}_2(A \cap B) \subseteq \underline{IP}_2(A) \cap \underline{IP}_2(B)$;
(7) $\underline{IP}_2(A \cup B) \supseteq \underline{IP}_2(A) \cup \underline{IP}_2(B)$;
(8) $A \subseteq B \Rightarrow \overline{IP}_2(A) \subseteq \overline{IP}_2(B)$;
(9) $A \subseteq B \Rightarrow \underline{IP}_2(A) \subseteq \underline{IP}_2(B)$;
(10) $\underline{IP}_2(\{e\}) \neq \emptyset$ *for all* $e \in E$.

Proof. From Definition 3.12, we can easily prove (1), (2) and (3).
(4) From Definition 3.12, we have $\overline{IP}_2(A) = \cup\{IMd_{IP}(a) : a \in A\}$ and $\overline{IP}_2(B) = \cup\{IMd_{IP}(a) : a \in B\}$. So $\overline{IP}_2(A) \cup \overline{IP}_2(B) = \big(\cup \{IMd_{IP}(a) : a \in A\}\big) \cup \big(\cup \{IMd_{IP}(a) : a \in B\}\big) = \cup\{IMd_{IP}(a) : a \in A \cup B\} = \overline{IP}_2(A \cup B)$.
(5) From Definition 3.12, we have $\underline{IP}_2(A) = \cup_{x \in X}\{F(x) : F(x) \subseteq A\}$ and $\underline{IP}_2(B) = \cup_{x \in X}\{F(x) : F(x) \subseteq B\}$. So $\underline{IP}_2(A) \cup \underline{IP}_2(B) = \big(\cup_{x \in X} \{F(x) : F(x) \subseteq A\}\big) \cap \big(\cup_{x \in X} \{F(x) : F(x) \subseteq B\}\big) \supseteq \cup_{x \in X}\{F(x) : F(x) \subseteq (A \cap B)\} = \underline{IP}_2(A \cap B)$.
The proofs of (6) and (7) are obtained similar to (4) and (5).
(8) By the definition of $\overline{IP}_2(A) = \cup\{IMd_{IP}(a) : a \in A\}$, $a_1, a_2, ...a_m \in A$, and $F(x_1), F(x_2), ..., F(x_m) \in S = (F, X)$ such that $F(a_i) \in IMd_{IP}(a_i)$, $1 \le i \le m$, $\overline{IP}_2(A)$ is expressed as $\overline{IP}_2(A) = F(x_1) \cup F(x_2) \cup ... \cup F(x_m)$. It is obvious that $F(x_i) \subseteq \overline{IP}_2(A)$. Since $A \subseteq B$ and $a_1, a_2, ...a_m \in A$, we obtain $a_i \in B$. For $1 \le i \le m$, $F(x_i) \subseteq \overline{IP}_2(B)$. Therefore, $\overline{IP}_2(A) \subseteq \overline{IP}_2(B)$.
The proof of (9) is obtained similar to (8).
The proof of (10) is a direct consequence of Definition 3.12. □

Now, we give examples to show that reverse inclusions of (5), (6) and (7) in Proposition 3.15 do not hold.

Example 3.16. Consider Example 3.14. For $B_1 = \{e_1, e_3, e_4, e_5\}$, we have

$$\overline{IP}_2(B_1) = \underline{IP}_2(B_1) \cup \{Md_p(a) : a \in B_1 - \underline{IP}_2(B_1)\} = \{e_1, e_3, e_4, e_5, e_6\}.$$

We have calculated the approximations of $A_1 \cap B_1 = A_2 = \{e_3, e_5\}$ in Example 3.14. Here, let us check (5)

$$\{e_3, e_5\} = \overline{IP}_2(A_1 \cap B_1) \not\supseteq \overline{IP}_2(A_1) \cap \overline{IP}_2(B_1) = \{e_3, e_5, e_6\}.$$

For $B_2 = \{e_1, e_4, e_6\}$ and $B_3 = \{e_2, e_6\}$, we have

$$\underline{IP}_2(B_2) = \{F(u) : u \in U \wedge F(u) \subseteq\ B_2\} = \{e_1, e_4, e_6\},$$
$$\underline{IP}_2(B_3) = \{F(u) : u \in U \wedge F(u) \subseteq\ B_3\} = \{e_2, e_6\}.$$

