J. Appl. Math. & Informatics Vol. **38***(2020), No. 3 - 4, pp. 211 - 220*
https://doi.org/10.14317/jami.2020.211

A NEW CLASS OF GENERALIZED POLYNOMIALS ASSOCIATED WITH HERMITE-BERNOULLI POLYNOMIALS

MOULOUD GOUBI

Abstract. In this paper, we introduce and investigate a new class of generalized polynomials associated with Hermite-Bernoulli polynomials of higher order. This generalization is a unification formula of Bernoulli numbers, Bernoulli polynomials, Hermite-Bernoulli polynomials of Dattoli, generalized Hermite-Bernoulli polynomials for two variables of order α and new other families of polynomials depending on any generating function f.

AMS Mathematics Subject Classification : 33C45, 33C99.
Key words and phrases : Hermite polynomials, Bernoulli polynomials, Hermite-Bernoulli polynomials.

1. Introduction

The Hermite polynomials $H_n(x)$ (see [1]) are defined by

$$H_n(x) = n! \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^r (2x)^{n-2r}}{r!(n-2r)!}. \tag{1}$$

We recall that the two variables Hermite Kampé de Fériet polynomials $H_n(x,y)$ (see [2]) sometimes called the higher order Hermite polynomials (see [10]) are given by

$$H_n(x,y) = n! \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \frac{y^r x^{n-2r}}{r!(n-2r)!} \tag{2}$$

and generated by the function

$$e^{xt+yt^2} = \sum_{n\geq 0} H_n(x,y) \frac{t^n}{n!}. \tag{3}$$

C.S. Ryoo (see [14]) studied differential equations arising from the generating function (3) and gave explicit identities for these polynomials. According to the

Received July 6, 2019. Revised January 14, 2020. Accepted January 16, 2020.

identities (1) and (2); the polynomials $H_n(x)$ and $H_n(x,y)$ are connected by the following relationship

$$H_n(2x,-1) = H_n(x).$$

For all real number $c>0$, $H_n(x,y)$ can be extended naturally to polynomials $H_n(x,y;c)$ by considering the generating function

$$c^{xt+yt^2} = \sum_{n\geq 0} H_n(x,y,c)\frac{t^n}{n!}.$$

One remarks that

$$H_n(x,y;c) = H_n\left(x\ln c, y\ln c\right).$$

And for $x=0$ one obtains

$$c^{yt^2} = \sum_{n\geq 0} (\ln c)^n y^n \frac{t^{2n}}{n!}.$$

Furthermore

$$H_n(0,y;c) = \begin{cases} \frac{(2k)!}{k!}(\ln c)^k y^k , & \text{if } n=2k, \\ 0 , & \text{otherwise.} \end{cases} \tag{4}$$

It is well-known that the generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$ of order $\alpha \in \mathbb{C}\setminus\{0\}$ are defined by the generating function

$$\left(\frac{t}{e^t-1}\right)^\alpha e^{xt} = \sum_{n\geq 0} B_n^{(\alpha)}(x)\frac{t^n}{n!}. \tag{5}$$

The generalized Hermite-Bernoulli polynomials ${}_HB_n^{(\alpha)}(x,y)$ for two variables of order α which were introduced and investigated by Pathan [12], are given by

$$\left(\frac{t}{e^t-1}\right)^\alpha e^{xt+yt^2} = \sum_{n\geq 0} {}_HB_n^{(\alpha)}(x,y)\frac{t^n}{n!}. \tag{6}$$

These polynomials are a generalization of Bernoulli numbers, Bernoulli polynomials, Hermite polynomials and Hermite-Bernoulli polynomials introduced and studied by Dattoli and al. [4]; which are given by the generating function

$$\left(\frac{t}{e^t-1}\right) e^{xt+yt^2} = \sum_{n\geq 0} {}_HB_n(x,y)\frac{t^n}{n!} \tag{7}$$

different of degenerate Hermite-Bernoulli polynomials studied by H. Haroon and A. K. Waseem (see [9]), and Hermite-Bernoulli Polynomials attached to a Dirichlet character studied by A. Serkan and al. in [15].

