

A NOTE OF THE MODIFIED BERNOULLI POLYNOMIALS AND IT'S THE LOCATION OF THE ROOTS[†]

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ABSTRACT. This type of polynomial is a generating function that substitutes $e^{\lambda t}$ for e^t in the denominator of the generating function for the Bernoulli polynomial, but polynomials by using this generating function has interesting properties involving the location of the roots. We define these generation functions and observe the properties of the generation functions.

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1. Introduction

Bernoulli numbers were discovered by Jakob Bernoulli in the 17th century. As it is well known, Bernoulli numbers are related to many important properties appearing in mathematics and physics. Thereby many mathematicians have studied the Bernoulli numbers and polynomials, Euler numbers and polynomials, Genocchi numbers and polynomials, and tangent numbers and polynomials(see [1-12]).

In this paper, we are going to talk about the Bernoulli polynomials with an some modified generation function. This type of polynomial is a generating function that substitutes e^{λ} for e^t in the denominator of the Bernoulli polynomial generation function, but polynomials using by this generating function has interesting properties involving the location of the roots. We define these generation functions and observe the properties of the generation functions. Also, we compare the structure of the roots of the modified Bernoulli polynomials defined here with the classical Bernoulli polynomials.

Throughout this paper, we will use the following notations: \mathbb{N} denotes the set of natural numbers, \mathbb{Z} denotes the ring of rational integers, \mathbb{Q} denotes the field

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of rational numbers, \mathbb{C} denotes the set of complex numbers, $\mathbb{Z}_0^+ = \mathbb{N} \cup \{0\}$ and $\mathbb{Z}_0^- = \{0, -1, -2, \dots\}$.

As a well known definition, the Bernoulli polynomials $B_n(x)$ is defined by the following generating function(see 3, 4, 7, 9, 11):

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad (|t| < 2\pi).$$

If $x = 0$, $B_n = B_n(0)$ is called the Bernoulli numbers.

We plot the zeros of the Bernoulli polynomials $B_n(x)$ (Figure 1).

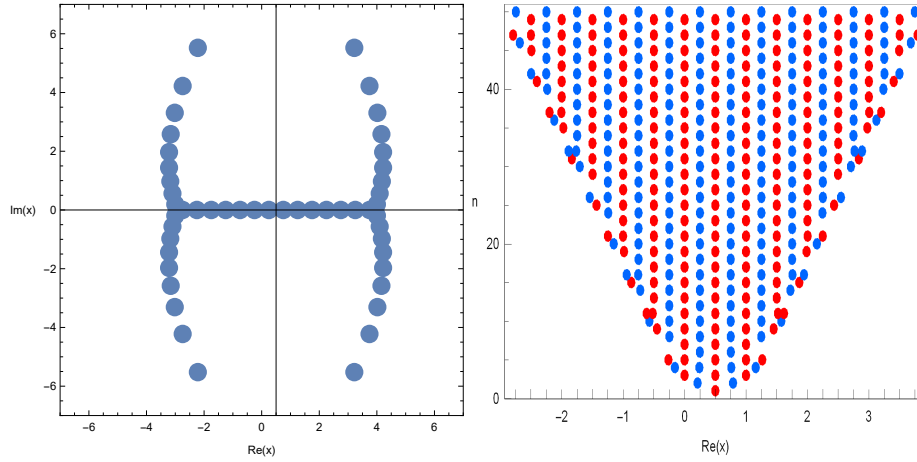


FIGURE 1. Plot the zeros of the Bernoulli polynomials

In Figure 1(left), plot the zeros of the Bernoulli polynomials for $n = 50$ and $x \in \mathbb{C}$. In Figure 1(right), plot of real zeros of the Bernoulli polynomials for $1 \leq n \leq 50$ structure are presented

We observe that $B_n(x)$, $x \in \mathbb{C}$, has $Re(x) = 1/2$ reflection symmetry in addition to the usual $Im(x) = 0$ reflection symmetry analytic complex functions. The obvious corollary is that the zeros of $B_n(x)$ will also inherit these symmetries.

$$\text{If } B_n(x_0) = 0, \text{ then } B_n(1 - x_0) = 0 = B_n(x_0^*) = B_n(1 - x_0^*).$$

Here, $*$ denotes complex conjugation. Prove that $B_n(x) = 0$ has n distinct solutions.

