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LOCAL SPECTRAL THEORY

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ABSTRACT. For any Banach spaces X and Y, let $L(X, Y)$ denote the set of all bounded linear operators from X to Y. Let $A \in L(X, Y)$ and $B, C \in L(Y, X)$ satisfying operator equation $ABA = ACA$. In this paper, we prove that AC and BA share the local spectral properties such as a finite ascent, a finite descent, property (K), localizable spectrum and invariant subspace.

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1. Introduction and preliminaries

For any Banach spaces X and Y, let $L(X, Y)$ denote the set of all bounded linear operators from X to Y. For given $A \in L(X)$, we use $\sigma(A)$, $\sigma_p(A)$, $\sigma_{ap}(A)$, $\rho(A)$ and $r(A)$ to denote the spectrum, the point spectrum, the approximate point spectrum, the resolvent set and the spectral radius of A, respectively. Recall that the *point spectrum* of $A \in L(X)$ is given by $\sigma_p(A) := \{\lambda \in \mathbb{C} : \lambda I - A$ is not injective$\}$. As usual, given $A \in L(X)$, let $ker(A)$ and $A(X)$ stand for the kernel and the range of an operator $A \in L(X)$, respectively.

The single-valued extension property(SVEP for brevity) dates back to the early days of local spectral theory and first appears in Dunford [7] and [8]. The localized version of SVEP considered in this paper was introduced in Finch [10] and has become an important tool in local spectral theory and Fredholm theory for operators on Banach spaces, see [1] and [16].

Definition 1.1. ([10]) An operator $A \in L(X)$ is said to have the *single-valued extension property* at a point $\lambda \in \mathbb{C}$ (SVEP at λ for brevity) if, for every open disc $U \subseteq \mathbb{C}$ centered at λ the only analytic function $f : U \to X$ satisfies the equation

$$(\mu I - A)f(\mu) = 0 \quad \text{for all} \quad \mu \in U$$

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is the constant function $f \equiv 0$ on U. In addition, the operator $A \in L(X)$ is said to have the SVEP if A has the SVEP at every point $\lambda \in \mathbb{C}$.

Evidently, an operator A has SVEP at a point λ precisely when $\lambda I - A$ has SVEP at 0. Moreover, SVEP at a point is inherited by restrictions to closed invariant subspaces. It is clear that $A \in L(X)$ has the SVEP at every point of the resolvent $\rho(A) := \mathbb{C} \setminus \sigma(A)$ and from the identity theorem for analytic function it follows that $A \in L(X)$ has the SVEP at every point of the boundary $\partial\sigma(A)$ of the spectrum. Moreover, it is clear that $A \in L(X)$ has the SVEP at every isolated point of the spectrum.

For $A \in L(X)$, the *local resolvent set* $\rho_A(x)$ of A at the point $x \in X$ is defined as the set of all $\lambda \in \mathbb{C}$ for which there exist an open neighborhood U of λ and analytic function $f : U \to X$ such that

$$(\mu I - A)f(\mu) = x \quad \text{for all } \mu \in U.$$

The *local spctrum* $\sigma_A(x)$ at x is the set defined by $\sigma_A(x) := \mathbb{C}\setminus\rho_A(x)$. It is obvious that the analytic solutions occurring in the defintion of the local resolvent set is unique if and only if A has the SVEP.