Otherwise let the generating function $f(t) = \sum_{n\geq 0} b_n \frac{t^n}{n!}$, with $f(0)=b_0\neq 0$. Then $F(t) = \frac{t}{f(t)-b_0}$ is a generating function too and generates numbers $B_n^{(f)}$

i.e,

$$F(t) = \sum_{n\geq 0} B_n^{(f)} \frac{t^n}{n!}. \tag{8}$$

But we have

$$(f(t) - b_0)\, F(t) = t$$

and then by using the Cauchy product (for more details about this product we refer to [7]) we conclude that $B_0^{(f)} = b_1^{-1}$ and

$$\sum_{k=0}^{n-1} \binom{n}{k} b_{n-k} B_k^{(f)} = 0,\ n \geq 2. \tag{9}$$

For $b_0 = 1$; $-F(t)/t$ can be seen as a special case of the notion of generating function of the function f ($x = 1$ and $|f(t)| < 1$) introduced in Definition 3.1 and Example 3.1 of our recent work [8]. Since $B_0^{(f)} \neq 0$; $F^\alpha(t)$ is a generating function too, and generates numbers $B_n^{(f,\alpha)}$. Then we have

$$\left(\frac{t}{f(t) - b_0}\right)^\alpha = \sum_{n\geq 0} B_n^{(f,\alpha)} \frac{t^n}{n!}.$$

We did everything to introduce a new generalization of Hermite-Bernoulli polynomials and other related polynomials.

Definition 1.1. The generalized Hermite-Bernoulli polynomials ${}_HB_n^{(f,\alpha)}(x, y; c)$ depending on real number $c > 0$ and the function f are given in means of the generating function

$$\left(\frac{t}{f(t) - b_0}\right)^\alpha c^{xt+yt^2} = \sum_{n\geq 0} {}_HB_n^{(f,\alpha)}(x, y; c) \frac{t^n}{n!}. \tag{10}$$

The family of polynomials ${}_HB_n^{(f,\alpha)}(x, y; c)$ includes the family of generalized Bernoulli polynomials introduced by Pathan and Khan (see [13, Definition 2.2 p.56]) which are defined in means of generating function

$$\left(\frac{t}{a^t - b^t}\right)^\alpha c^{xt+yt^2} = \sum_{n\geq 0} B_n^{(\alpha)}(x, y; a, b, c) \frac{t^n}{n!},\ a \neq b, c > 0$$

just taking $f(t) = a^t - b^t + 1$.

In this work, we give the explicit formula of ${}_HB_n^{(f,\alpha)}(x, y; c)$ and apply the result to some special case of the function f. Which goes alone to obtain an improvement of [13, Theorem 2.7, p. 57] and other important results.

2. Main results

For any complex number α the extended binomial coefficient is given by

$$\binom{\alpha}{k} = \frac{(\alpha)_k}{k!}, \text{ with } (\alpha)_k = \alpha(\alpha-1)\cdots(\alpha-k+1).$$

If $\alpha \in \mathbb{N}$ we obtain the standard binomial coefficient

$$\binom{\alpha}{k} = \begin{cases} \frac{\alpha!}{k!(\alpha-k)!}, & \text{if } k \leq \alpha, \\ 0, & \text{otherwise.} \end{cases}$$

And the multinomial coefficients of order n are defined by

$$\binom{k}{k_1 \cdots k_n} = \frac{k!}{k_1! \cdots k_n!}$$

where $k_1 + \cdots + k_n = k$. Which is identical with binomial coefficient for $n = 2$.

Theorem 2.1. *Let $\alpha \in \mathbb{C} \setminus \{0\}$ and the set*

$$\pi_n(k) = \{(k_1, \cdots, k_n) \in \mathbb{N}^n \setminus k_1 + \cdots + k_n = k, k_1 + 2k_2 + \cdots + nk_n = n\}$$

then we have

$$\begin{aligned} &\frac{{}_HB_n^{(f,\alpha)}(x,y;c)}{n!} \\ = &\sum_{k=0}^{n} \sum_{(k_1,\cdots,k_n)\in\pi_n(k)} \binom{-\alpha}{k}\binom{k}{k_1\cdots k_n} b_1^{-\alpha-k} \\ &\times \prod_{i=1}^{n}\left(\sum_{j=0}^{i}\frac{b_{j+1}H_{i-j}\left(-\frac{x\ln c}{\alpha}, -\frac{y\ln c}{\alpha}\right)}{(j+1)!(i-j)!}\right)^{k_i}. \end{aligned} \tag{11}$$

