# NOTES ON THE PARAMETRIC POLY-TANGENT POLYNOMIALS ${ }^{\dagger}$ 

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#### Abstract

Recently, M. Masjed-Jamai et al. in ([6]-[7]) and Srivastava et al. in ([15]-[16]) considered the parametric type of the Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials. They proved some theorems and gave some identities and relations for these polynomials.

In this work, we define the parametric poly-tangent numbers and polynomials. We give some relations and identities for the parametric polytangent polynomials.

AMS Mathematics Subject Classification : 11B68, 11B83, 11S40, 11S80. Key words and phrases : Bernoulli polynomials and numbers, Tangent numbers and polynomials, Stirling numbers of the second kind, Polylogarithm functions, Poly-Bernoulli polynomials, Poly-Euler polynomials, Polytangent polynomials, Parametric Poly-Tangent numbers and polynomials.


## 1. Introduction

It is well known that Bernoulli polynomials $B_{n}(x)$, Tangent polynomials $T_{n}(x)$, Apostol-Bernoulli polynomials $B_{n}^{(\alpha)}(x ; \lambda)$, poly-Bernoulli polynomials $B_{n}^{(k)}(x)$ and poly-Tangent polynomials $T_{n}^{(k, \alpha)}(x)$ are fundamental importance in several parts of analysis and have applications in various other fields such as combinatorics, numerical analysis and so on.

The Bernoulli polynomials $B_{n}(x)$ are usually defined ([5], [13]) by means of the generating function

$$
\begin{equation*}
\sum_{n=0}^{\infty} B_{n}(x) \frac{t^{n}}{n!}=\frac{t}{e^{t}-1} e^{x t},|t|<2 \pi \tag{1}
\end{equation*}
$$

[^0]and the Bernoulli number $B_{n}:=B_{n}(0)$ by the corresponding equation
$$
\sum_{n=0}^{\infty} B_{n} \frac{t^{n}}{n!}=\frac{t}{e^{t}-1},|t|<2 \pi
$$

The $B_{n}$ are rational numbers. We have in particular

$$
\begin{gathered}
B_{0}=1, B_{1}=-\frac{1}{2}, B_{2}=\frac{1}{6} \text { and } B_{2 k+1}=0 \text { for } k=1,2, \cdots, \\
B_{0}(x)=1, B_{1}(x)=x-\frac{1}{2}, B_{2}(x)=x^{2}-x+\frac{1}{6}, \cdots
\end{gathered}
$$

The following properties are well known:

$$
\begin{gathered}
B_{n}(x)=\sum_{k=0}^{n}\binom{n}{k} B_{k} x^{n-k} \\
B_{n}^{\prime}(x)=n B_{n-1}(x)
\end{gathered}
$$

and

$$
\sum_{k=0}^{n-1}\binom{n}{k} B_{k}(x)=n x^{n-1}, n=2,3, \cdots
$$

These polynomials have been extensively studied by many mathematicians ([3]-[7], [13]-[16]).

The generalized Apostol-Bernoulli polynomials $B_{n}^{(\alpha)}(x ; \lambda)$ of order $\alpha$ were defined by Luo and Srivastava by means of the following generating functions (see [5], [13], [14])

$$
\begin{gather*}
\sum_{n=0}^{\infty} B_{n}^{(\alpha)}(x ; \lambda) \frac{t^{n}}{n!}=\left(\frac{t}{\lambda e^{x t}-1}\right)^{\alpha} e^{x t}  \tag{2}\\
\lambda \in \mathbb{C},|t|<2 \pi, \text { when } \lambda=1 ;|t|<|\log \lambda| \text { when } \lambda \neq 1
\end{gather*}
$$

The Stirling numbers of the second kind are defined by the following generating function

$$
\begin{equation*}
\sum_{n=m}^{\infty} S_{2}(n, m) \frac{t^{n}}{n!}=\frac{\left(e^{t}-1\right)^{m}}{m!} \tag{3}
\end{equation*}
$$

The tangent polynomials $T_{n}(x)$ in ([8], [11]) are defined by the generating function:

$$
\begin{equation*}
\sum_{n=0}^{\infty} T_{n}(x) \frac{t^{n}}{n!}=\left(\frac{2}{e^{2 t}+1}\right) e^{x t},|2 t|<\pi \tag{4}
\end{equation*}
$$