For every subset F of $\mathbb{C}$, the *analytic spectral subspace* of A associated with F is the set

$$X_A(F) := \{x \in X : \sigma_A(x) \subseteq F\}.$$

Evidently, $X_A(F)$ is a $A-$invariant linear subspace of X and that $X_A(F) \subseteq X_A(G)$ whenever $F \subseteq G$. Even for a closed subset F of $\mathbb{C}$, the analytic spectral subspace $X_A(F)$ need not be closed. It follows from Proposition 1.2.16 of [16] that for every closed F of subset of $\mathbb{C}$, we have

$$(\lambda I - A)X_A(F) = X_A(F) \text{ for all } \lambda \in \mathbb{C} \setminus F.$$

For every closed subset F of $\mathbb{C}$, the *glocal spectral subspace* $\mathcal{X}_A(F)$ is defined as the set of all $x \in X$ for which there exists an anlytic function $f : \mathbb{C} \setminus F \to X$ that satisfies $(\lambda I - A)f(\lambda) = x$ for all $\lambda \in \mathbb{C} \setminus F$. Note that $X_{\lambda I - A}(\mathbb{C} \setminus \{0\}) = X_A(\mathbb{C} \setminus \{\lambda\})$ for every $\lambda \in \mathbb{C}$.

The following elementary lemma will be useful in the sequel.

Lemma 1.2. *Let $A \in L(X, Y)$ and $B, C \in L(Y, X)$ satisfying operator equation $ABA = ACA$. Then $A(BA)^{n-1} = (AC)^{n-1}A$ and $(BA)^n B = B(AC)^{n-1}AB$ for all $n = 1, 2, \cdots$.*

Lemma 1.3. *Let $T \in L(X)$, $S \in L(Y)$ and $A \in L(X, Y)$ be operators on Banach spaces X and Y with $AT = SA$. Then $\sigma_S(Ax) \subseteq \sigma_T(x)$ and $AX_T(F) \subseteq Y_S(F)$ for all subsets F of $\mathbb{C}$.*

Proof. We have to show that $\rho_T(x) \subseteq \rho_S(Ax)$ for all $x \in X$. Suppose that $\lambda_0 \in \rho_T(x)$. Then there exists an analytic function $f : U \to X$ defined on some open neighborhood U of λ_0 such that $(\mu I - T)f(\mu) = x$ for all $\mu \in U$. Clearly,

$Af(\mu)$ is analytic. Since $AT = SA$, it is easily seen that $(\mu I - S)Af(\mu) = Ax$ for all $\mu \in U$, so that $\lambda_0 \in \rho_S(Ax)$, as desired. □

Corollary 1.4. *Let $A \in L(X,Y)$ and $B, C \in L(Y,X)$ satisfying operator equation $ABA = ACA$. Then we have the followings:*
(a) $\sigma_{AC}(Ax) \subseteq \sigma_{BA}(x)$ and $\sigma_{AB}(Ax) \subseteq \sigma_{CA}(x)$ for all $x \in X$. Moreover, if A is injective then $\sigma_{BA}(x) = \sigma_{CA}(x)$ for every $x \in X$.
(b) $AX_{BA}(F) \subseteq Y_{AC}(F)$ and $AX_{CA}(F) \subseteq Y_{AB}(F)$ for all subsets F of $\mathbb{C}$.

Proof. By Lemma 1.3, $\sigma_{AC}(Ax) \subseteq \sigma_{BA}(x)$ and $\sigma_{AB}(Ax) \subseteq \sigma_{CA}(x)$ for all $x \in X$. Since A is injective, by Proposition 3.1 of [5] we have $\sigma_{BA}(x) = \sigma_{AB}(Ax)$ and $\sigma_{CA}(x) = \sigma_{AC}(Ax)$ for every $x \in X$. Thus for every $x \in X$

$$\sigma_{BA}(x) = \sigma_{AB}(Ax) \subseteq \sigma_{CA}(x) = \sigma_{AC}(Ax) \subseteq \sigma_{BA}(x)$$

and hence $\sigma_{BA}(x) = \sigma_{CA}(x)$ for every $x \in X$. □

Corollary 1.5. *([5]) Let $A \in L(X,Y)$ and $B \in L(Y,X)$. Then we have the followings:*
(a) $\sigma_{AB}(Ax) \subseteq \sigma_{BA}(x) \subseteq \sigma_{AB}(Ax) \cup \{0\}$ for all $x \in X$;
(b) $\sigma_{BA}(By) \subseteq \sigma_{AB}(y) \subseteq \sigma_{BA}(By) \cup \{0\}$ for all $y \in Y$.