For $B_2 \cap B_3 = \{e_6\}$, we have

$$\underline{IP}_2(B_2 \cap B_3) = \{F(u) : u \in U \wedge F(u) \subseteq \ B_2 \cap B_3\} = \emptyset.$$

Here, let us check (6)

$$\emptyset = \underline{IP}_2(B_2 \cap B_3) \not\supseteq \underline{IP}_2(B_2) \cap \underline{IP}_2(B_3) = \{e_6\}.$$

For $B_4 = \{e_1, e_2, e_4\}$, we have

$$\underline{IP}_2(B_4) = \{F(u) : u \in U \wedge F(u) \subseteq \ B_4\} = \emptyset.$$

For $A_1 \cup B_4 = \{e_1, e_2, e_4\}$, we have

$$\underline{IP}_2(A_1 \cup B_4) = \{F(u) : u \in U \wedge F(u) \subseteq \ A_1 \cup B_4\} = \{e_1, e_3, e_4, e_5\}.$$

Here, let us check (7)

$$\{e_1, e_3, e_4, e_5\} = \underline{IP}_2(A_1 \cup B_4) \not\subseteq \underline{IP}_2(A_1) \cup \underline{IP}_2(B_4) = \{e_3, e_5\}.$$

3.3. Third Type of Inverse Soft Covering Based Rough Sets.

Definition 3.17. Let $IP = (E, IC_S)$ be a inverse soft covering approximation space. For a set $A \subseteq E$, the inverse soft covering lower and upper approximations are respectively defined as

$$\underline{IP}_3(A) = \bigcup\{F(u) : u \in X \wedge F(u) \subseteq \ A\},$$

$$\overline{IP}_3(A) = \underline{IP}_3(A) \cup \{IMd_{IP}(a) : a \in A - \underline{IP}_3(A)\}.$$

In addition,

$$IPos_{IP}(A) = \underline{IP}_3(A),$$

$$INeg_{IP}(A) = E - \overline{IP}_3(A),$$

$$IBnd_{IP}(A) = \overline{IP}_3(A) - \underline{IP}_3(A)$$

are called the soft covering positive, negative, and boundary regions of A, respectively.

Definition 3.18. Let $IP = (E, IC_S)$ be a inverse soft covering approximation space. A subset $A \subseteq E$ is called inverse soft covering based definable if $\underline{IP}_3(A) = \overline{IP}_3(A)$; in the opposite case, i.e., if $\underline{IP}_3(A) \neq \overline{IP}_3(A)$, A is said to be a inverse soft covering based rough set.

Example 3.19. Let $IP = (E, IC_S)$ be a soft covering approximation space, where $U = \{u_1, u_2, u_3, u_4, u_5\}$, $A = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\} \subseteq E$, $F(u_1) = \{e_4, e_5\}$, $F(u_2) = \{e_3, e_5, e_7\}$, $F(u_3) = \{e_2, e_6\}$, $F(u_4) = \{e_1\}$ and $F(u_5) = \{e_1, e_8\}$. For $A_1 = \{e_4, e_5, e_7\}$, we have

$$\underline{IP}_3(A) = \{F(u) : u \in X \wedge F(u) \subseteq \ A\} = \{e_4, e_5\},$$

$$\overline{IP}_3(A) = \underline{IP}_3(A) \cup \{IMd_{IP}(a) : a \in A - \underline{IP}_3(A)\} = \{e_3, e_4, e_5, e_7\}.$$

Thus, $\underline{IP}_3(A) \neq \overline{IP}_3(A)$ and A_1 is a inverse soft covering based rough set. For $A_2 = \{e_1, e_2, e_6\}$, we have

$$\underline{IP}_3(A) = \{F(u) : u \in X \wedge F(u) \subseteq A\} = \{e_1, e_2, e_6\},$$

$$\overline{IP}_3(A) = \underline{IP}_3(A) \cup \{IMd_{IP}(a) : a \in A - \underline{IP}_3(A)\} = \{e_1, e_2, e_6\}.$$

Thus, $\underline{IP}_3(A) = \overline{IP}_3(A)$ and A_2 is a inverse soft covering based definable set.