For $x=0$, we define the tangent numbers $T_{n}:=T_{n}(0)$.
$T_{n}(x)$ are polynomials of degree $n$. Here is the list of the first tangent's polynomials:

$$
T_{0}(x)=1, T_{1}(x)=x-1, T_{2}(x)=x^{2}-2 x, T_{3}(x)=x^{3}-3 x^{2}+2, \cdots
$$

The classical $k$-th polylogarithm function $L i_{k}(z)([1],[2],[3],[10])$ is defined the generating function

$$
L i_{k}(z)=\sum_{n=0}^{\infty} \frac{z^{n}}{n^{k}}, k \in \mathbb{Z}
$$

This function is convergent for $|z|<1$. When $k=1$, we get

$$
\begin{equation*}
L i_{1}(z)=-\log (1-z) \tag{5}
\end{equation*}
$$

Hamahata in [1] defined the poly-Euler polynomials by

$$
\begin{equation*}
\sum_{n=0}^{\infty} \mathcal{E}_{n}^{(k)}(x) \frac{t^{n}}{n!}=\frac{2 L i_{k}\left(1-e^{-t}\right)}{t\left(e^{t}+1\right)} e^{x t} \tag{6}
\end{equation*}
$$

For $k=1$, by (5), we write as

$$
\begin{gathered}
\sum_{n=0}^{\infty} \mathcal{E}_{n}^{(1)}(x) \frac{t^{n}}{n!}=\frac{2 L i_{1}\left(1-e^{-t}\right)}{t\left(e^{t}+1\right)} e^{x t} \\
=\frac{2 t e^{x t}}{t\left(e^{t}+1\right)}=\sum_{n=0}^{\infty} E_{n}(x) \frac{t^{n}}{n!}
\end{gathered}
$$

From here, we get $E_{n}^{(1)}(x)=E_{n}(x)$, where $E_{n}(x)$ are Euler polynomials.
Definition 1. A new definition the poly-tangent polynomials $T_{n}^{(k, \alpha)}(x)$ of order $\alpha$ are the following generating functions:

$$
\begin{equation*}
\sum_{n=0}^{\infty} T_{n}^{(k, \alpha)}(x) \frac{t^{n}}{n!}=\left(\frac{2 L i_{k}\left(1-e^{-t}\right)}{t\left(e^{2 t}+1\right)}\right)^{\alpha} e^{x t} \tag{7}
\end{equation*}
$$

For $\alpha=1$ in (7) and $k=1$ in (5), we get $T_{n}^{(1,1)}(x)=T_{n}(x)$. This definition is different Ryoo's definition [12].

Srivastava et al. ([15], [16]) defined two parametric kind of special cases of the Apostol-Bernoulli polynomials $\mathcal{B}_{n}^{(\alpha, c)}(x, y ; \lambda)$ and $\mathcal{B}_{n}^{(\alpha, s)}(x, y ; \lambda)$ of order $\alpha$;

$$
\begin{equation*}
\sum_{n=0}^{\infty} \mathcal{B}_{n}^{(\alpha, c)}(x, y ; \lambda) \frac{t^{n}}{n!}=\left(\frac{t}{\lambda e^{t}-1}\right)^{\alpha} e^{x t} \cos (y t) \tag{8}
\end{equation*}
$$

and

$$
\begin{equation*}
\sum_{n=0}^{\infty} \mathcal{B}_{n}^{(\alpha, s)}(x, y ; \lambda) \frac{t^{n}}{n!}=\left(\frac{t}{\lambda e^{t}-1}\right)^{\alpha} e^{x t} \sin (y t) \tag{9}
\end{equation*}
$$

For $x, y \in \mathbb{R}$, it was proved in ([7], [16]) that the Taylor expansions of the two functions $e^{x t} \cos (y t)$ and $e^{x t} \sin (y t)$ are given, respectively, by

$$
\begin{equation*}
e^{x t} \cos (y t)=\sum_{n=0}^{\infty} C_{n}(x, y) \frac{t^{n}}{n!} \tag{10}
\end{equation*}
$$

and

$$
\begin{equation*}
e^{x t} \sin (y t)=\sum_{n=0}^{\infty} S_{n}(x, y) \frac{t^{n}}{n!}, \tag{11}
\end{equation*}
$$

where

$$
C_{n}(x, y)=\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}(-1)^{k}\binom{n}{2 k} x^{n-2 k} y^{k}
$$

and

$$
S_{n}(x, y)=\sum_{k=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}(-1)^{k}\binom{n}{2 k+1} x^{n-2 k-1} y^{2 k+1} .
$$