For an operator $A \in L(X)$, let $\mathcal{S}(A) := \{\lambda \in \mathbb{C} : A \text{ fails to have SVEP at } \lambda\}$. Obviously, $\mathcal{S}(A)$ is empty precisely when T has SVEP. It follows from the identity theorem for analytic functions that $\mathcal{S}(A)$ is open, and therefore contained in the interior of the spectrum $\sigma(A)$. In general, SVEP is not preserved under quotients and duality, see [1] and [16].

Proposition 1.6. *Let $A \in L(X,Y)$ and $B, C \in L(Y,X)$ satisfying operator equation $ABA = ACA$. Then $\mathcal{S}(AC) = \mathcal{S}(BA)$.*

Proof. If $\lambda_0 \notin \mathcal{S}(AC)$, then for every open subset V_{λ_0} centered at λ_0 the only analytic function $f : V_{\lambda_0} \to Y$ which satisfies the equation

$$(\mu I - AC)f(\mu) = 0$$

is the function $f \equiv 0$. Let U_{λ_0} be open and analytic function $g : U_{\lambda_0} \to X$ such that

$$(\mu I - BA)g(\mu) = 0 \quad \text{for all} \quad \mu \in U_{\lambda_0}.$$

Then $(\mu I - AC)Ag(\mu) = A(\mu I - BA)g(\mu) = 0$ for all $\mu \in U_{\lambda_0}$. It follows that $Ag(\mu) = 0$ for all $\mu \in U_{\lambda_0}$, thus $\mu g(\mu) = 0$ for all $\mu \in U_{\lambda_0}$ and $g(\mu) = 0$ for every $\mu \in U_{\lambda_0}$ $\mu \neq \lambda_0$, and from the continuity of g at λ_0, we conclude that $g(\lambda_0) = 0$. Hence $g \equiv 0$ on U_{λ_0} and therefore BA has the SVEP at λ_0. This implies that $\lambda_0 \notin \mathcal{S}(BA)$. The converse implication is similar. □

It is clear that if $A \in L(X, Y)$ and $B \in L(Y, X)$ satisfying operator equation $ABA = ACA$ then AC has SVEP if and only if BA has SVEP.

Proposition 1.7. *Let $F \subseteq \mathbb{C}$ be a closed and let $G \subseteq \mathbb{C}$ be a finite set with $F \cap G = \phi$. Suppose that $A \in L(X)$ has SVEP. If $X_A(F \cup G)$ is closed, then $X_A(F)$ is closed.*

Proof. Let $G := \bigcup_{i=1}^{n} \{\lambda_i\}$ and let $Z := X_A(F \cup G)$ and $S := A|Z \in L(Z)$. Then Z is closed and S has SVEP. It follows from Proposition 1.2.20 of [16] that

$$\sigma(S) \subseteq F \cup G$$

and $Z_S(F) = Z_S(F \cap \sigma(S)) = Z_S(\sigma(S)) = Z$. Thus by Lemma 2.4 of [2], we have

$$Z_S(W) = X_A(W) \text{ for every closed } W \subseteq F \cup G.$$

Case (I). Suppose that $\lambda_i \notin \sigma(S)$ for all $i = 1, 2, \cdots, n$. Then $\sigma(S) \subseteq F$ and $Z_S(F) = Z$ and hence $X_A(F) = Z_S(F) = Z$ is closed.