Corollary 2.2.

$$\frac{B_n^{(f,\alpha)}}{n!} = \sum_{k=0}^{n} \sum_{(k_1,\cdots,k_n)\in\pi_n(k)} \binom{-\alpha}{k}\binom{k}{k_1\cdots k_n} b_1^{-\alpha-k} \prod_{i=1}^{n}\left(\frac{b_{i+1}}{(i+1)!}\right)^{k_i}. \tag{12}$$

$$\frac{B_n^{(f)}}{n!} = \sum_{k=0}^{n} \sum_{(k_1,\cdots,k_n)\in\pi_n(k)} (-1)^k \binom{k}{k_1\cdots k_n} b_1^{-1-k} \prod_{i=1}^{n}\left(\frac{b_{i+1}}{(i+1)!}\right)^{k_i}. \tag{13}$$

2.1. Proof of main results.

Let $h(t) = \sum_{n\geq 0} a_n t^n$ be a generating function with $a_0 \neq 0$. Then $h^\alpha(t)$ is a generating function too. Denoting $h^\Delta(n,\alpha)$ (see [11]) the numbers generated by $h^\alpha(t)$ then their explicit formula is given by the following lemma.

Lemma 2.3. *We have $h^\Delta(0,\alpha) = a_0^\alpha$ and*

$$h^\Delta(n,\alpha) = \sum_{k=0}^{n} \sum_{(k_1,\cdots,k_n)\in\pi_n(k)} \binom{\alpha}{k}\binom{k}{k_1\cdots k_n} a_1^{k_1}\cdots a_n^{k_n} a_0^{\alpha-k}, \ n \geq 1. \tag{14}$$

Proof. We consider the auxiliary function $g(t) = t^\alpha$ then $g \circ h(t) = h^\alpha(t)$ is a generating function. Since

$$h^\alpha(t) = \sum_{n\geq 0} h^\Delta(n,\alpha) t^n$$

we deduce that

$$\frac{d^n h^\alpha(t)}{dt^n}|_{t=0} = h^\Delta(n,\alpha) n!.$$

But from the Faà di Bruno formula (see [5]) we have $(g \circ h)^{(0)}(t) = g \circ h(t)$ and

$$\begin{aligned}&(g\circ h)^{(n)}(t)\\ &= \sum_{k=0}^{n} \sum_{(k_1,\cdots,k_n)\in\pi_n(k)} \frac{n!}{k_1!\cdots k_n!}\left(g^{(k)}\circ h(t)\right)\prod_{i=1}^{n}\left(\frac{h^{(i)}(t)}{i!}\right)^{k_i},\ n\geq 1.\end{aligned}$$

Furthermore

$$\begin{aligned}&(g\circ h)^{(n)}(t)\\ &= \sum_{k=0}^{n} \sum_{(k_1,\cdots,k_n)\in\pi_n(k)} \frac{n!}{k_1!\cdots k_n!}(\alpha)_k h^{\alpha-k}(t)\prod_{i=1}^{n}\left(\frac{h^{(i)}(t)}{i!}\right)^{k_i},\ n\geq 1.\end{aligned}$$

which means that

$$(g\circ h)^{(n)}|_{t=0} = \sum_{k=0}^{n} \sum_{(k_1,\cdots,k_n)\in\pi_n(k)} \frac{n!}{k_1!\cdots k_n!}(\alpha)_k a_0^{\alpha-k}\prod_{i=1}^{n} a_i^{k_1},\ n\geq 1.$$

Finally $h^\Delta(0,\alpha) = (g\circ h)^{(0)}(0) = a_0^\alpha$ and

$$h^\Delta(n,\alpha) = \sum_{k=0}^{n}\frac{1}{k!}\sum_{(k_1,\cdots,k_n)\in\pi_n(k)}\binom{k}{k_1\cdots k_n}(\alpha)_k a_0^{\alpha-k} a_1^{k_1}\cdots a_n^{k_n},\ n\geq 1.$$

