Bull. Korean Math. Soc. **57** (2020), No. 3, pp. 803–813
https://doi.org/10.4134/BKMS.b190531
pISSN: 1015-8634 / eISSN: 2234-3016

FINITELY GENERATED G-PROJECTIVE MODULES OVER PVMDS

Kui Hu, Jung Wook Lim, and Shiqi Xing

Abstract. Let M be a finitely generated G-projective R-module over a PVMD R. We prove that M is projective if and only if the canonical map $\theta : M \bigotimes_R M^* \to \mathrm{Hom}_R(\mathrm{Hom}_R(M,M),R)$ is a surjective homomorphism. Particularly, if $G\text{-}gldim(R) \leqslant \infty$ and $\mathrm{Ext}^i_R(M,M) = 0$ $(i \geqslant 1)$, then M is projective.

1. Introduction

Throughout this note, all rings are commutative with identity element and all modules are unitary. For convenience, we denote the R-module $\mathrm{Hom}_R(X,R)$ and $\mathrm{Hom}_R(\mathrm{Hom}_R(X,R),R)$ by X^* and X^{**} respectively for any R-module X.

Let R be a domain with quotient field K. For a fractional ideal J of R, J^{-1} is defined as follows:

$$J^{-1} = \{x \in K \mid xJ \subset R\}.$$

A finitely generated ideal J is said to be a GV-ideal (denoted by $J \in GV(R)$) if $J^{-1} = R$. An element x of an R-module M is said to be GV-torsion if there exists some $J \in GV(R)$ such that $Jx = 0$. An R-module M is said to be GV-torsion if every element of M is GV-torsion. If for any $x \in M$ and any $J \in GV(R)$, $Jx = 0$ implies $x = 0$, then M is said to be GV-torsion-free. The w-envelope M_w of a GV-torsion-free module M is defined by $M_w = \{x \in E(M) \mid Jx \subset M \text{ for some } J \in GV(R)\}$.

In 1997, Wang and McCasland [19] introduced the concept of w-modules over a domain. A GV-torsion-free module is a w-module if and only if $M_w = M$. A fractional ideal I of a domain R is said to be w-invertible when $(II^{-1})_w = R$. A domain R is said to be a PVMD when any nonzero finitely generated ideal of R is w-invertible. Equivalently, R is a PVMD if and only if $R_{\mathtt{m}}$ is a valuation domain for any maximal w-ideal $\mathtt{m}$. Since coherent domains are finite conductor domains, it follows from [22, Theorem 2] that coherent integrally closed domains are PVMDs.

Received May 28, 2019; Accepted September 5, 2019.
2010 *Mathematics Subject Classification.* 13G05, 13D03.
Key words and phrases. Gorenstein projective module, projective module, PVMD.

It was proved in [3, Corollary 2.3] that the class of all Gorenstein projective (*G-projective* for short) R-modules and the class of all projective R-modules are the same class when $\mathrm{wgldim}(R) \leqslant \infty$. Particularly, if R is a Prüfer domain, then any G-projective R-module is projective. Since PVMDs are a kind of generalization of Prüfer domains, it is natural to ask whether G-projective modules over PVMDs are projective or not. In this paper, we prove that if M is a finitely generated G-projective R-module over a PVMD R, then M is projective if and only if the canonical map $\theta : M \bigotimes_R M^* \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, M), R)$ is a surjective homomorphism.

Recall a left R-module M is called FP-injective [16] if $\mathrm{Ext}^1_R(F, M) = 0$ for any finitely presented left R-module F. Accordingly the FP-injective dimension of M, denoted by $\text{FP-id}_R(M)$, is defined to be the smallest $n \geqslant 0$ such that $\mathrm{Ext}^{n+1}_R(F, M) = 0$ for all finitely presented left R-modules F (if no such n exists, set $\text{FP-id}_R(M) = \infty$). An n-FC ring is a coherent ring with self-FP-injective dimension n, i.e., $\text{FP-id}_R(R) \leqslant n$. These rings were introduced and studied by Ding and Chen in [8] and [9]. An n-FC ring is also called "a Ding-Chen ring" in [12]. In this paper, we prove that if R is a coherent and integrally closed domain such that $\text{FP-id}_R(R) \leqslant n$ (or $\mathrm{wGgldim}(R) \leqslant n$) and if M is a finitely generated G-projective R-module such that $\mathrm{Ext}^i_R(M, M) = 0$ $(i = 1, 2, \ldots, n-1)$, then M is projective. We also prove that if R is a PVMD such that $\text{G-gldim}(R) \leqslant \infty$, then finitely generated self-orthogonal G-projective R-modules are projective. For unexplained concepts and notations, one can refer to [1, 10, 13, 15].