Case (II). Suppose that $G \subseteq \sigma(S)$. Let $F_0 := \sigma(S) \cap F$. Then F_0 is closed and $\sigma(S) = F_0 \cup G$. Thus $\lambda_i \notin F_0$ for all $i = 1, 2, \cdots, n$. It follows from Proposition 3.3.3 of [16] that

$$Z = Z_S(\sigma(S)) = Z_S(F_0) \oplus Z_S(\{\lambda_1\}) \oplus Z_S(\{\lambda_2\}) \oplus \cdots \oplus Z_S(\{\lambda_n\})$$

is closed. Thus $Z_S(F_0)$ is closed and $Z_S(F_0) = Z_S(\sigma(S) \cap F) = Z_S(F) = X_A(F)$, and hence $X_A(F)$ is closed.

Case (III). Suppose that $\lambda_1 \in \sigma(S)$ and $\lambda_i \notin \sigma(S)$ for all $i = 2, 3, \cdots, n$. Then it is clear that $\sigma(S) \subseteq F \cup (\bigcup_{i=2}^{n} \{\lambda_i\})$. Let $F_0 := \sigma(S) \cap F$. Then F_0 is closed. Also $\sigma(S) = F_0 \cup \{\lambda_1\}$ and F_0 and $\{\lambda_1\}$ are disjoint. It follows from Proposition 3.3.3 of [16] that $Z = Z_S(\sigma(S)) = Z_S(F_0) \oplus Z_S(\{\lambda_1\})$ is closed. Thus $Z_S(F_0)$ is closed. Also, $Z_S(F_0) = Z_S(F \cap \sigma(S)) = Z_S(F) = X_A(F)$, and hence $X_A(F)$ is closed. $\square$

Proposition 1.8. *Let $A \in L(X, Y)$ and $B, C \in L(Y, X)$ satisfy operator equation $ABA = ACA$. Then we have the followings;*
(a) $\sigma_p(AC) \setminus \{0\} = \sigma_p(AB) \setminus \{0\}$;
(b) if A is surjective, then $\sigma_{sur}(AC) \subseteq \sigma_{sur}(BA)$ and $\sigma_{sur}(AB) \subseteq \sigma_{sur}(CA)$.

Proof. (a) Let $\lambda \in \sigma_p(AC) \setminus \{0\}$. Then $ACy = \lambda y$ for some nonzero $y \in Y$. It is clear that

$$\lambda ABy = (ABA)Cy = (ACA)Cy = AC(ACy) = \lambda ACy = \lambda^2 y.$$

Thus we have $\lambda(AB - \lambda I)y = (AB - \lambda I)ACy = 0$. Hence $\lambda \in \sigma_p(AB) \setminus \{0\}$. The converse implication is similar.

(b) Let $\lambda \notin \sigma_{sur}(BA)$. Because of A is surjective, we have

$$Y = A(X) = A(\lambda I - BA)(X) = (\lambda I - AC)(Y).$$

Thus $\lambda \notin \sigma_{sur}(AC)$ and hence $\sigma_{sur}(AC) \subseteq \sigma_{sur}(BA)$. The converse implication is similar. $\square$

2. Main results

It is well known from Theorem 3.8 of [1] that SVEP is intimately related to certain conditions from classical operator theory. Let $p(A)$ denote the *ascent* of an operator $A \in L(X)$, i.e., $p(A)$ is the smallest non-negative integer p for which $ker(A^p) = ker(A^{p+1})$, if such an integer exists and otherwise $p(A) = \infty$. Analogously, let $q(A)$ denote the *descent* of an operator $A \in L(X)$, i.e., $q(A)$ is the smallest non-negative integer q for which $A^q(X) = A^{q+1}(X)$, if such an integer exists and otherwise $q(A) = \infty$. It follows from Theorem 2.4 of [4] or Theorem 3.8 of [1] that if $p(\lambda I - A)$ is finite then A has SVEP at λ, and dually, for the adjoint operator $A^* \in L(X^*)$ if $q(\lambda I - A)$ is finite then A^* has SVEP at λ, see [1], [11], [16].

Proposition 2.1. *Let $A \in L(X, Y)$ and $B, C \in L(Y, X)$ satisfy operator equation $ABA = ACA$. Then we have the followings;*
(a) Suppose A is injective and AC has a finite ascent. Then BA has a finite ascent;
(b) Suppose that A is surjective and BA has a finite descent. Then AC has a finite descent.