□

2.2. Proof of Theorem.2.1. For $\alpha \in \mathbb{C}\setminus\{0\}$ we have

$$\left(\frac{t}{f(t)-b_0}\right)^\alpha c^{xt+yt^2} = \left(\frac{f(t)-b_0}{t}e^{-\frac{x\ln c}{\alpha}t-\frac{y\ln c}{\alpha}t^2}\right)^{-\alpha}.$$

But

$$\frac{f(t)-b_0}{t} = \sum_{n\geq 0} b_{n+1}\frac{t^n}{(n+1)!}$$

and

$$e^{-\frac{x\ln c}{\alpha}t-\frac{y\ln c}{\alpha}t^2} = \sum_{n\geq 0} H_n\left(-\frac{x\ln c}{\alpha}, -\frac{y\ln c}{\alpha}\right)\frac{t^n}{n!}.$$

The Cauchy product of the last two functions conducts to

$$\frac{f(t)-b_0}{t}e^{-\frac{x\ln c}{\alpha}t-\frac{y\ln c}{\alpha}t^2} = \sum_{n\geq 0}\sum_{k=0}^{n}\frac{b_{k+1}H_{n-k}\left(-\frac{x\ln c}{\alpha}, -\frac{y\ln c}{\alpha}\right)}{(k+1)!(n-k)!}t^n.$$

Let us $h(t) = \frac{f(t)-b_0}{t} e^{-\frac{x \ln c}{\alpha} t - \frac{y \ln c}{\alpha} t^2}$, then we remark that $a_0 = b_1 \neq 0$ and for $n \geq 1$,

$$a_n = \sum_{k=0}^{n} \frac{b_{k+1} H_{n-k}\left(-\frac{x \ln c}{\alpha}, -\frac{y \ln c}{\alpha}\right)}{(k+1)(n-k)!}.$$

Substituting these values in the identity (14) Lemma 2.3 we obtain the identity (11) Theorem 2.1.

2.3. Proof of Corollary 2.2. One takes $x = y = 0$ in the identity (11) Theorem 2.1 to deduce the identity (12) Corollary 2.2.

For the second identity (16) Corollary 2.2 we take $x = y = 0$ and $\alpha = 1$ in the identity (11) Theorem 2.1 and we use the fact that $\binom{-1}{k} = (-1)^k$ to conclude.

3. Applications

3.1. Exponential function. In the special case $f(t) = e^t$ and $c = e$, the sequence b_n is constant and equal 1. We have already proved the following formula for generalized Hermite-Bernoulli polynomials (the proof is left as an exercise).

Theorem 3.1.

$$\frac{{}_H B_n^{(\alpha)}(x,y)}{n!} = \sum_{k=0}^{n} \sum_{(k_1,\cdots,k_n)\in\pi_n(k)} \binom{-\alpha}{k} \binom{k}{k_1 \cdots k_n} \prod_{i=1}^{n} \left(\sum_{j=0}^{i} \frac{H_{i-j}\left(-\frac{x}{\alpha}, -\frac{y}{\alpha}\right)}{(j+1)!(i-j)!} \right)^{k_i}. \quad (15)$$

This identity conducts directly to explicit formula of well-known Bernoulli numbers B_n^{α} and B_n.

Corollary 3.2.

$$\frac{B_n^{(\alpha)}}{n!} = \sum_{k=0}^{n} \sum_{(k_1,\cdots,k_n)\in\pi_n(k)} \binom{-\alpha}{k} \binom{k}{k_1 \cdots k_n} \prod_{i=1}^{n} \left(\frac{1}{(i+1)!} \right)^{k_i}. \quad (16)$$

$$\frac{B_n}{n!} = \sum_{k=0}^{n} \sum_{(k_1,\cdots,k_n)\in\pi_n(k)} (-1)^k \binom{k}{k_1 \cdots k_n} \prod_{i=1}^{n} \left(\frac{1}{(i+1)!} \right)^{k_i}. \quad (17)$$

The proof consists to take $x = y = 0$ and remark that $H_{i-j}(0,0) = 1$ for $j = i$ and zero otherwise. The identity (17) is an advanced expression of B_n which help us to compute directly the Bernoulli numbers without using the well-known recurrence formula of these numbers.

