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HOMOLOGICAL PROPERTIES OF SEMI-WAKAMATSU-TILTING MODULES

Dajun Liu and Jiaqun Wei

Abstract. For a fixed semi-Wakamatsu-tilting module ${}_AT$, we generalize the concepts of Auslander class, Bass class, and investigate many homological properties of such classes. Moreover, we establish an equivalence between the class of ∞-T-cotorsionfree modules and a subclass of the class of T-adstatic modules. Finally, a similar version of Auslander-Bridger approximation theorem and a nice property of relative cotranspose are obtained.

1. Introduction

It is generally known that Auslander-Bridge transpose [1] plays an important role in the classical Auslander-Reiten theory. Auslander and Bridge showed how Auslander-Bridge transpose can be used to investigate n-torsionfree modules over two-sided rings. A faithfully balanced self-orthogonal bimodule is called a generalized tilting bimodule by Wakamatsu [16]. Following [10], generalized tilting bimodules are Wakamatsu-tilting modules. Clearly, tilting modules are Wakamatsu-tilting modules and Wakamatsu-tilting modules are generalized from tilting modules without the restriction of projective dimension [13]. A semidualizing bimodule defined in [11] is also a Wakamatsu-tilting module.

Dibaei and Sadeghi [8] introduced the C-transpose $\mathrm{Tr}_C M$ for a module M respect to a semidualizing module ${}_RC$ (See [8, Definition 1.3]). Dually, Tang and Huang [15] introduced and demonstrated the cotranspose of modules with respect to a semidualizing bimodule ${}_RC_S$. Moreover, they introduced n-C-cotorsionfree modules and verified that n-C-cotorsionfree modules have many dual properties of n-torsionfree modules.

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Xi [19] gave the notion of relative transpose of a given module M. This inspires us in [12] to introduce the dual concept, n-T-cotorsionfree modules (See Definition 4.2) defined by the above dual conception relative cotranspose (See Definition 4.1). Based on these previous work, and in order to further study ∞-T-cotorsionfree modules (See Definition 4.2), we replace semidualizing modules with semi-Wakamatsu-tilting modules ${}_AT_B$ to redefine some conceptions of classes of modules (See [11]). In particular, we define T-Auslander class $\mathcal{A}_T(B)$, T-Bass class $\mathcal{B}_T(A)$ with respect to ${}_AT_B$ (See Definition 3.1), and T-projective $\mathcal{P}_T(A)$ (See Definition 3.5). It turn out that many important results on Auslender and Bass classes with respect to a semidualizing module are still true in our more general setting.

We mainly prove the following conclusions:

Theorem 1.1. *Let $U, V \in B$-mod, $S, W \in A$-mod, and $S' \in$ mod-A.*

(1) *If $U \in \mathcal{A}_T(B)$ and $\mathrm{Tor}^B_{i\geqslant 1}(T,\ V) = 0$, then*

$$\mathrm{Ext}^{i\geqslant 1}_B(V,\ U) \cong \mathrm{Ext}^{i\geqslant 1}_A(T \otimes_B V,\ T \otimes_B U).$$

(2) *If $S \in \mathcal{B}_T(A)$ and $\mathrm{Ext}^{i\geqslant 1}_A(T,\ W) = 0$, then*

$$\mathrm{Ext}^{i\geqslant 1}_A(S,\ W) \cong \mathrm{Ext}^{i\geqslant 1}_B(\mathrm{Hom}_A(T,\ S),\ \mathrm{Hom}_A(T,\ W)).$$

(3) *If $S \in \mathcal{A}_T(B)$ and $\mathrm{Tor}^A_{i\geqslant 1}(S',\ T) = 0$, then*

$$\mathrm{Tor}^A_{i\geqslant 1}(S',\ S) \cong \mathrm{Tor}^A_{i\geqslant 1}(S' \otimes_A T,\ \mathrm{Hom}_A(T,\ S)).$$

Moreover, these isomorphisms are natural isomorphisms of abelian groups.

Indeed, the above theorem is an essential characterization on modules in $\mathcal{A}_T(B), \mathcal{B}_T(A)$.

We denote that $\mathcal{S} = \{M \in B\text{-mod} \mid \mathrm{Ext}^{i\geqslant 1}_B(M, T^+) = 0\}$ and $\mathcal{T} = \mathrm{Adst}(T) \cap \mathcal{S}$ (See Remark 2.6), where $(-)^+ = \mathrm{Hom}_{\mathbf{Z}}(-, \mathbf{Q}/\mathbf{Z})$ with $\mathbf{Z}$ the additive group of integers and $\mathbf{Q}$ the additive group of rational numbers. The class of ∞-T-cotorsionfree modules is denoted by $\triangle(A)$. If $\triangle(A) \subseteq \mathrm{Copre}({}_AT)$ (resp., ${}_AT^{\perp} \subseteq \mathrm{Copre}({}_AT)$), we have:

Theorem 1.2. *Let ${}_AT_B$ be a semi-Wakamatsu-tilting module. There is an equivalence of categories $(-)^* : \triangle(A) \leftrightarrows \mathcal{T} : T \otimes_B -$.*

Thus, the theorem provides an equivalence between a subcategory of A-mod associated to ${}_AT$ and a subcategory of B-mod associated with the character module T^+.