2. Weak Gorenstein projective modules over PVMDs

The author in [11] introduced weak Gorenstein projective modules. Recall that an R-module M is called weak Gorenstein projective if there exists an exact sequence of projective R-modules $P = \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$ such that: $M = \ker(P^0 \to P^1)$. Obviously, any Gorenstein projective module is weak Gorenstein projective. Let R be a PVMD and M be a weak Gorenstein projective R-module. Next we will prove that M_m is a projective R_m module for any maximal w-ideal m. First we need the following lemma.

Lemma 2.1. *Let R be a ring such that* $\mathrm{wgldim}(R) \leqslant \infty$. *Then any weak Gorenstein projective R-module is projective.*

Proof. Let M be a weak Gorenstein projective R-module. There exists the following exact sequence of projective R-modules:

$$\cdots \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} P_0 \xrightarrow{d_0} P_{-1} \xrightarrow{d_{-1}} P_{-2} \longrightarrow \cdots$$

such that: $M = \mathrm{Im}(d_0)$. Denote the image of d_i by K_i, we get the following short exact sequences:

$$0 \longrightarrow K_i \longrightarrow P_{i-1} \longrightarrow K_{i-1} \longrightarrow 0, \ (i \in \mathbb{Z}).$$

By adding such sequences, we get the following short exact sequence:

$$(1) \qquad 0 \longrightarrow \bigoplus K_i \longrightarrow \bigoplus P_i \longrightarrow \bigoplus K_i \longrightarrow 0.$$

By combining such sequences one by one, we can get an exact sequence of sufficient length:

$$0 \longrightarrow \bigoplus K_i \longrightarrow \bigoplus P_i \longrightarrow \cdots \longrightarrow \bigoplus P_i \longrightarrow \bigoplus K_i \longrightarrow 0\ .$$

Since $\text{wgldim}(R) \leqslant \infty$, it is easy to see that $\bigoplus K_i$ is flat. Considering sequence (1), we will get that $\bigoplus K_i$ is projective by [7, Theorem 2.5]. Because $M = \text{Im}(d_0) = K_0$, we get that M is also projective. □

Theorem 2.2. *Let R be a PVMD and $\mathfrak{m}$ be a maximal w-ideal. If M is a G-projective R-module, then $M_{\mathfrak{m}}$ is a projective $R_{\mathfrak{m}}$-module.*

Proof. Since R is a PVMD and $\mathfrak{m}$ is a maximal w-ideal, $R_{\mathfrak{m}}$ is a valuation domain, and hence $\text{wgldim}(R_{\mathfrak{m}}) \leqslant 1$. By Lemma 2.1, we only need to prove that $M_{\mathfrak{m}}$ is a weak Gorenstein projective $R_{\mathfrak{m}}$-module. Because M is a Gorenstein projective R-module, there exists an exact sequence of projective R-modules:

$$\cdots \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} P_0 \xrightarrow{d_0} P_{-1} \xrightarrow{d_{-1}} P_{-2} \longrightarrow \cdots$$

such that: $M = \text{Im}(d_0)$. Since $R_{\mathfrak{m}}$ is a flat R-module, we can get the following exact sequence of projective $R_{\mathfrak{m}}$-modules:

$$\cdots \xrightarrow{d'_n} (P_{n-1})_{\mathfrak{m}} \xrightarrow{d'_{n-1}} \cdots \xrightarrow{d'_1} (P_0)_{\mathfrak{m}} \xrightarrow{d'_0} (P_{-1})_{\mathfrak{m}} \xrightarrow{d'_{-1}} (P_{-2})_{\mathfrak{m}} \longrightarrow \cdots$$

such that: $M_{\mathfrak{m}} = \text{Im}(d'_0)$. Therefore $M_{\mathfrak{m}}$ is a weak Gorenstein projective $R_{\mathfrak{m}}$-module. □

In 2010, Yin et al. [21] defined w-modules over general commutative rings. A finitely generated ideal J of R is called a Glaz-Vasconcelos ideal (*GV-ideal* for short), denoted by $J \in GV(R)$, if the natural homomorphism $\varphi : R \to J^* = \text{Hom}_R(J, R)$ is an isomorphism. Equivalently, $J \in GV(R)$ if and only if $\text{Hom}_R(R/J, R) = 0$ and $\text{Ext}^1_R(R/J, R) = 0$. A module M is called a w-module if $\text{Hom}_R(R/J, M) = 0$ and $\text{Ext}^1_R(R/J, M) = 0$ for any $J \in GV(R)$. This definition is consistent with that over domains. It is easy to see that free modules, hence projective modules are w-modules. Next we show that weak Gorenstein projective R-modules are w-modules.