3.2. Geometric function. We consider for example the generating function

$$f(t)=\frac{1}{1-t}=\sum_{n\geq 0}t^n,\ \ |t|<1$$

then $f(t)-1=\frac{t}{1-t}$ furthermore $\frac{t}{f(t)-1}=1-t$ which means that $B_0^{(f)}=1$, $B_1^{(f)}=-1$ and $B_n^{(f)}=0$ for $n\geq 1$. In means of the identity (14) Lemma 2.3 we conclude that

$$\left(\frac{t}{f(t)-1}\right)^{\alpha}=(1-t)^{\alpha}=\sum_{n\geq 0}\binom{\alpha}{n}(-1)^n t^n.$$

Furthermore by using Cauchy product [7, 6] we obtain

$$\left(\frac{t}{f(t)-1}\right)^{\alpha}c^{xt+yt^2}=\sum_{n\geq 0}\sum_{k=0}^{n}\binom{\alpha}{k}(-1)^k H_{n-k}\left(x\ln c,y\ln c\right)t^n$$

and then

$$\frac{{}_HB_n^{(f,\alpha)}(x,y;c)}{n!}=\sum_{k=0}^{n}\binom{\alpha}{k}(-1)^k H_{n-k}\left(x\ln c,y\ln c\right).$$

According to the identity (11) Theorem 2.1, we have already proved the following theorem.

Theorem 3.3.

$$\begin{aligned}&\sum_{k=0}^{n}\binom{\alpha}{k}(-1)^k H_{n-k}\left(x\ln c,y\ln c\right)\\ &=\sum_{k=0}^{n}\sum_{(k_1,\cdots,k_n)\in\pi_n(k)}\binom{-\alpha}{k}\binom{k}{k_1\cdots k_n}\\ &\qquad\times\prod_{i=1}^{n}\left(\sum_{j=0}^{i}\frac{H_{i-j}\left(-\frac{x\ln c}{\alpha},-\frac{y\ln c}{\alpha}\right)}{(i-j)!}\right)^{k_i}.\end{aligned}\tag{18}$$

Furthermore for $c=e$ we obtain the following result

Corollary 3.4.

$$\begin{aligned}&\sum_{k=0}^{n}\sum_{(k_1,\cdots,k_n)\in\pi_n(k)}\binom{-\alpha}{k}\binom{k}{k_1\cdots k_n}\prod_{i=1}^{n}\left(\sum_{j=0}^{i}\frac{H_{i-j}\left(-\frac{x}{\alpha},-\frac{y}{\alpha}\right)}{(i-j)!}\right)^{k_i}\\ =&\sum_{k=0}^{n}\binom{\alpha}{k}(-1)^k H_{n-k}\left(x,y\right).\end{aligned}\tag{19}$$

and if $x=y=0$ we get

Corollary 3.5.

$$\binom{\alpha}{n} = (-1)^n \sum_{k=0}^{n} \sum_{(k_1,\cdots,k_n)\in\pi_n(k)} \binom{-\alpha}{k}\binom{k}{k_1\cdots k_n}. \tag{20}$$

This identity shows that $\binom{\alpha}{k}$ is a linear combination of numbers $\binom{-\alpha}{k}$ for $1 \leq k \leq n$. And gives for example $\binom{-1}{n} = (-1)^n$. Furthermore if $n \geq 2$ we obtain the identity

$$\sum_{k=0}^{n} (-1)^k \left(\sum_{(k_1,\cdots,k_n)\in\pi_n(k)} \binom{k}{k_1\cdots k_n} \right) = 0.$$

4. Generalized Bernoulli polynomials

The generalized Bernoulli polynomials $B_n^{(\alpha)}(x,y;a,b,c)$ introduced by MA. Pathan and WA. Khan (see [13]) admit the following explicit formula

Theorem 4.1.

$$\begin{aligned} &\frac{B_n^{(\alpha)}(x,y;a,b,c)}{n!} \\ = &\sum_{k=0}^{n} \sum_{(k_1,\cdots,k_n)\in\pi_n(k)} \binom{-\alpha}{k}\binom{k}{k_1\cdots k_n}\left(\ln\frac{a}{b}\right)^{-\alpha-k} \\ &\times \prod_{i=1}^{n}\left(\sum_{j=0}^{i} \frac{\left((\ln a)^{j+1}-(\ln b)^{j+1}\right) H_{i-j}\left(-\frac{x\ln c}{\alpha}, -\frac{y\ln c}{\alpha}\right)}{(j+1)!(i-j)!} \right)^{k_i}. \end{aligned} \tag{21}$$