2. Definition for the modified Bernoulli numbers and polynomials and its basic properties

Definition 2.1. For $\lambda \neq 0$, the modified Bernoulli polynomials $B_{n,\lambda}(x)$ are defined by means of the generaing function:

$$\frac{t}{e^{\lambda t} - 1} e^{xt} = \sum_{n=0}^{\infty} B_{n,\lambda}(x) \frac{t^n}{n!}, \quad (|t| < \frac{2\pi}{|\lambda|}). \quad (2.1)$$

If $x = 0$, then $B_{n,\lambda}(0) = B_{n,\lambda}$ and we call it a modified Bernoulli numbers and as $\lambda \rightarrow 1$, $B_{n,\lambda}(x) = B_n(x)$.

From (2.1), we get the following form.

$$te^{xt} = e^{(\lambda + B_{\lambda}(x))t} - e^{B_{\lambda}t}. \quad (2.2)$$

The left side and right side of Equation (2.2) are changed by the Taylor series as follows.

The left side is $\sum_{n=0}^{\infty} nx^{n-1} \frac{t^n}{n!}$.

The right side is $\sum_{n=1}^{\infty} ((\lambda + B_{\lambda}(x))^n - B_{n,\lambda}(x)) \frac{t^n}{n!}$.

The following theorem is obtained by comparing the coefficients of $\frac{t^n}{n!}$ on the left and right sides.

Theorem 2.2. For the nonnegative integer n and $\lambda \in \mathbb{C}$,

$$\begin{aligned} nx^{n-1} &= (\lambda + B_{\lambda}(x))^n - B_{n,\lambda}(x) \\ &= \sum_{l=0}^{n-1} \binom{n}{l} \lambda^{n-l} B_{l,\lambda}(x). \end{aligned}$$

In paticula, if $x=1$, then

$$\begin{aligned} n &= (\lambda + B_{\lambda}(1))^n - B_{n,\lambda}(1) \\ &= \sum_{l=0}^{n-1} \binom{n}{l} \lambda^{n-l} B_{l,\lambda}(1). \end{aligned}$$

If $x = 0$, then

$$\sum_{l=0}^{n-1} \binom{n}{l} \lambda^{n-l} B_{l,\lambda} = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1. \end{cases}$$

Theorem 2.3. For nonnegative integer n and $\lambda \in \mathbb{C}$, we get

$$\frac{B_{n,\lambda}(m\lambda + \lambda) - B_{n,\lambda}(\lambda)}{n} = m^{n-1}.$$

Proof. From Definition 2.1, we get the following:

$$\begin{aligned} & \sum_{n=0}^{\infty} (B_{n,\lambda}(m+\lambda) - B_{n,\lambda}(\lambda)) \frac{t^n}{n!} \\ &= te^{mt} \\ &= \sum_{n=0}^{\infty} nm^{n-1} \frac{t^n}{n!}. \end{aligned} \tag{2.3}$$

Comparing the coefficient of $\frac{t^n}{n!}$ on both sides of (2.3), we obtain as below:

$$B_{n,\lambda}(m+\lambda) - B_{n,\lambda}(\lambda) = nm^{n-1}. \tag{2.4}$$

From (2.4), we get the theorem.

Also, if $m = 1$, then we get

$$B_{n,1}(1+\lambda) - B_{n,1}(\lambda) = n.$$

□

In equation (2.4), replace $-m$ by λ . Then, we get a equation as follows:

$$B_{n,-m} - B_{n,-m}(-m) = (-1)^{n-1} nm^{n-1}. \tag{2.5}$$

In equation (2.5), replace $-m$ by m . Then, we get an another equation as follows:

$$B_{n,m} - B_{n,m}(m) = nm^{n-1}. \tag{2.6}$$

From equations (2.5) and (2.6), we get a following corollary.