By motions, we define the following generalized parametric cosine-poly-tangent polynomials ${ }_{C} T_{n}^{[k, \alpha]}(x, y)$ of order $\alpha$ and parametric sine-poly-tangent polynomials ${ }_{S} T_{n}^{[k, \alpha]}(x, y)$ of order $\alpha$ as, respectively;

$$
\begin{equation*}
\sum_{n=0}^{\infty}{ }_{C} T_{n}^{[k, \alpha]}(x, y) \frac{t^{n}}{n!}=\left(\frac{2 L i_{k}\left(1-e^{-t}\right)}{t\left(e^{2 t}+1\right)}\right)^{\alpha} e^{x t} \cos (y t) \tag{12}
\end{equation*}
$$

and

$$
\begin{equation*}
\sum_{n=0}^{\infty}{ }_{S} T_{n}^{[k, \alpha]}(x, y) \frac{t^{n}}{n!}=\left(\frac{2 L i_{k}\left(1-e^{-t}\right)}{t\left(e^{2 t}+1\right)}\right)^{\alpha} e^{x t} \sin (y t) . \tag{13}
\end{equation*}
$$

For $k=\alpha=1, y=0$ in relation (12) reduces the classical tangent polynomials.

## 2. Explicit Relations For The Parametric Tangent Numbers and Polynomials

In this section, we show some properties of the parametric tangent numbers and polynomials. By applying a similar technique, other ones can be determined. We can deduce some interesting relations between the parametric tangent numbers and polynomials, the Stirling numbers of the second kind.

By using (10) and (11) in (12) and (13), respectively, we have

$$
{ }_{C} T_{n}^{[k, \alpha]}(x, y)=\sum_{m=0}^{n}\binom{n}{m} T_{n-m}^{[k, \alpha]}(0,0) C_{m}(x, y)
$$

and

$$
S_{n}^{[k, \alpha]}(x, y)=\sum_{m=0}^{n}\binom{n}{m} T_{n-m}^{[k, \alpha]}(0,0) S_{m}(x, y) .
$$

Theorem 2.1. For $n \in \mathbb{Z}^{+}$, we have

$$
{ }_{C} T_{n}^{[k, \alpha]}(x, y)=\sum_{m=0}^{n}\binom{n}{m}{ }_{C} T_{n-m}^{[k, \alpha]}(0, y) x^{m}
$$

and

$$
{ }_{S} T_{n}^{[k, \alpha]}(x, y)=\sum_{m=0}^{n}\binom{n}{m}{ }_{S} T_{n-m}^{[k, \alpha]}(0, y) x^{m}
$$

Proof. By (12), we write

$$
\begin{gathered}
\sum_{n=0}^{\infty}{ }_{C} T_{n}^{[k, \alpha]}(x, y) \frac{t^{n}}{n!}=\left(\frac{2 L i_{k}\left(1-e^{-t}\right)}{t\left(e^{2 t}+1\right)}\right)^{\alpha} \cos (y t) e^{x t} \\
=\sum_{l=0}^{\infty} C_{C} T_{l}^{[k, \alpha]}(0, y) \frac{t^{l}}{l!} \sum_{m=0}^{\infty} x^{m} \frac{t^{m}}{m!}
\end{gathered}
$$

By using Cauchy product and comparing the coefficients of $\frac{t^{n}}{n!}$, we have first equation.

The proof of second equation is similar to first equation proof, we omit it.
Theorem 2.2. The following equations hold true:

$$
\begin{gathered}
{ }_{C} T_{n}^{[k, \alpha]}(x+z, y)=\sum_{m=0}^{n}\binom{n}{m}{ }_{C} T_{m}^{[k, \alpha]}(x, y) z^{n-m}, \\
{ }_{S} T_{n}^{[k, \alpha]}(x+z, y)=\sum_{m=0}^{n}\binom{n}{m}{ }_{S} T_{m}^{[k, \alpha]}(x, y) z^{n-m} \\
\frac{d}{d x}{ }_{C} T_{n}^{[k, \alpha]}(x, y)=n_{C} T_{n-1}^{[k, \alpha]}(x, y)
\end{gathered}
$$

and

$$
\frac{d}{d y}{ }_{S} T_{n}^{[k, \alpha]}(x, y)=-n{ }_{S} T_{n-1}^{[k, \alpha]}(x, y)
$$

Proof. The proof of the equations in this theorem obtain from (12) and (13), easily.