Proposition 3.20. *Let $S = (F, X)$ be a inverse soft covering set over E, $IP = (E, IC_S)$ be a inverse soft covering approximation space and $A, B \in E$. Then the third type of inverse soft covering lower and upper approximations have the following properties:*
(1) $\underline{IP}_3(E) = \overline{IP}_3(E) = E$;
(2) $\underline{IP}_3(\emptyset) = \overline{IP}_3(\emptyset) = \emptyset$;
(3) $\underline{IP}_3(A) \subseteq A \subseteq \overline{IP}_3(A)$,
(4) $\forall u \in X, \underline{IP}_3(F(u)) = F(u)$;
(5) $\forall u \in X, \overline{IP}_3(F(u)) = F(u)$.

Proof. From Definition 3.17, we can easily prove (1), (2) and (3).
(4) From Definition 3.17, we have $\underline{IP}_3(F(u)) = \bigcup_{x \in X}\{F(x) : F(x) \subseteq F(u)\}$. The proof is clear, since $F(u) \subseteq F(u)$.
The proof of (5) is obtained similar to (4). □

4. Decision making problem based on Inverse soft rough sets

Algorithm.
Step 1: Choose a inverse (covering) soft set on E, for $X \subseteq U$.
Step 2: Let us determine how many elements in the parameter set.
Step 3: Experiments are made for the most appropriate parameter set.
Step 4: Choose the parameter subset $A \subseteq E$, where $IBnd_{IP}(A)$ is the minimal comparatively in all may be considered as the optimal choice to parameter set most qualified for the evaluation.

Example 4.1. Assume that a company wants to fill a position. There are 10 candidates for the position. The decision-making group is asked to indicate the most appropriate criteria for the applicants. Ten of the most appropriate evaluation criteria are selected. The three most appropriate alternatives for these candidates will be determined.
Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}\}$,
$E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}\}$ and $X = \{u_2, u_3, u_6, u_8, u_9\} \subseteq U$. For $i = 1, 2, ..., 10$, the parameters e_i stand for "experience", "computer knowledge", "foreign language knowledge", "effective speech", "working discipline", "training", "higher education", "young age", "marital status" and "good health", respectively. Let $T = (F, X)$ be a nverse covering soft set over E given by Table 2 and the inverse soft covering approximation space $IP = (E, IC_T)$.

Table 2. Tabular representation of the inverse covering soft set T.

T	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}
u_2	0	0	0	1	0	1	0	1	0	0
u_3	1	0	1	0	0	0	0	0	0	0
u_6	0	0	0	0	1	0	0	0	0	1
u_8	0	1	0	0	0	0	0	0	1	0
u_9	1	0	0	0	0	0	1	0	0	0

Let us use the upper and lower approximations defined for the first type of inverse soft covering based rough set model.

For this, we use the inverse soft covering approximation space we gave in Definition 3.4 and Proposition 3.6.
For $A_1 = \{e_1, e_5, e_7\} \subseteq E$, $\underline{IP}_1(A_1) = \{e_1, e_7\}$ and $\overline{IP}_1(A_1) = \{e_1, e_3, e_5, e_7, e_{10}\}$. Thus $\underline{IP}_1(A_1) \neq \overline{IP}_1(A_1)$ and A_1 is a inverse soft covering based $IP-$rough set. Moreover, it is easy to see that $IBnd_{IP}(A_1) = \{e_3, e_5, e_{10}\}$.

For $A_2 = \{e_1, e_9, e_{10}\} \subseteq E$, $\underline{IP}_1(A_2) = \emptyset$ and $\overline{IP}_1(A_2) = \{e_1, e_2, e_3, e_5, e_7, e_9, e_{10}\}$. Thus $\underline{IP}_1(A_2) \neq \overline{IP}_1(A_2)$ and A_2 is a inverse soft covering based $IP-$rough set. Moreover, it is easy to see that $IBnd_{IP}(A_2) = \{e_1, e_2, e_3, e_5, e_7, e_9, e_{10}\}$.