Corollary 2.4. *For the non-negative integer n*

if n is even, then, we get

$$B_{n,-m} + B_{n,m} = B_{n,-m}(-m) + B_{n,-m}(m)$$

and

if n is odd, then, we get

$$B_{n,-m} + B_{n,m}(m) = B_{n,m} + B_{n,-m}(-m).$$

Let $F(x, \lambda, t) = \frac{t}{e^{\lambda t} - 1} e^{xt} = \sum_{n=0}^{\infty} B_{n,\lambda}(x) \frac{t^n}{n!}$. Then

$$\begin{aligned} F(\lambda - x, \lambda, -t) &= \frac{-t}{e^{-\lambda t} - 1} e^{-(\lambda-x)t} \\ &= \frac{t}{e^{\lambda t} - 1} e^{xt} \\ &= F(x, \lambda, t) \end{aligned} \tag{2.7}$$

and

$$F(\lambda - x, \lambda, -t) = \sum_{n=0}^{\infty} (-1)^n B_{n,\lambda}(\lambda - x) \frac{t^n}{n!}. \quad (2.8)$$

From (2.7) and (2.8), we get the following property.

Theorem 2.5. For $n \equiv 2 \pmod{2}$ and $\lambda \in \mathbb{C}$

$$B_{n,\lambda}(x) = B_{n,\lambda}(\lambda - x).$$

It means that the polynomials $B_{n,\lambda}(x)$ is symmetric about $x = \frac{\lambda}{2}$.

By the Definition 2.1

$$\begin{aligned} \sum_{b=0}^{p-1} \left(\sum_{n=0}^{\infty} B_{n,\lambda} \left(\alpha + \frac{b}{p} \lambda \right) \frac{t^n}{n!} \right) &= \sum_{b=0}^{p-1} \frac{t}{e^{\lambda t} - 1} e^{(\alpha + \frac{b}{p} \lambda)t} \\ &= \frac{t}{e^{\frac{\lambda}{p} t} - 1} e^{\alpha t} \\ &= p \sum_{n=0}^{\infty} B_{n,\lambda}(p\alpha) \frac{1}{p^n} \frac{t^n}{n!}. \end{aligned}$$

Hence, we get the following property.

Theorem 2.6. For non-negative integer n and $\lambda \in \mathbb{C}$,

$$B_{n,\lambda}(p\alpha) = p^{n-1} \sum_{b=0}^{p-1} B_{n,\lambda} \left(\alpha + \frac{b}{p} \lambda \right).$$

We consider the partial derivation for x at $\sum_{n=0}^{\infty} B_{n,\lambda}(x) \frac{t^n}{n!} = \frac{t}{e^{\lambda t} - 1} e^{xt}$.

$$\begin{aligned} \frac{\partial}{\partial x} \sum_{n=0}^{\infty} B_{n,\lambda}(x) \frac{t^n}{n!} &= \frac{\partial}{\partial x} \frac{t}{e^{\lambda t} - 1} e^{xt} \\ &= t \sum_{n=0}^{\infty} B_{n,\lambda}(x) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} n B_{n-1,\lambda}(x) \frac{t^n}{n!}. \end{aligned} \quad (2.9)$$

Therefore we get as below property.