Proposition 2.3. *If M is a weak Gorenstein projective R-module, then M is a w-module.*

Proof. Since M is weak Gorenstein projective, there exists a short exact sequence:

$$0 \longrightarrow M \longrightarrow P \longrightarrow N \longrightarrow 0,$$

where P is projective and N is a submodule of some free module. Because free modules are GV-torsion-free, N is also GV-torsion-free. Notice that P is also a w-module, we get that M is a w-module by [18, Theorem 6.1.17]. □

Since G-projective modules are weak Gorenstein projective, we surely get that any G-projective modules are w-modules.

Let S be a multiplicative closed set in the ring R, M and N be R-modules. Since we need localization, we consider the following homomorphism of R_S-modules:

$$\varphi : \mathrm{Hom}_R(M,N)_S \to \mathrm{Hom}_{R_S}(M_S,N_S)$$

such that:

$$\varphi(\frac{f}{s})(\frac{x}{1}) = \frac{f(x)}{s},$$

where $s \in S$, $x \in M$ and $f \in \mathrm{Hom}_R(M,N)$. The following lemma is (6) and (8) of [17, Theorem 3.4.8].

Lemma 2.4. *Let S be a multiplicative closed set which is consisted of some nonzero-divisors in the ring R, M and N be R-modules, N be torsion-free.*

(1) *If M is finitely generated, then φ is an isomorphism.*

(2) *If M is a submodule of some finitely presented module F and $(\frac{F}{M})_S$ is a projective R_S-module, then φ is an isomorphism.*

Let R be a domain and M be a finitely generated G-projective R-module. Immediately we get the following isomorphism of $R_{\mathfrak{m}}$-modules:

$$\varphi_M : \mathrm{Hom}_R(M,M)_{\mathfrak{m}} \to \mathrm{Hom}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}},M_{\mathfrak{m}}),$$

where $\mathfrak{m}$ is any prime ideal. Furthermore, we can prove the following proposition.

Proposition 2.5. *Let R be a PVMD and M a finitely generated G-projective R-module and $\mathfrak{m}$ be a maximal w-ideal. Then the canonical homomorphism of $R_{\mathfrak{m}}$-modules*:

$$\eta : \mathrm{Hom}_R(\mathrm{Hom}_R(M,M),R)_{\mathfrak{m}} \to \mathrm{Hom}_{R_{\mathfrak{m}}}(\mathrm{Hom}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}},M_{\mathfrak{m}}),R_{\mathfrak{m}})$$

is an isomorphism.

Proof. By Lemma 2.4, we only need to prove that the module $\mathrm{Hom}_R(M,M)$ is a submodule of some finitely presented module F and $(\frac{F}{\mathrm{Hom}_R(M,M)})_{\mathfrak{m}}$ is a projective $R_{\mathfrak{m}}$-module. Since M is finitely generated G-projective, there exists a short exact sequence of R-modules:

$$0 \longrightarrow M \longrightarrow P \longrightarrow G \longrightarrow 0,$$

where P is finitely generated and projective and G is G-projective by [20, Proposition 2.6]. Without loss of generality, we can assume that $M \subset P$ and P is finitely generated free. Applying the functor $\mathrm{Hom}_R(M,-)$ on this sequence, we get that the following exact sequence:

$$0 \longrightarrow \mathrm{Hom}_R(M,M) \longrightarrow \mathrm{Hom}_R(M,P) \longrightarrow \mathrm{Hom}_R(M,G) \longrightarrow \mathrm{Ext}^1_R(M,M) \longrightarrow 0.$$

Since P is finitely generated free, $\mathrm{Hom}_R(M,P)$ is isomorphic to a finite direct sum of some copies of $M^* = \mathrm{Hom}_R(M,R)$. Since M^* is super finitely presented by [20, Proposition 2.6], $\mathrm{Hom}_R(M,P)$ is finitely presented. So we can let $F=\mathrm{Hom}_R(M,P)$. In order to prove that $(\frac{F}{\mathrm{Hom}_R(M,M)})_{\mathfrak{m}}$ is a projective $R_{\mathfrak{m}}$-module, we apply the functor $\mathrm{Hom}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}},-)$ on the short exact sequence of $R_{\mathfrak{m}}$-modules:

$$0\longrightarrow M_{\mathfrak{m}}\longrightarrow P_{\mathfrak{m}}\longrightarrow G_{\mathfrak{m}}\longrightarrow 0.$$