For $A_3 = \{e_1, e_3, e_4\} \subseteq E$, $\underline{IP}_1(A_3) = \{e_1, e_3\}$ and $\overline{IP}_1(A_3) = \{e_1, e_3, e_4, e_6, e_7, e_8\}$. Thus $\underline{IP}_1(A_3) \neq \overline{IP}_1(A_3)$ and A_3 is a inverse soft covering based $IP-$rough set. Moreover, it is easy to see that $IBnd_{IP}(A_3) = \{e_6, e_7, e_8\}$.

For $A_4 = \{e_2, e_5, e_{10}\} \subseteq E$, $\underline{IP}_1(A_4) = \{e_2, e_5\}$ and $\overline{IP}_1(A_4) = \{e_2, e_5, e_9, e_{10}\}$. Thus $\underline{IP}_1(A_4) \neq \overline{IP}_1(A_4)$ and A_4 is a inverse soft covering based $IP-$rough set. Moreover, it is easy to see that $IBnd_{IP}(A_4) = \{e_9, e_{10}\}$.

For $A_5 = \{e_1, e_3, e_7\} \subseteq E$, $\underline{IP}_1(A_5) = \{e_1, e_3, e_7\}$ and $\overline{IP}_1(A_5) = \{e_1, e_3, e_7\}$. Thus $\underline{IP}_1(A_5) = \overline{IP}_1(A_5)$ and A_6 is a inverse soft covering based $IP-$definable set. Moreover, it is easy to see that $IBnd_{IP}(A_5) = \emptyset$.

According to the above parameter sets, the most appropriate set are $A_5 > A_4 > A_1 \equiv A_3 > A_2$, respectively.

Let us use the upper and lower approximations defined for the second type of inverse soft covering based rough set model.

For this, we use the inverse soft covering approximation space we gave in Definition 3.11 and Definition 3.12.
For $A_1 = \{e_1, e_5, e_7\} \subseteq E$, $\underline{IP}_2(A_1) = \{e_1, e_7\}$ and $\overline{IP}_2(A_1) = \{e_1, e_5, e_7, e_{10}\}$. Thus $\underline{IP}_2(A_1) \neq \overline{IP}_2(A_1)$ and A_1 is a inverse soft covering based $IP-$rough set. Moreover, it is easy to see that $IBnd_{IP}(A_1) = \{e_5, e_{10}\}$.

For $A_2 = \{e_1, e_9, e_{10}\} \subseteq E$, $\underline{IP}_2(A_2) = \emptyset$ and $\overline{IP}_2(A_2) = \{e_2, e_5, e_9, e_{10}\}$. Thus $\underline{IP}_2(A_2) \neq \overline{IP}_2(A_2)$ and A_2 is a inverse soft covering based $IP-$rough set. Moreover, it is easy to see that $IBnd_{IP}(A_2) = \{e_2, e_5, e_9, e_{10}\}$.

For $A_3 = \{e_1, e_3, e_4\} \subseteq E$, $\underline{IP}_2(A_3) = \{e_1, e_3\}$ and $\overline{IP}_2(A_3) = \{e_1, e_3, e_4, e_6, e_8\}$. Thus $\underline{IP}_2(A_3) \neq \overline{IP}_2(A_3)$ and A_3 is a inverse soft covering based $IP-$rough set. Moreover, it is easy to see that $IBnd_{IP}(A_3) = \{e_6, e_8\}$.

For $A_4 = \{e_2, e_5, e_{10}\} \subseteq E$, $\underline{IP}_2(A_4) = \{e_2, e_5\}$ and $\overline{IP}_2(A_4) = \{e_2, e_5, e_9, e_{10}\}$. Thus $\underline{IP}_2(A_4) \neq \overline{IP}_2(A_4)$ and A_4 is a inverse soft covering based $IP-$rough set. Moreover, it is easy to see that $IBnd_{IP}(A_4) = \{e_9, e_{10}\}$.