Proof. To get the proof, just take $f(t) = a^t - b^t + 1$ and then

$$f(t) = 1 + \sum_{n\geq 1} \left((\ln a)^n - (\ln b)^n\right) \frac{t^n}{n!}$$

thus $b_0 = 1$ and $b_n = (\ln a)^n - (\ln b)^n$ for $n \geq 0$. Furthermore $B_n^{(\alpha)}(x,y;a,b,c) = B_n^{(f,\alpha)}(x,y;c)$. ☐

We have

$$\begin{aligned} &\sum_{j=0}^{i} \frac{\left((\ln a)^{j+1}-(\ln b)^{j+1}\right) H_{i-j}\left(-\frac{x\ln c}{\alpha}, -\frac{y\ln c}{\alpha}\right)}{(j+1)!(i-j)!} \\ = &\sum_{j=0}^{i} \frac{\left((\ln a)^{i-j+1}-(\ln b)^{i-j+1}\right) H_{j}\left(-\frac{x\ln c}{\alpha}, -\frac{y\ln c}{\alpha}\right)}{(i-j+1)!j!} \end{aligned}$$

, then as a consequence of the identity (21) Theorem 4.1 and the expression (4) of $H_n(0,y;c)$ we obtain

$$\begin{aligned} &\frac{B_n^{(\alpha)}(0,y;a,b,c)}{n!} \\ = &\sum_{k=0}^{n} \sum_{(k_1,\cdots,k_n)\in\pi_n(k)} \binom{-\alpha}{k}\binom{k}{k_1\cdots k_n}\left(\ln\frac{a}{b}\right)^{-\alpha-k} \\ &\times\prod_{i=1}^{n}\left(\sum_{j=0}^{\lfloor\frac{i}{2}\rfloor}\frac{\left((\ln a)^{i-2j+1}-(\ln b)^{i-2j+1}\right)(\ln c)^j(-y)^j}{(i-2j+1)!j!\alpha^j}\right)^{k_i}. \end{aligned}$$

which is an improvement of the identity

$$B_n^{(\alpha)}(0,y;a,b,c)=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{n!}{k!(n-2k)!}(\ln c)^k B_{n-2k}^{(\alpha)}(a,b)y^k$$

showed in the work [13].

5. Conclusion

In this work we introduced a new family of polynomials attached to a any generating function f not vanishing on zero. This family is a unification formula of Bernoulli numbers, Bernoulli polynomials, Hermite-Bernoulli polynomials, generalized Hermite-Bernoulli polynomials for two variables of higher order.

In section 2, we studied the family ${}_HB_n^{(f,\alpha)}(x,y;c)$ and stated its explicit formula (Theorem 2.1). Furthermore explicit formula for numbers $B_n^{(f,\alpha)}$ and $B_n^{(f)}$ are deduced (Corollary 2.2).

In section 3 we apply this result to some special cases in order to give a closed formula for the polynomials ${}_HB_n^{(\alpha)}(x,y)$. Which goes alone to give a new reformulation of Bernoulli numbers B_n without using their recurrence formula (Corollary 3.2); based on a special partition of the number n.

Finally, in the last section we revisit polynomials $B_n^{(\alpha)}(x,y;a,b,c)$ introduced by MA. Pathan and WA. Khan and get their explicit formula; which includes an improvement of [13, Theorem 2.7, p.57].

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Mouloud Goubi received M.Sc. (2001) from USTHB University Algeirs and Ph.D. (2016) at the same University. Since 2002 he has been at Mentouri University and after 2006 at UMMTO University. His research interests include Riemann zeta function, Dedekind sums and generating functions.

Department of Mathematics, Faculty of Sciences, UMMTO University, Tizi-Ouzou 15000, Algeria.
Laboratory of Algebra and Number Theory USTHB Algeirs.
e-mail: `mouloud.goubi@ummto.dz`