Because $M_{\mathfrak{m}}$ is a projective $R_{\mathfrak{m}}$-module by Theorem 2.2, we get the following exact sequence:

$$0\longrightarrow \mathrm{Hom}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}},M_{\mathfrak{m}})\longrightarrow \mathrm{Hom}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}},P_{\mathfrak{m}})\longrightarrow \mathrm{Hom}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}},G_{\mathfrak{m}})\longrightarrow 0.$$

Since $G_{\mathfrak{m}}$ is also a projective $R_{\mathfrak{m}}$-module by Theorem 2.2, the $R_{\mathfrak{m}}$-module $\mathrm{Hom}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}},G_{\mathfrak{m}})$ is projective. Therefore by Lemma 2.4,

$$\begin{aligned}(\frac{F}{\mathrm{Hom}_R(M,M)})_{\mathfrak{m}}=(\frac{\mathrm{Hom}_R(M,P)}{\mathrm{Hom}_R(M,M)})_{\mathfrak{m}}&\cong\frac{\mathrm{Hom}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}},P_{\mathfrak{m}})}{\mathrm{Hom}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}},M_{\mathfrak{m}})}\\&\cong \mathrm{Hom}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}},G_{\mathfrak{m}})\end{aligned}$$

is also a projective $R_{\mathfrak{m}}$-module. □

Lemma 2.6 ([21, Theorem 2.8])**.** *Let A be an R-module and M a w-module. Then* $\mathrm{Hom}_R(A,M)$ *is a w-module. In particular, A^* and A^{**} are w-modules. Therefore, reflexive modules are w-modules.*

Since G-projective modules are w-modules, we get that $\mathrm{Hom}_R(M,M)$ is a w-module for any G-projective module M.

Lemma 2.7 ([17, Theorem 8.3.2])**.** *Let A, B be w-modules and $f:A\to B$ a homomorphism. Then f is an isomorphism if and only if $f_{\mathfrak{m}}:A_{\mathfrak{m}}\to B_{\mathfrak{m}}$ is an isomorphism for any maximal w-ideal $\mathfrak{m}$ of R.*

Let M be an R-module. Define

$$\theta: M\otimes_R M^*\to \mathrm{Hom}_R(\mathrm{Hom}_R(M,M),R)$$

by

$$\theta(a\otimes f)(g)=f(g(a)),\ a\in M,\ f\in M^*,\ g\in \mathrm{Hom}_R(M,M).$$

If M is a finitely generated projective module, then θ is an isomorphism by [17, Theorem 3.4.5]. Next we prove that if M is a finitely generated G-projective module over a PVMD, then the converse also holds.

Let R be a domain. For any module X, denote the GV-torsion submodule of X by

$$\mathrm{Tor}_{\mathrm{GV}}(X)=\{x\in X\mid Jx=0 \text{ for some } GV \text{ ideal } J \text{ of } R\}.$$

$\mathrm{Tor}_{\mathrm{GV}}(X)$ is the maximal GV-torsion submodule of X.

Lemma 2.8. *Let R be a PVMD and M be a finitely generated G-projective R-module.*

$$\theta: M\otimes_R M^*\to \mathrm{Hom}_R(\mathrm{Hom}_R(M,M),R)$$

is defined as above. Then both $\mathrm{coker}(\theta)$ *and* $\ker(\theta)$ *are GV-torsion. Furthermore,* $\ker(\theta)$ *is exactly* $\mathrm{Tor}_{\mathrm{GV}}(M\otimes_R M^*)$.

Proof. Since a module N is GV-torsion if and only if $N_{\mathfrak{m}} = 0$ for any maximal w-ideal $\mathfrak{m}$ of R [18, Theorem 6.2.15], we only need to prove that

$$\theta_{\mathfrak{m}} : (M \otimes_R M^*)_{\mathfrak{m}} \to (\mathrm{Hom}_R(\mathrm{Hom}_R(M, M), R))_{\mathfrak{m}}$$

is an isomorphism. Because $M_{\mathfrak{m}}$ is a projective $R_{\mathfrak{m}}$-module, the canonical map:

$$(M_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} M_{\mathfrak{m}}^*) \to (\mathrm{Hom}_{R_{\mathfrak{m}}}(\mathrm{Hom}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}}, M_{\mathfrak{m}}), R_{\mathfrak{m}}))$$