Theorem 2.7. For non-negative integer n and $\lambda \in \mathbb{C}$,

$$\frac{\partial}{\partial x} B_{n,\lambda}(x) = n B_{n-1,\lambda}(x).$$

Corollary 2.8. For non-negative integer n and $\lambda \in \mathbb{C}$,

$$\frac{\partial^n}{\partial x^n} B_{n,\lambda}(x) = \frac{n!}{\lambda}.$$

From Definition 2.1 and Cauchy product, we get

$$\sum_{n=0}^{\infty} B_{n,\lambda}(x) \frac{t^n}{n!} = \frac{t}{e^{\lambda t} - 1} e^{xt} = \sum_{n=0}^{\infty} \left(\sum_{l=0}^n \binom{n}{l} B_{l,\lambda} x^{n-l} \right) \frac{t^n}{n!}. \quad (2.10).$$

Comparing both sides of (2.10) with respect to $\frac{t^n}{n!}$, we have the following:

$$B_{n,\lambda}(x) = \sum_{l=0}^n \binom{n}{l} B_{l,\lambda} x^{n-l} = (B_{\lambda} + x)^n.$$

From $F(x, \lambda, t) = \frac{t}{e^{\lambda t} - 1} e^{xt}$, differential operator D and identity differential operator I ,

$$\begin{aligned} e^{\lambda t} F(x, \lambda, t) - F(x, \lambda, t) &= t e^{xt} \\ \Leftrightarrow D^k (e^{\lambda t} F(x, \lambda, t) - F(x, \lambda, t)) &= D^k (t e^{xt}) \\ \Leftrightarrow D^{k-1} (e^{\lambda t} (D + \lambda I) F(x, \lambda, t) - D^k F(x, \lambda, t)) &= (k x^{k-1} + x^k t) e^{xt} \\ &\vdots \end{aligned}$$

$$\begin{aligned} &\Leftrightarrow (D + \lambda I)^k F(x, \lambda, t) - e^{-\lambda t} D^k F(x, \lambda, t) = e^{-\lambda t} (k x^{k-1} + x^k t) e^{xt} \\ &\Leftrightarrow D^m (D + \lambda I)^k F(x, \lambda, t) - e^{-\lambda t} (D - \lambda I)^m D^k F(x, \lambda, t) \\ &= [(k + xt)(x - \lambda) + m] x^{k-1} e^{(x-\lambda)t} (x - \lambda)^{m-1} \\ &\Leftrightarrow e^{\lambda t} D^m (D + \lambda I)^k F(x, \lambda, t) - (D - \lambda I)^m D^k F(x, \lambda, t) \\ &= [(k + xt)(x - \lambda) + m] x^{k-1} e^{(x-\lambda)t} (x - \lambda)^{m-1} e^{\lambda t} \\ &\Leftrightarrow \sum_{l=0}^k \binom{k}{l} e^{\lambda t} \lambda^{k-l} D^{m+l} F(x, \lambda, t) - \sum_{l=0}^m \binom{m}{l} (-\lambda)^{m-l} D^{k+l} F(x, \lambda, t) \\ &= [(k + xt)(x - \lambda) + m] x^{k-1} e^{(x-\lambda)t} (x - \lambda)^{m-1} e^{\lambda t}. \end{aligned}$$

Since $D^{k+l} F(x, \lambda, t)|_{t=0} = B_{k+l,\lambda}(x)$ and $e^{\lambda t} D^{m+l} F(x, \lambda, t)|_{t=0} = B_{m+l,\lambda}(x)$, we get the following Proposition.

Theorem 2.9. For non-negative integer n and $\lambda \in \mathbb{C}$,

$$\begin{aligned} &\sum_{l=0}^{\max\{k,m\}} \left[\binom{k}{l} \lambda^{k-l} B_{m+l,\lambda}(x) - (-\lambda)^{m-l} \binom{m}{l} B_{l+k,\lambda}(x) \right] \\ &= \{(k + xt)(x - \lambda) + m\} x^{k-1} (x - \lambda)^{m-1}. \end{aligned}$$

3. Distribution of zeros of the modified Bernoulli polynomials

This section aims to demonstrate the benefit of using numerical investigation to support theoretical prediction and to discover new interesting pattern of the zeros of the modified Bernoulli polynomials $B_{n,\lambda}(x)$. We investigate the zeros of the $B_{n,\lambda}(x)$ by using a computer. We plot the zeros of the modified Bernoulli polynomials $B_{n,\lambda}(x)$ for $n = 50, \lambda = 1, 3, 5, 7$ and $x \in \mathbb{C}$ (Figure 2).