For $A_5 = \{e_1, e_3, e_7\} \subseteq E$, $\underline{IP}_2(A_5) = \{e_1, e_3, e_7\}$ and $\overline{IP}_2(A_5) = \{e_1, e_3, e_7\}$. Thus $\underline{IP}_2(A_5) = \overline{IP}_2(A_5)$ and A_6 is a inverse soft covering based $IP-$definable set.Moreover, it is easy to see that $IBnd_{IP}(A_5) = \emptyset$.

According to the above parameter sets, the most appropriate set are $A_5 > A_4 \equiv A_1 \equiv A_3 > A_2$, respectively.

Let us use the upper and lower approximations defined for the third type of inverse soft covering based rough set model.

Finally, we use the inverse soft covering approximation space we gave in Definition 3.17.
For $A_1 = \{e_1, e_5, e_7\} \subseteq E$, $\underline{IP}_3(A_1) = \{e_1, e_7\}$ and $\overline{IP}_3(A_1) = \{e_1, e_5, e_7, e_{10}\}$. Thus $\underline{IP}_3(A_1) \neq \overline{IP}_3(A_1)$ and A_1 is a inverse soft covering based $IP-$rough set. Moreover, it is easy to see that $IBnd_{IP}(A_1) = \{e_5, e_{10}\}$.

For $A_2 = \{e_1, e_9, e_{10}\} \subseteq E$, $\underline{IP}_3(A_2) = \emptyset$ and $\overline{IP}_3(A_2) = \{e_2, e_5, e_9, e_{10}\}$. Thus $\underline{IP}_3(A_2) \neq \overline{IP}_3(A_2)$ and A_2 is a inverse soft covering based $IP-$rough set. Moreover, it is easy to see that $IBnd_{IP}(A_2) = \{e_2, e_5, e_9, e_{10}\}$.

For $A_3 = \{e_1, e_3, e_4\} \subseteq E$, $\underline{IP}_3(A_3) = \{e_1, e_3\}$ and $\overline{IP}_3(A_3) = \{e_1, e_3, e_4, e_6, e_8\}$. Thus $\underline{IP}_3(A_3) \neq \overline{IP}_3(A_3)$ and A_3 is a inverse soft covering based $IP-$rough set. Moreover, it is easy to see that $IBnd_{IP}(A_3) = \{e_6, e_8\}$.

For $A_4 = \{e_2, e_5, e_{10}\} \subseteq E$, $\underline{IP}_3(A_4) = \{e_2, e_5\}$ and $\overline{IP}_3(A_4) = \{e_2, e_5, e_{10}\}$. Thus $\underline{IP}_3(A_4) \neq \overline{IP}_3(A_4)$ and A_4 is a inverse soft covering based $IP-$rough set. Moreover, it is easy to see that $IBnd_{IP}(A_4) = \{e_{10}\}$.

For $A_5 = \{e_1, e_3, e_7\} \subseteq E$, $\underline{IP}_3(A_5) = \{e_1, e_3, e_7\}$ and $\overline{IP}_3(A_5) = \{e_1, e_3, e_7\}$. Thus $\underline{IP}_3(A_5) = \overline{IP}_3(A_5)$ and A_6 is a inverse soft covering based $IP-$definable set. Moreover, it is easy to see that $IBnd_{IP}(A_5) = \emptyset$.

According to the above parameter sets, the most appropriate set are $A_5 > A_4 > A_1 \equiv A_3 > A_2$, respectively.

We see that the results obtained from three different approximations are similar. Now, let us decide which type is best out of three types to use for the best evaluation on the approximations. If we compare all three types in the above example,we note that $IBnd_{IP_3}(A) \subseteq IBnd_{IP_2}(A) \subseteq IBnd_{IP_1}(A)$. Thus the decision is made that type three is best out of all three types.

Furthermore, we observe in Example 4.1 that the number of elements for the lower approximations are not changed while the number of elements can be reduced in the upper approximation expressions.

5. Results and discussion

In this study, we introduce inverse soft rough sets by using soft rough sets and inverse soft sets. In addition, some basic properties were studied by defining different approaches on the given soft set. Thus, the relationships between these approaches examined more easily. We propose a decision-making algorithm using inverse soft rough sets and an application of this algorithm is given in solving a decision-making problem. Finally, the results obtained in each of the three approaches given in this application are compared and discussed.

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