Theorem 2.3. The parametric poly-tangent polynomials satisfy the following relation

$$
\begin{align*}
& \left\{(n+1){ }_{C} T_{p}^{[k, 1]}(x, y)-\sum_{m=0}^{n}\binom{n}{m}(m+1){ }_{C} T_{m}^{[k, 1]}(x, y)(-1)^{m}\right. \\
& \left.-(1+x){ }_{C} T_{n}^{[k, 1]}(x, y)+y_{S} T_{n}^{[k, 1]}(x, y)\right\} \\
& +\sum_{p=0}^{n}\binom{n}{p}\left\{(p+1){ }_{C} T_{p}^{[k, 1]}(x, y)-\sum_{m=0}^{p}\binom{p}{m}(m+1){ }_{C} T_{m}^{[k, 1]}(x, y)(-1)^{p-m}\right. \\
& \left.-(1+x){ }_{C} T_{m}^{[k, 1]}(x, y)+y{ }_{S} T_{n}^{[k, 1]}(x, y)\right\} 2^{n-p} \\
= & -2{ }_{C} T_{n}^{[k, 1]}(x, y) \tag{14}
\end{align*}
$$

Proof. For $\alpha=1$, by differentiating the equation (12) with respect to $t$ and thereafter comparing the coefficients of similar powers of $t$ in the result equation gives the equation (14).

Theorem 2.4. The following relations hold true:

$$
\begin{equation*}
n_{C} T_{n-1}^{[k, 1]}(x, y)=\sum_{p=0}^{\infty} \frac{1}{(p+1)^{k}} \sum_{i=0}^{p+1}\binom{p+1}{i}(-1)^{i}{ }_{C} T_{n}^{[k, 1]}(x-i, y) \tag{15}
\end{equation*}
$$

Proof. By using (12) and for $\alpha=1$, we write as

$$
\begin{aligned}
& \sum_{n=0}^{\infty}{ }_{C} T_{n}^{[k, 1]}(x, y) \frac{t^{n}}{n!}=\frac{2 L i_{k}\left(1-e^{-t}\right)}{t\left(e^{2 t}+1\right)} e^{x t} \cos (y t) \\
& \sum_{n=0}^{\infty} n_{C} T_{n-1}^{[k, 1]}(x, y) \frac{t^{n}}{n!}=\sum_{p=0}^{\infty} \frac{\left(1-e^{-t}\right)^{p+1}}{(p+1)^{k}} \frac{2 e^{x t}}{\left(e^{2 t}+1\right)} \cos (y t) \\
&= \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{1}{(p+1)^{k}} \sum_{i=0}^{p+1}\binom{p+1}{i}(-1)^{i}{ }_{C} T_{n}^{[k, 1]}(x-i, y) \frac{t^{n}}{n!} .
\end{aligned}
$$

Comparing the both sides of this equation, we have (15).
Corollary 2.5. For $n \in \mathbb{Z}^{+}$, we have

$$
n{ }_{S} T_{n-1}^{[k, 1]}(x, y)=\sum_{p=0}^{\infty} \frac{1}{(p+1)^{k}} \sum_{i=0}^{p+1}\binom{p+1}{i}(-1)^{i}{ }_{S} T_{n}^{[k, 1]}(x-i, y)
$$

Theorem 2.6. There is the following relation between the Bernoulli numbers and the parametric tangent polynomials as:

$$
\begin{equation*}
{ }_{C} T_{n}^{[2,1]}(x, y)=\sum_{m=0}^{n}\binom{n}{m} m{ }_{C} T_{n-m}^{[2,1]}(x, y) B_{m-1} \tag{16}
\end{equation*}
$$