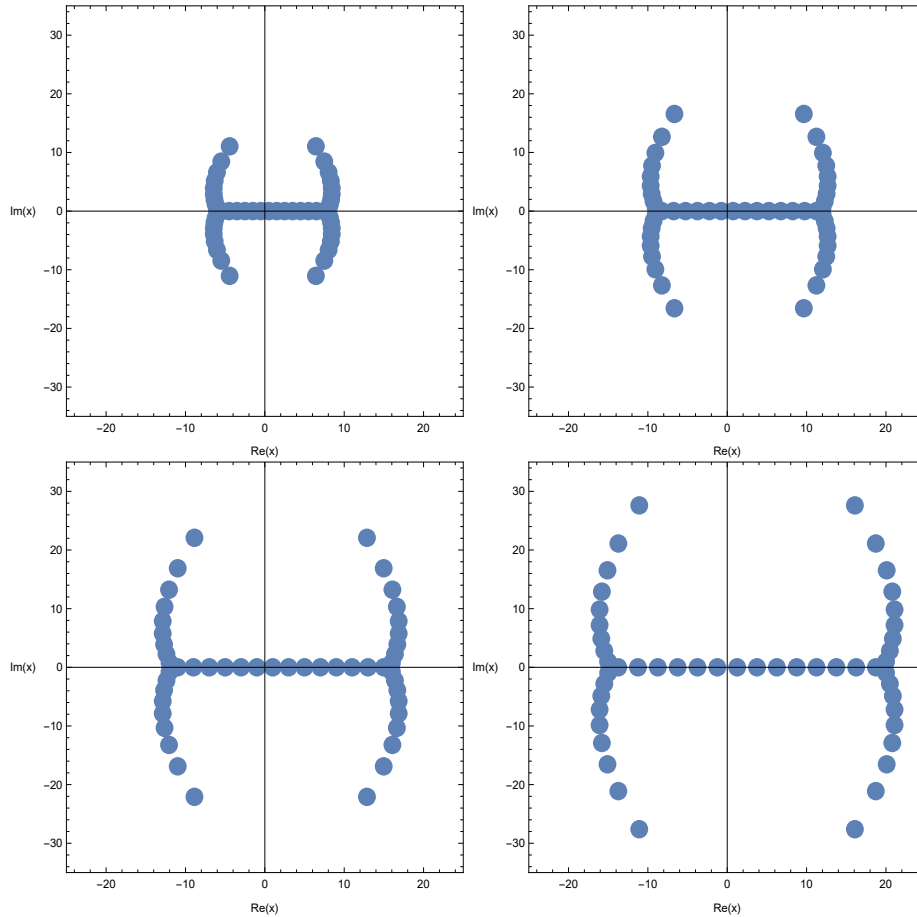
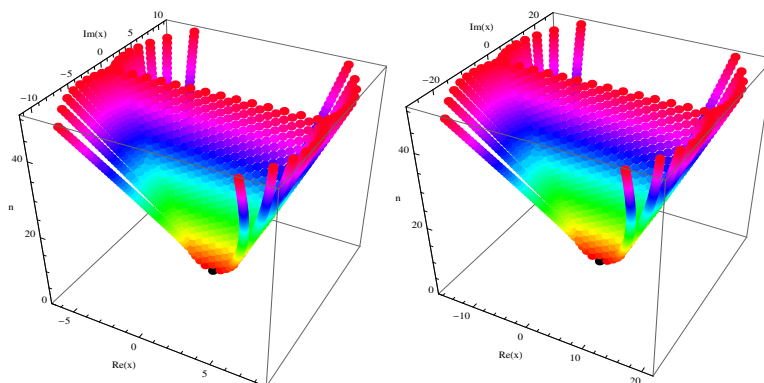


FIGURE 2. Zeros of $B_{n,\lambda}(x)$

In Figure 2(top-left), we choose $n = 50, \lambda = 2$. In Figure 2(top-right), we choose $n = 50, \lambda = 3$. In Figure 2(bottom-left), we choose $n = 50, \lambda = 4$. In Figure 2(bottom-right), we choose $n = 50, \lambda = 5$.