Proof. (a) Suppose that AC has a finite ascent p. It is clear from Lemma 1.2 that $A(BA)^{n+1} = (AC)^{n+1}A$ for all $n = 1, 2, \cdots$. If $(BA)^{p+1}x = 0$ then $(AC)^{p+1}Ax = A(BA)^{p+1}x = 0$ and $Ax \in ker(AC)^{p+1} = ker(AC)^p$. Thus $A(BA)^p x = (AC)^p Ax = 0$. Since A is injective, $(BA)^p x = 0$ and hence BA has a finite ascent p.

(b) Suppose that BA has a finite descent q. Since A is surjective, $Y = A(X)$. It follows from Lemma 1.2 that $(AC)^{q+1}A(X) = A(BA)^{q+1}(X) = A(BA)^q(X)$. Thus we have

$$(AC)^{q+1}(Y) = (AC)^{q+1}A(X) = A(BA)^q(X) = (AC)^q A(X) = (AC)^q(Y).$$

Hence AC has a finite descent q. $\square$

In [12] M. Mbekhta introduced two important subspaces of X.

Definition 2.2. ([12]) The quasi-nilpotent part of an operator $A \in L(X)$ is the set

$$H_0(A) := \{x \in X : \lim_{n\to\infty} \|A^n x\|^{\frac{1}{n}} = 0\},$$

and the analytic core of $A \in L(X)$ is the set $K(A)$ of all $x \in X$ such that there exist a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ and $c > 0$ for which $x = x_0$, $Ax_{n+1} = x_n$ and $\|x_n\| \leq c^n \|x\|$ for every $n = 0, 1, \cdots$.

It is clear that both $H_0(A)$ and $K(A)$ are $A-$hyperinvariant linear subspaces of X and fail to be closed in general. Moreover, $H_0(A) = \mathcal{X}_A(\{0\})$, $K(A) = X_A(\mathbb{C} \setminus \{0\})$, $T(K(A)) = K(A)$ and if A is quasi-nilpotent then $K(A) = \{0\}$ and the identity $K(A) = X$ holds precisely when A is surjective, see Theorem 1.18 and Theorem 2.22 of [1].

Proposition 2.3. *Let $A \in L(X,Y)$ and $B \in L(Y,X)$. Then we have*
(a) $H_0(AB)$ is closed if and only if $H_0(BA)$ is closed;
(b) if A and B are injective, then $K(AB)$ is closed if and only if $K(BA)$ is closed.

Proof. It is direct consequence of Corollary 3.3 and Corollary 3.7 of [22]. □

Definition 2.4. ([19]) An operator $A \in L(X)$ is said to have the property (K) at a point $\lambda_0 \in \mathbb{C}$ if both $H_0(\lambda_0 I - A)$ and $K(\lambda_0 I - A)$ are closed, and

$$X = H_0(\lambda_0 I - A) + K(\lambda_0 I - A).$$

Moreover, A is said to have the property (K) if A has the property (K) at every point $\lambda \in \mathbb{C}$.

It is clear that if A has the property (K), then A has SVEP and $H_0(\lambda I - A) = X_A(\{\lambda\})$ is closed for every $\lambda \in \mathbb{C}$. Clearly, if $A \in L(X)$ has the property (K) at λ_0, then $H_0(\lambda_0 I - A) \cap K(\lambda_0 I - A) = \{0\}$. Obviously, if $A \in L(X)$ has the property (K) at λ_0 then both A and A^* have the SVEP at λ_0. Also, it is well known that A has the property (K) at every $\lambda \in \rho(A)$ and if $A \in L(X)$ is a Riesz operator, then A has the property (K) at every point $\lambda \in \sigma(A) \setminus \{0\}$. Moreover, A has the property (K) at the point λ_0 if and only if $\lambda_0 \in \sigma(A)$ is an isolated point of the spectrum $\sigma(A)$, see [19] and [20].