is an isomorphism by Lemma [17, Theorem 3.4.5]. Notice that

$$\eta : \mathrm{Hom}_R(\mathrm{Hom}_R(M, M), R)_{\mathfrak{m}} \to \mathrm{Hom}_{R_{\mathfrak{m}}}(\mathrm{Hom}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}}, M_{\mathfrak{m}}), R_{\mathfrak{m}})$$

is an isomorphism by Proposition 2.5. It follows from the canonical isomorphism $(M \otimes_R M^*)_{\mathfrak{m}} \to (M_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} M_{\mathfrak{m}}^*)$ that $\theta_{\mathfrak{m}}$ is also an isomorphism. □

Let M be an R-module. Define

$$\xi : M \otimes_R M^* \to \mathrm{Hom}_R(M, M)$$

by

$$\xi(x \otimes f)(y) = f(y)x, \ x \in M, \ y \in M, \ f \in M^* = \mathrm{Hom}_R(M, R).$$

The following result can be found in [14, Exercise 2.20].

Lemma 2.9. *Let R be any ring and*

$$\xi : M \otimes_R M^* \to \mathrm{Hom}_R(M, M) = \mathrm{End}(M)$$

be defined as above. Then the following statements are equivalent:

(1) *M is a finitely generated projective module.*
(2) *ξ is an isomorphism.*
(3) *ξ is a surjective homomorphism.*

We sill need the following lemma.

Lemma 2.10. *Let R be a PVMD and M be a finitely generated G-projective R-module.*

$$\xi : M \otimes_R M^* \to \mathrm{Hom}_R(M, M)$$

is defined as above. Then both $\mathrm{coker}(\xi)$ *and* $\ker(\xi)$ *are GV-torsion. Furthermore,* $\ker(\xi)$ *is exactly* $\mathrm{Tor}_{\mathrm{GV}}(M \otimes_R M^*)$.

Proof. Since

$$\varphi_M : \mathrm{Hom}_R(M, M)_{\mathfrak{m}} \to \mathrm{Hom}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}}, M_{\mathfrak{m}})$$

is an isomorphism for any maximal w-ideal m by Lemma 2.4, the proof is similar to that of Lemma 2.8. □

Now we can prove the main theorem of this section.

Theorem 2.11. *Let R be a PVMD and M be a finitely generated G-projective R-module. If θ and ξ are defined as in Lemma 2.8 and Lemma 2.9 respectively, then there exists an isomorphism $\psi : \mathrm{Hom}_R(\mathrm{Hom}_R(M, M), R) \to \mathrm{Hom}_R(M, M)$ and we have the following commutative diagram with exact rows:*

$$\begin{CD}
0 @>>> T @>>> M \otimes M^* @>\theta>> \mathrm{Hom}_R(\mathrm{Hom}_R(M, M), R) @>>> \mathrm{coker}\theta @>>> 0 \\
@. @| @| @VV\psi V @VV\alpha V @. \\
0 @>>> T @>>> M \otimes M^* @>\xi>> \mathrm{Hom}_R(M, M) @>>> \mathrm{coker}\xi @>>> 0
\end{CD}$$

where $T = \mathrm{Tor}_{\mathrm{GV}}(M \otimes_R M^*)$ and α is also an isomorphism.

Proof. By Lemma 2.8 and Lemma 2.10, we have $\ker\theta = \ker\xi = T$. Thus we have the following commutative diagram with exact rows:

$$\begin{CD}
0 @>>> T @>>> M \otimes M^* @>>> \mathrm{Im}\theta @>>> 0 \\
@. @| @| @VV\delta V @. \\
0 @>>> T @>>> M \otimes M^* @>>> \mathrm{Im}\xi @>>> 0
\end{CD}$$

where δ is also an isomorphism by Five Lemma. If we denote the inclusion map $\mathrm{Im}\xi \hookrightarrow \mathrm{Hom}_R(M, M)$ by λ, then $\lambda\delta$ is a homomorphism from $\mathrm{Im}\theta$ to $\mathrm{Hom}_R(M, M)$. Because $\mathrm{Hom}_R(M, M)$ is a w-module and $\mathrm{coker}\theta$ is GV-torsion, $\mathrm{Ext}^1_R(\mathrm{coker}\theta, \mathrm{Hom}_R(M, M)) = 0$ by [18, Theorem 6.2.7]. Now, we denote the module $\mathrm{Hom}_R(M, M)$ by H. By using the functor $\mathrm{Hom}_R(-, H)$ on the short exact sequence

$$0 \longrightarrow \mathrm{Im}\theta \xrightarrow{\ i\ } \mathrm{Hom}_R(H, R) \longrightarrow \mathrm{coker}\theta \longrightarrow 0,$$

we get the following exact sequence:

$$\mathrm{Hom}_R(\mathrm{Hom}_R(H, R), H) \longrightarrow \mathrm{Hom}_R(\mathrm{Im}\theta, H) \longrightarrow 0.$$