Stacks of zeros of $B_{n,\lambda}(x, \lambda)$ for $1 \leq n \leq 50$ from a 3-D structure are presented(Figure 3).

FIGURE 3. Stacks of zeros of $B_n(x, \lambda)$ for $1 \leq n \leq 50$

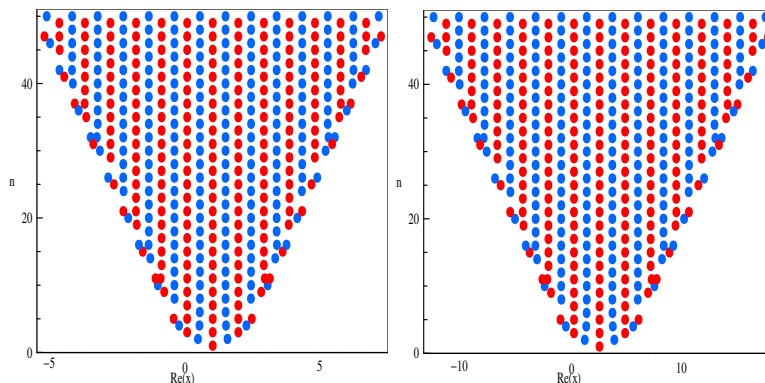
In Figure 3(left), we choose $1 \leq n \leq 50$ and $\lambda = 2$. In Figure 3(right), we choose $1 \leq n \leq 50$ and $\lambda = 5$. Our numerical results for approximate solutions of real zeros of $B_n(x, \lambda)$ are displayed (Tables 1, 2).

Table 1. Numbers of real and complex zeros of $B_n(x, \lambda)$

degree n	$\lambda = 2$		$\lambda = 5$	
	real zeros	complex zeros	real zeros	complex zeros
1	1	0	1	0
2	2	0	2	0
3	3	0	3	0
4	4	0	4	0
5	5	0	5	0
6	2	4	2	4
7	3	4	3	4
8	4	4	4	4
9	5	4	5	4
10	6	4	6	4
11	7	4	7	4
12	4	8	4	8
13	5	8	5	8
14	6	8	6	8
15	7	8	7	8
16	8	8	8	8
17	5	12	5	12

For $\lambda = 2, 5$, plot of real zeros of $B_n(x, \lambda)$ for $1 \leq n \leq 50$ structure are presented (Figure 4).

In Figure 4(left), we choose $1 \leq n \leq 50$ and $\lambda = 2$. In Figure 4(right), we choose $1 \leq n \leq 50$ and $\lambda = 5$.

FIGURE 4. Real zeros of $B_n(x, \lambda)$ for $1 \leq n \leq 50$

We observe a remarkably regular structure of the complex roots of the modified Bernoulli polynomials $B_{n,\lambda}(x)$ (see Table 1). Next, we calculated an approximate solution satisfying $B_n(x, \lambda) = 0$ for $\lambda = 2, x \in \mathbb{C}$. The results are given in Table 2.

Table 2. Approximate solutions of $B_n(x, \lambda) = 0, x \in \mathbb{R}$

degree n	x
1	1.0000
2	0.42265, 1.5774
3	0, 1.0000, 2.0000
4	-0.31541, 0.48067, 1.5193, 2.3154
5	-0.52753, 0, 1.000, 2.0000, 2.5275
6	0.49508, 1.5049
7	0, 1.000, 2.0000
8	-0.49443, 0.49876, 1.5012, 2.4944
9	-0.89821, 0, 1.0000, 2.0000, 2.8982
10	-1.1479, -0.49985, 0.49969, 1.5003, 2.4998, 3.1479
11	-1.2361, -1.0374, 0, 1.0000, 2.0000, 3.0374, 3.2361

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