The *surjectivity spectrum* $\sigma_{sur}(A)$ of $A \in L(X)$ is defined as the set of all $\lambda \in \mathbb{C}$ such that the operator $\lambda I - A$ is not surjective. It is well known that $\sigma_{sur}(A)$ is a compact subset of $\mathbb{C}$ that contains the boundary of $\sigma(A)$, and $X = \mathcal{X}_A(\sigma_{sur}(A)) = X_A(\sigma_{sur}(A))$. The *approximate point spectrum* $\sigma_{ap}(A)$ of $A \in L(X)$ is defined as the set of all $\lambda \in \mathbb{C}$ such that the operator $\lambda I - A$ is not bounded below.

Theorem 2.5. *Let $A \in L(X)$ and let $0 \in \sigma_{sur}(A)$. If 0 is isolated in $\sigma_{sur}(A)$ then A has the property (K) at 0.*

Proof. Suppose that 0 is an isolated in $\sigma_{sur}(A)$. Then by Theorem 1 of [18], $H_0(A)$ and $K(A)$ are closed. It is clear that $H_0(A) + K(A) \subseteq X$ and

$$\mathcal{X}(\sigma_{sur}(A) \setminus \{0\}) \subseteq X_A(\mathbb{C} \setminus \{0\}) = K(A).$$

It follows from Proposition 3.3.1 of [16] that

$$X = \mathcal{X}_A(\sigma_{sur}(A)) = X_A(\{0\}) + X_A(\sigma_{sur}(A) \setminus \{0\}).$$

It follows that $X = H_0(A) + K(A)$. Hence A has the property (K) at 0. □

It is well known that $\sigma_{ap}(A)$ is a compact subset of $\mathbb{C}$ and $\sigma_{ap}(A) = \sigma_{sur}(A^*)$, where A^* denote the conjugate operator of $A \in L(X)$.

Corollary 2.6. *Let $A \in L(X)$. Suppose that $0 \in \sigma_{ap}(A)$ is isolated in $\sigma_{ap}(A)$. Then A^* has the property (K) at 0.*

Corollary 2.7. *Let $A \in L(X)$ and let $Z(x)$ be the set of all accumulation points of $\sigma_A(x)$. If $0 \notin \bigcap_{x\in X} Z(x)$ then A has the property (K) at 0.*

Corollary 2.8. *Let $A \in L(X)$. If $\sigma(A)$ is a finite set, then A has the property (K) at 0.*

Theorem 2.9. *Let $A \in L(X, Y)$ and $B \in L(Y, X)$. Suppose that both A and B are bijective. Then BA has the property (K) at 0 if and only if AB has the property (K) at 0.*

Proof. Suppose that BA has the property (K) at 0. Then both $H_0(BA)$ and $K(BA)$ are closed and $X = H_0(BA) + K(BA)$. Note that $H_0(BA) = X_{BA}(\{0\})$ and $K(BA) = X_{BA}(\mathbb{C} \setminus \{0\})$. It follows from Proposition 2.3 that both $H_0(AB)$ and $K(AB)$ are closed. Because A is bijective, $Y = A(X) = A(X_{BA}(\{0\}) \oplus X_{BA}(\mathbb{C} \setminus \{0\}))$. If $y \in Y$, then $y = Aa + Ab$ for some $a \in X_{BA}(\{0\})$ and $b \in X_{BA}(\mathbb{C} \setminus \{0\})$. It follows from Corollary 1.5 that

$$\sigma_{AB}(Aa) \subseteq \sigma_{BA}(a) \subseteq \{0\} \text{ and } \sigma_{AB}(Ab) \subseteq \sigma_{BA}(b) \subseteq \mathbb{C} \setminus \{0\}.$$