This means that for the homomorphism $\lambda\delta : \mathrm{Im}\theta \to H$, there exists a homomorphism ψ from $\mathrm{Hom}_R(H, R)$ to H such that $\lambda\delta = \psi i$. Therefore, we have the following commutative diagram with exact rows:

$$\begin{CD}
0 @>>> \mathrm{Im}\theta @>i>> \mathrm{Hom}_R(H, R) @>>> \mathrm{coker}\theta @>>> 0 \\
@. @V\delta VV @VV\psi V @VV\alpha V @. \\
0 @>>> \mathrm{Im}\xi @>\lambda>> H @>>> \mathrm{coker}\xi @>>> 0
\end{CD}$$

where α is induced by the left commutative square. Now, combining the above two horizontal ladders, we get the desired commutative diagram. Let $\mathtt{m}$ be any maximal w-ideal of R. Then $M_{\mathtt{m}}$ is a projective $R_{\mathtt{m}}$ module by Lemma 2.2. Since $\mathrm{coker}\theta$ and $\mathrm{coker}\xi$ are GV-torsion, $\xi_{\mathtt{m}} = \psi_{\mathtt{m}}\theta_{\mathtt{m}}$. It can be seen from the proof of Lemma 2.8 and Lemma 2.10 that $\xi_{\mathtt{m}}$ and $\theta_{\mathtt{m}}$ are isomorphisms. Therefore $\psi_{\mathtt{m}}$ is also an isomorphism. Since G-projective modules are w-modules, $H^* =$

$\mathrm{Hom}_R(H,R)$ and $H = \mathrm{Hom}_R(M,M)$ are w-modules by Lemma 2.6. Thus ψ is an isomorphism by Lemma 2.7. So α is also an isomorphism by Five Lemma. □

Corollary 2.12. *Let R be a PVMD and M be a finitely generated G-projective R-module. If θ and ξ are defined as in Lemma 2.8 and Lemma 2.9 respectively, then M is projective if and only if θ is a surjective homomorphism.*

Proof. By Lemma 2.9, M is projective if and only if ξ is a surjective homomorphism, that is, if and only if $\mathrm{coker}\xi = 0$. By Lemma 2.11, $\mathrm{coker}\xi \cong \mathrm{coker}\theta$. Therefore, M is projective if and only if $\mathrm{coker}\theta = 0$, that is to say, if and only if θ is a surjective homomorphism. □

3. Some sufficient conditions for *G*-projective modules over PVMDs to be projective

Let R be a ring. An R-module M is called FP-injective (or absolutely pure) if $\mathrm{Ext}_R^1(N,M) = 0$ for all finitely presented left R-modules N. The FP-injective dimension of M, denoted by FP-id(M), is defined to be the smallest nonnegative integer n such that $\mathrm{Ext}_R^{n+1}(F,M) = 0$ for every finitely presented R-module F (if no such n exists, set FP-id$(M) = \infty$).

Lemma 3.1. *Let R be an n-FC ring and M be a finitely generated G-projective R-module. If $\mathrm{Ext}_R^i(M,M) = 0$ $(i = 1, 2, \dots, n-1)$, then*

$$\theta : M \otimes_R M^* \to \mathrm{Hom}_R(\mathrm{Hom}_R(M,M), R)$$

is a surjective homomorphism.

Proof. As a finitely generated submodule of a free module over a coherent ring, M is in fact finitely presented. So we have the following exact sequence

$$\cdots \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \longrightarrow M \longrightarrow 0,$$

where the F_i's are finitely generated free modules. Denote the image of d_i by N_i. Then we have $\mathrm{Ext}_R^1(N_i,M) \cong \mathrm{Ext}_R^{i+1}(M,M) = 0$ $(i = 1, 2, \dots, n-2)$. Using the functor $\mathrm{Hom}_R(-,M)$ on this sequence, we get the following two exact sequences

$$0 \to \mathrm{Hom}_R(N_1,M) \to \mathrm{Hom}_R(F_1,M) \to \cdots \to \mathrm{Hom}_R(F_n,M) \to D \to 0$$