Specialized to tilting modules and Wakamatsu-tilting modules, the above results can be obtained from the generalized Brenner-Butler theorem for tilting modules (See [13, Theorem 1.16]) and for Wakamatsu-tilting modules (See [17, Proposition 2.5]). In this sense, the above theorems are partial generalizations of the classical results for tilting theory and Wakamatsu-tilting theory.

Finally, using the concept of T-cograde of N with respect to semi-Wakamatsu-tilting modules (See [12, Definition 3.2]), we get a similar version of Auslander-Bridger approximation theorem [9, Theorem 3.8]:

Theorem 1.3. *Let ${}_AT_B$ be a semi-Wakamatsu-tilting module, $M \in \mathrm{Copre}(T)$ and $n \geqslant 1$. If* Tor-*cograde*$_T$ $\mathrm{Ext}^i_A(T, M) \geqslant i$ *for any $1 \leqslant i \leqslant n$, then there exist a module $U \in A$-mod and a homomorphism $f : U \to M$ satisfying the following conditions*:

(1) $\mathcal{P}_T(A)$-$id_AU \leqslant n$;

(2) $\mathrm{Ext}^i_A(T, f)$ *is bijective for any $1 \leqslant i \leqslant n$.*

As we know, tilting modules over a ring have many good properties. For example, tilting modules induce generally recollements of triangulated categories (see [4], [5]). If they, in addition, have smaller projective dimension or other homological properties (see [6], [7]), then they give rise to even recollements of derived categories of rings. In this paper, we want to obtain results analogous to those for tilting modules.

This paper is organized as follows. In Section 3, we introduce some important classes of modules by replacing semidualizing modules with semi-Wakamatsu-tilting modules. For instance, we define the concepts of T-Auslander class $\mathcal{A}_T(B)$, T-Bass class $\mathcal{B}_T(A)$, and T-flat, etc. An equivalence between $\mathcal{A}_T(B)$ and $\mathcal{B}_T(A)$ is obtained. Another key point is that we give a nice property on $\mathcal{A}_T(B)$ and $\mathcal{B}_T(A)$. In Section 4, we establish an equivalence between the class of ∞-T-cotorsionfree modules and a subclass of the class of T-adstatic modules (i.e., Theorem 4.5). Not only that, but we study many homological properties on $\mathcal{P}_T(A)$-pd_AM. In Section 5, we give a similar version of Auslander-Bridger approximation theorem and obtain a nice property of relative cotranspose (See Definition 4.1).

2. Preliminaries

Let A be an Artin R-algebra, that is, R is a commutative Artin ring and A is an R-algebra which is finitely generated as an R-module. The category of finitely generated left A-modules will be denoted by A-mod, respectively, the category of finitely generated right A-modules will be denoted by mod-A. Throughout this paper, all modules are invariably finitely generated.

Let $\mathcal{X}$ be a subcategory of A-mod and M be a left A-module. A homomorphism f: $X \to M$ with $X \in \mathcal{X}$ is called a right $\mathcal{X}$-approximation (or, $\mathcal{X}$-precover) of M if the induced morphism $\mathrm{Hom}(X', f)$ is surjective for all $X' \in \mathcal{X}$. Dually, a homomorphism f: $M \to X$ with $X \in \mathcal{X}$ is called a left $\mathcal{X}$-approximation (or, $\mathcal{X}$-preenvelope) of M if the induced morphism $\mathrm{Hom}(f, X')$ is surjective for all $X' \in \mathcal{X}$. For further details, see [2, 3]. An $\mathcal{X}$-resolution of M is an exact sequence

$$\cdots \longrightarrow X^n \longrightarrow X^{n-1} \longrightarrow \cdots \longrightarrow X^1 \longrightarrow X^0 \longrightarrow M \longrightarrow 0$$

with $X^i \in \mathcal{X}$ for all $i \geqslant 0$. In addition, if the exact sequence is $\mathrm{Hom}(\mathcal{X}, -)$-exact, then the exact sequence is called a proper $\mathcal{X}$-resolution of M. Dually, we can define the notion of $\mathcal{X}$-coresolution and proper $\mathcal{X}$-coresolution. We say that

M has $\mathcal{X}$-projective dimension at most m, denoted by $\mathcal{X}$-pd $(M) \leqslant m$, if there is an $\mathcal{X}$-resolution of M of the form $0 \to X^m \to \cdots \to X^1 \to X^0 \to M \to 0$.

Let T be a module in A-mod. We denoted by B the endomorphism algebra of T, thus T is an A-B bimodule in the natural manner.