Proof. By using (12), for $\alpha=1$;

$$
\begin{aligned}
& \sum_{n=0}^{\infty}{ }_{C} T_{n}^{[k, 1]}(x, y) \frac{t^{n}}{n!}=\frac{2}{t\left(e^{2 t}+1\right)} e^{x t} \cos (y t) L i_{k}\left(1-e^{-t}\right) \\
= & \sum_{n=0}^{\infty}{ }_{C} T_{n}^{[k, 1]}(x, y) \frac{t^{n}}{n!} \underbrace{\int_{0}^{t} \frac{1}{e^{y}-1} \int_{0}^{y} \frac{1}{e^{y}-1} \cdots \int_{0}^{y} \frac{y}{e^{y}-1}}_{(k-2) \text { times }} d y \cdots d y .
\end{aligned}
$$

For $k=2$,

$$
\sum_{n=0}^{\infty} C_{n} T_{n}^{[k, 1]}(x, y) \frac{t^{n}}{n!}=\sum_{n=0}^{\infty}{ }_{C} T_{n}^{[2,1]}(x, y) \frac{t^{n}}{n!} \int_{0}^{t} \sum_{m=0}^{\infty} B_{m} \frac{t^{m+1}}{m!}
$$

By using Cauchy product and comparing the coefficients, we have (16).

Theorem 2.7. There is the following relation the parametric tangent polynomials and the Stirling numbers of the second kind as

$$
\begin{equation*}
{ }_{C} T_{n}^{[k, \alpha]}(x, y)=\sum_{p=0}^{\infty} \sum_{i=p}^{n}\binom{n}{i} S_{2}(i, p){ }_{C} T_{n-i}^{[k, \alpha]}(-p, y)\langle x\rangle_{p}, \tag{17}
\end{equation*}
$$

where $\langle x\rangle_{p}=x(x+1) \cdots(x+p-1),(p \geq 1)$ with $\langle x\rangle_{0}=1$.
Proof. By (12),

$$
\begin{gathered}
\sum_{n=0}^{\infty}{ }_{C} T_{n}^{[k, \alpha]}(x, y) \frac{t^{n}}{n!}=\left(\frac{2 L i_{k}\left(1-e^{-t}\right)}{t\left(e^{2 t}+1\right)}\right)^{\alpha}\left(1-\left(1-e^{-t}\right)\right)^{-x} \cos (y t) \\
=\left(\frac{2 L i_{k}\left(1-e^{-t}\right)}{t\left(e^{2 t}+1\right)}\right)^{\alpha} \cos (y t) \sum_{p=0}^{\infty}\binom{x+p-1}{p}\left(1-e^{-t}\right)^{p} \\
=\sum_{n=0}^{\infty}\langle x\rangle_{p} \frac{\left(e^{t}-1\right)^{p}}{p!}\left(\frac{2 L i_{k}\left(1-e^{-t}\right)}{t\left(e^{2 t}+1\right)}\right)^{\alpha} e^{-p t} \cos (y t) \\
=\sum_{n=0}^{\infty}\langle x\rangle_{p} \sum_{n=0}^{\infty} S_{2}(n, p) \frac{t^{n}}{n!} \sum_{n=0}^{\infty}{ }_{C} T_{n}^{[k, \alpha]}(-p, y) \frac{t^{n}}{n!}
\end{gathered}
$$

By using Cauchy product and comparing the coefficients of $\frac{t^{n}}{n!}$, we have (17).

Corollary 2.8. The following relation holds true:

$$
{ }_{S} T_{n}^{[k, \alpha]}(x, y)=\sum_{p=0}^{\infty} \sum_{i=p}^{n}\binom{n}{i} S_{2}(i, p){ }_{S} T_{n-i}^{[k, \alpha]}(-p, y)\langle x\rangle_{p} .
$$

Theorem 2.9. There is the following the other relation between the parametric tangent polynomials and the Stirling numbers of the second kind as:

$$
\begin{equation*}
{ }_{C} T_{n}^{[k, \alpha]}(x, y)=\sum_{p=0}^{\infty} \sum_{i=p}^{n}\binom{n}{i}(x)_{p} S_{2}(i, p){ }_{C} T_{n-i}^{[k, \alpha]}(0, y) \tag{18}
\end{equation*}
$$