and

$$0 \longrightarrow \mathrm{Hom}_R(M,M) \longrightarrow \mathrm{Hom}_R(F_0,M) \longrightarrow \mathrm{Hom}_R(N_1,M) \longrightarrow 0,$$

where D is the cokernel of the map $\mathrm{Hom}_R(F_{n-1},M) \to \mathrm{Hom}_R(F_n,M)$. Denote $\mathrm{Hom}_R(N_1,M)$ by L. Then we have that $\mathrm{Ext}_R^1(L,R) \cong \mathrm{Ext}_R^{n+1}(D,R)$ since all these $\mathrm{Hom}(F_i,M)$ are Gorenstein projective. It can be seen that all the modules that appeared in these sequences are finitely presented. Because FP-$\mathrm{id}_R(R) \leqslant n$, $\mathrm{Ext}_R^{n+1}(D,R) = 0$ holds. Therefore, $\mathrm{Ext}_R^1(L,R) = 0$. Now, using the functor $\mathrm{Hom}(-,R)$ on the short exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(M,M) \longrightarrow \mathrm{Hom}_R(F_0,M) \longrightarrow L \longrightarrow 0,$$

we get the following exact sequence.

$$\mathrm{Hom}_R(\mathrm{Hom}_R(F_0,M),R) \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(M,M),R) \longrightarrow 0.$$

Thus, we get the following commutative diagram with exact rows:

$$\begin{array}{ccccc} F_0 \otimes_R \mathrm{Hom}_R(M,R) & \longrightarrow & M \otimes_R M^* & \longrightarrow & 0 \\ \downarrow{\scriptstyle \theta_{F_0}} & & \downarrow{\scriptstyle \theta} & & \\ \mathrm{Hom}_R(\mathrm{Hom}_R(F_0,M),R) & \longrightarrow & \mathrm{Hom}_R(\mathrm{Hom}_R(M,M),R) & \longrightarrow & 0. \end{array}$$

Since θ_{F_0} is an isomorphism by [17, Theorem 3.4.5], θ is a surjective homomorphism. □

Recall that the weak Gorenstein global dimensions ($\mathrm{wGgldim}(R)$) of rings [5] is defined as follows: $\mathrm{wGgldim}(R) = \sup\{\mathrm{Gfd}_R(M) \mid M \text{ is an } R\text{-module}\}$. From [9, Theorem 7] (see also [2] and [4]), for a positive integer n and a commutative coherent ring R, $\mathrm{wGgldim}(R) = n$ if and only if R is an n-FC ring. Recall that a finitely generated R-module M is called self-orthogonal [6] if $\mathrm{Ext}_R^i(M,M) = 0$ for any $i \geqslant 1$.

Lemma 3.2. *Let R be a ring such that* $\text{G-gldim}(R) < \infty$ *and M be a finitely generated G-projective R-module. If M is self-orthogonal, then*

$$\theta : M \otimes_R M^* \to \mathrm{Hom}_R(\mathrm{Hom}_R(M,M),R)$$

is a surjective homomorphism.

Proof. Since $\text{G-gldim}(R) < \infty$, M is *super finitely presented* by [20, Corollary 3.4]. Also notice that the module L which appears in the proof of Lemma 3.1 is G-projective, and so $\mathrm{Ext}_R^1(L,R) = 0$. Thus the rest of the proof is similar to that of Lemma 3.1. □

Theorem 3.3. *Let R be a coherent and integrally closed domain such that* FP-$\mathrm{id}_R(R) \leqslant n$ *or* ($\mathrm{wGgldim}(R) \leqslant n$). *If M is a finitely generated G-projective module such that* $\mathrm{Ext}_R^i(M,M) = 0$ $(i = 1, 2, \ldots, n-1)$, *then M is projective.*

Proof. Since R is a coherent integrally closed domain, R is a PVMD. In order to prove that M is projective, we only need to show that θ is a surjective homomorphism by Corollary 2.12. But this follows from Lemma 3.1. □

Theorem 3.4. *Let R be a PVMD with* $\text{G-gldim}(R) < \infty$. *Then any finitely generated self-orthogonal G-projective module over R is projective.*

Proof. The result follows from Corollary 2.12 since θ is a surjective homomorphism by Lemma 3.2. □

Acknowledgements. This work was partially supported by the Department of Mathematics of Kyungpook National University and National Natural Science Foundation of China(Grant No. 11671283). The second author was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2017R1C1B1008085).

References

[1] F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*, Springer-Verlag, New York, 1974.