Thus $Aa \in X_{AB}(\{0\})$ and $Ab \in X_{AB}(\mathbb{C} \setminus \{0\})$. We have

$$y \in X_{AB}(\{0\}) + X_{AB}(\mathbb{C} \setminus \{0\}).$$

This implies that $Y = X_{AB}(\{0\}) + X_{AB}(\mathbb{C} \setminus \{0\}) = H_0(AB) + K(AB)$. Hence AB has the property (K) at 0. The reverse implication is obtaind by symmetry. $\square$

Definition 2.10. Let $A \in L(X)$ be a bounded linear operator on a complex Banach space X. The *localizable spectrum* $\sigma_{loc}(A)$ of an operator $A \in L(X)$ is defined as a set of all $\lambda \in \mathbb{C}$ for which $X_A(\overline{V}) \neq \{0\}$ for every open neighborhood V of λ.

Obviously, $\sigma_{loc}(A)$ is a closed subset of $\sigma(A)$, and $\sigma_{loc}(A)$ contains the point spectrum $\sigma_p(A)$ and is included in the approximate point spectrum $\sigma_{ap}(A)$ of A. It is clear that if A does not have the SVEP, then $X_A(\phi) \subseteq X_A(\overline{V})$ for every open neighborhood V of $\lambda \in \mathbb{C}$, and hence $\sigma_{loc}(A) = \sigma(A)$. The localizable spectrum plays an important role in the theory of invariant subspaces, see [9], [14], [18], [19].

Theorem 2.11. *Let $A \in L(X, Y)$ and $B, C \in L(Y, X)$ satisfying operator equation $ABA = ACA$. If A and B are injective then $\sigma_{loc}(AC) = \sigma_{loc}(BA)$.*

Proof. Suppose that $\lambda \in \sigma_{loc}(BA)$. Then $X_{BA}(\overline{V}) \neq \{0\}$ for all open neighborhood V of λ. Thus there exists nonzero $x \in X$ such that $\sigma_{BA}(x) \subseteq \overline{V}$. It follows from Lemma 2.2 that $\sigma_{AC}(Ax) \subseteq \sigma_{BA}(x) \subseteq \overline{V}$. Since A is injective, $Ax \neq 0$ and so $Y_{AC}(\overline{V}) \neq \{0\}$ for every open neighborhood V of λ. Hence $\lambda \in \sigma_{loc}(AC)$.

Conversely, suppose that $\mu \in \sigma_{loc}(AC)$. Then $Y_{AC}(\overline{U}) \neq \{0\}$ for all open neighborhood U of μ. Thus there exists nonzero $y \in Y$ such that $\sigma_{AC}(y) \subseteq \overline{U}$. It follows from Lemma 2.2 that $\sigma_{BA}(BACy) \subseteq \sigma_{AC}(y) \subseteq \overline{U}$. Clearly, $BACy \neq 0$ and so $Y_{BA}(\overline{U}) \neq \{0\}$ for every open neighborhood U of μ. Hence $\mu \in \sigma_{loc}(BA)$. □

Recall that an operator $A \in L(X,Y)$ is said to be a *quasiaffinity* if A is injective and has dense range.

Theorem 2.12. *Let $A \in L(X,Y)$ and $B, C \in L(Y,X)$ satisfy operator equation $ABA = ACA$. Suppose that A, B and C are quasiaffinities. If BA has a non-trivial closed invariant subspace, then AC has a non-trivial closed invariant subspace.*

Proof. Suppose that BA has a non-trivial closed invariant subspace M. Let $N := \overline{ABA(M)}$. Then clearly N is closed and

$$AC(N) \subseteq AC(\overline{ABA(M)}) \subseteq \overline{ACABA(M)} \subseteq \overline{ACA(M)} = \overline{ABA(M)} = N.$$

Since A and B are injective and has dense range, we have $\{0\} \neq N \neq Y$. Hence AC has a non-trivial closed invariant subspace N. □

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