Proof. Using (12);

$$
\begin{gathered}
\sum_{n=0}^{\infty}{ }_{C} T_{n}^{[k, \alpha]}(x, y) \frac{t^{n}}{n!}=\left(\frac{2 L i_{k}\left(1-e^{-t}\right)}{t\left(e^{2 t}+1\right)}\right)^{\alpha}\left(\left(e^{t}-1\right)+1\right)^{x} \cos (y t) \\
=\left(\frac{2 L i_{k}\left(1-e^{-t}\right)}{t\left(e^{2 t}+1\right)}\right)^{\alpha} \sum_{p=0}^{\infty}\binom{x}{p}\left(e^{t}-1\right)^{p} \cos (y t) \\
=\sum_{p=0}^{\infty}(x)_{p} \sum_{n=0}^{\infty} S_{2}(n, p) \frac{t^{n}}{n!} \sum_{n=0}^{\infty}{ }_{C} T_{n}^{[k, \alpha]}(0, y) \frac{t^{n}}{n!}
\end{gathered}
$$

By using Cauchy product and comparing the coefficients of $\frac{t^{n}}{n!}$, we have (18).

Corollary 2.10. The following relation holds true:

$$
{ }_{S} T_{n}^{[k, \alpha]}(x, y)=\sum_{p=0}^{\infty} \sum_{i=p}^{n}\binom{n}{i}(x)_{p} S_{2}(i, p){ }_{S} T_{n-i}^{[k, \alpha]}(0, y)
$$

Theorem 2.11. There is the following relation between the Bernoulli numbers and Stirling numbers of the second kind and the parametric tangent polynomials:

$$
\begin{equation*}
{ }_{C} T_{n}^{[k, \alpha]}(x, y)=\sum_{p=0}^{n} \frac{\binom{n}{p}}{\binom{p+r}{r}} S_{2}(p+r, r) \sum_{i=0}^{n-p}\binom{n-p}{i} B_{i}^{(r)}{ }_{C} T_{n-p-i}^{[k, \alpha]}(x, y) \tag{19}
\end{equation*}
$$

Proof. Using the parametric tangent polynomials Definition's;

$$
\begin{gathered}
\sum_{n=0}^{\infty}{ }_{C} T_{n}^{[k, \alpha]}(x, y) \frac{t^{n}}{n!}=\frac{\left(e^{t}-1\right)^{r}}{r!} \frac{r!}{t^{r}}\left(\frac{t}{e^{t}-1}\right)^{r} \sum_{n=0}^{\infty}{ }_{C} T_{n}^{[k, \alpha]}(x, y) \frac{t^{n}}{n!} \\
=\frac{\left(e^{t}-1\right)^{r}}{r!}\left(\sum_{n=0}^{\infty} B_{n}^{(r)} \frac{t^{n}}{n!}\right)\left(\sum_{n=0}^{\infty} C_{n} T_{n}^{[k, \alpha]}(x, y) \frac{t^{n}}{n!}\right) \frac{r!}{t^{r}} \\
=\sum_{n=0}^{\infty}\left(\sum_{p=0}^{n} \frac{\binom{n}{p}}{\binom{p+r}{r}} S_{2}(p+r, r) \sum_{i=0}^{n-p}\binom{n-p}{i} B_{i}^{(r)}{ }_{C} T_{n-p-i}^{[k, \alpha]}(x, y)\right) \frac{t^{n}}{n!} .
\end{gathered}
$$

By comparing the coefficients of $\frac{t^{n}}{n!}$, we have (19).
Corollary 2.12. The following relation holds true:

$$
{ }_{S} T_{n}^{[k, \alpha]}(x, y)=\sum_{p=0}^{n} \frac{\binom{n}{p}}{\binom{p+r}{r}} S_{2}(p+r, r) \sum_{i=0}^{n-p}\binom{n-p}{i} B_{i}^{(r)}{ }_{S} T_{n-p-i}^{[k, \alpha]}(x, y)
$$

## 3. Conclusion

Hamahata [1], Kim et al. ([2], [3]) considered the poly-Bernoulli polynomials and the poly-Genocchi polynomials. Srivastava et al. ([13], [14]) introduced and investigated Bernoulli polynomials, Euler polynomials and related polynomials. Srivastava et al. ([14], [15]) considered parametric type of Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials. Ryoo et al. ([8]-[12]) introduced the tangent numbers and polynomials, $q$-poly-tangent polynomials. Ryoo gave some identities and relations for these polynomials.

In this work, we define the parametric poly-tangent numbers and polynomials. We give some identities and recurrence relations for the parametric poly-tangent polynomials.

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