[2] D. Bennis, *A note on Gorenstein global dimension of pullback rings*, Int. Electron. J. Algebra **8** (2010), 30–44.

[3] D. Bennis and N. Mahdou, *A generalization of strongly Gorenstein projective modules*, J. Algebra Appl. **8** (2009), no. 2, 219–227. https://doi.org/10.1142/S021949880900328X

[4] ______, *Gorenstein global dimensions and cotorsion dimension of rings*, Comm. Algebra **37** (2009), no. 5, 1709–1718. https://doi.org/10.1080/00927870802210050

[5] ______, *Global Gorenstein dimensions*, Proc. Amer. Math. Soc. **138** (2010), no. 2, 461–465. https://doi.org/10.1090/S0002-9939-09-10099-0

[6] D. Bennis and K. Ouarghi, *Self-orthogonality and Gorenstein projectivity*, Int. J. Contemp. Math. Sci. **5** (2010), no. 1-4, 61–66.

[7] D. J. Benson and K. R. Goodearl, *Periodic flat modules, and flat modules for finite groups*, Pacific J. Math. **196** (2000), no. 1, 45–67. https://doi.org/10.2140/pjm.2000.196.45

[8] N. Ding and J. Chen, *The flat dimensions of injective modules*, Manuscripta Math. **78** (1993), no. 2, 165–177. https://doi.org/10.1007/BF02599307

[9] ______, *Coherent rings with finite self-FP-injective dimension*, Comm. Algebra **24** (1996), no. 9, 2963–2980. https://doi.org/10.1080/00927879608825724

[10] E. E. Enochs and O. M. G. Jenda, *Gorenstein injective and projective modules*, Math. Z. **220** (1995), no. 4, 611–633. https://doi.org/10.1007/BF02572634

[11] Z. Gao, *Weak Gorenstein projective, injective and flat modules*, J. Algebra Appl. **12** (2013), no. 2, 1250165, 15 pp. https://doi.org/10.1142/S0219498812501654

[12] J. Gillespie, *Model structures on modules over Ding-Chen rings*, Homology Homotopy Appl. **12** (2010), no. 1, 61–73. http://projecteuclid.org/euclid.hha/1296223822

[13] I. Kaplansky, *Commutative Rings*, revised edition, The University of Chicago Press, Chicago, IL, 1974.

[14] T. Y. Lam, *Exercises in Modules and Rings*, Problem Books in Mathematics, Springer, New York, 2007. https://doi.org/10.1007/978-0-387-48899-8

[15] J. J. Rotman, *An Introduction to Homological Algebra*, second edition, Universitext, Springer, New York, 2009. https://doi.org/10.1007/b98977

[16] B. Stenström, *Coherent rings and FP-injective modules*, J. London Math. Soc. (2) **2** (1970), 323–329. https://doi.org/10.1112/jlms/s2-2.2.323

[17] F Wang, *Commutative Rings and Star Operation Theory*, (in Chinese), Beijing, Science Press, 2006.

[18] F. Wang and H. Kim, *Foundations of commutative rings and their modules*, Algebra and Applications, **22**, Springer, Singapore, 2016. https://doi.org/10.1007/978-981-10-3337-7

[19] F. Wang and R. L. McCasland, *On w-modules over strong Mori domains*, Comm. Algebra **25** (1997), no. 4, 1285–1306. https://doi.org/10.1080/00927879708825920

[20] F. Wang, L. Qiao, and H. Kim, *Super finitely presented modules and Gorenstein projective modules*, Comm. Algebra **44** (2016), no. 9, 4056–4072. https://doi.org/10.1080/00927872.2015.1087532

[21] H. Yin, F. Wang, X. Zhu, and Y. Chen, *w-modules over commutative rings*, J. Korean Math. Soc. **48** (2011), no. 1, 207–222. https://doi.org/10.4134/JKMS.2011.48.1.207

[22] M. Zafrullah, *On finite conductor domains*, Manuscripta Math. **24** (1978), no. 2, 191–204. https://doi.org/10.1007/BF01310053

Kui Hu
College of Science
Southwest University of Science and Technology
Mianyang, 621010, P. R. China
and
Department of Mathematics
Kyungpook National University
Daegu 41566, Korea
Email address: hukui200418@163.com

Jung Wook Lim
Department of Mathematics
Kyungpook National University
Daegu 41566, Korea
Email address: jwlim@knu.ac.kr

Shiqi Xing
College of Applied Mathematics
Chengdu University of Information Technology
Chengdu, Sichuan 610225, P. R. China
Email address: sqxing@yeah.net