We say a module ${}_AT$ is self-orthogonal if $\mathrm{Ext}^i(T,T) = 0$ for any $i \geqslant 1$. Recall that an A-module T is Wakamatsu-tilting [16] provided that

(1) $\mathrm{End}_B T \cong A$, where $B := \mathrm{End}_A T$ and,

(2) $\mathrm{Ext}^i_A(T,T) = 0 = \mathrm{Ext}^i_B(T,T) = 0$ for all $i > 0$.

In order to give more characteristics on n-T-cotorsionfree modules, we give the following definition:

Definition 2.1 ([12, Definition 2.11])**.** A module ${}_AT_B$ is called semi-Wakamatsu-tilting if $B := \mathrm{End}_A T$ and ${}_AT$ is self-orthogonal.

Setup: Throughout this paper, we shall fix such a semi-Wakamatsu-tilting module ${}_AT_B$, and add$({}_AT)$ stands for the category consisting of all modules isomorphic to direct summands of finite direct sums of copies of ${}_AT$ and Prod $({}_AT)$ the category consisting of all modules isomorphic to direct summands of direct products of copies of ${}_AT$.

We denote the following full subcategories of A-mod:

$\mathrm{Cogen}({}_AT) = \{M \in A\text{-mod} \mid$ there is an injective morphism from M to $T^n, n \in N\}$,

$\mathrm{Copre}({}_AT) = \{M \in A\text{-mod} \mid$ there is an exact sequence

$$0 \longrightarrow M \xrightarrow{f^0} T^0 \xrightarrow{f^1} T^1$$

with $T^i \in \mathrm{add}T$ for $i = 0{,}1\}$,

$\mathrm{Coapp}({}_AT) = \{M \in A\text{-mod} \mid$ there is an exact sequence

$$0 \longrightarrow M \xrightarrow{f^0} T^0 \xrightarrow{f^1} T^1$$

such that $\mathrm{Coker}(f^0) \in \mathrm{Cogen}(T)$ and f^0 is an add$T$-preenvelope of $M\}$,

Dually, we can define the subcategories Gen(T), Pre(T) and App(T).

For simplicity: We shall denote all left A modules by A-mod, all right A modules by mod-A and the functor $\mathrm{Hom}({}_AT,-)$ by $(-)_*$. Especially, $0 \to L \to M \to N \to 0$ is called $\mathrm{Hom}(T,-)$-exact exact sequence if $0 \to L_* \to M_* \to N_* \to 0$ is an exact sequence.

Lemma 2.2 ([19, Lemma 2.5(1)])**.** *Given modules ${}_AM, {}_AN_B$, ${}_AF$. If ${}_BF$ is a flat module, then the tensor evaluation homomorphism*

$$\mathrm{Hom}_A(M,N) \otimes_B F \longrightarrow \mathrm{Hom}_A(M, N \otimes_B F),$$

which induces an isomorphism of abelian groups

$$\mathrm{Ext}^i_A(M,N) \otimes_B F \cong \mathrm{Ext}^i(M, N \otimes_B F).$$

For convenience, we give a dual version of [14, Lemma 9.71].

Lemma 2.3. *Given modules M_A, ${}_BN_A$, ${}_BC$. If M_A is finite presented and ${}_BC$ is an injective module, then there exists an isomorphism*

$$M \otimes_B \mathrm{Hom}_A(N, C) \cong \mathrm{Hom}_B(\mathrm{Hom}_A(M, N), C),$$

which induces an isomorphism of abelian groups

$$\mathrm{Tor}_n^A(M, \mathrm{Hom}_B(N, C)) \cong \mathrm{Hom}_B(\mathrm{Ext}_A^n(M, N), C).$$

Wisbauer [18] pointed that for a module ${}_AU_B$, arbitrary modules ${}_BN$, ${}_AM$, there exist canonical homomorphisms $\mu_N : N \longrightarrow \mathrm{Hom}_A(U, U \otimes_B N)$ and $\theta_M : U \otimes_B \mathrm{Hom}_A(U, M) \longrightarrow M$.

The following lemma give a characterization on θ_M.

Lemma 2.4 ([19, Lemma 2.1(3)])**.** *If $M \in \mathrm{Gen}(U)$, then the evaluation map $\theta_M : U \otimes_B \mathrm{Hom}(U, M) \to M$ is surjective. If $M \in \mathrm{App}(U)$, then θ_M is bijective. Conversely, if θ_M is bijective, then $M \in \mathrm{App}(U)$. In particular, if $M \in \mathrm{add}U$, then θ_M is bijective.*

On the other hand, here is a characterization on μ_N. We denote Bild := $\{N \in B\text{-mod} \mid \mu_N \text{ is an isomorphism}\}$.

Lemma 2.5 ([19, Proposition 5.1])**.** *For any A-mod ${}_AU$, the functor*

$$\mathrm{Hom}_A({}_AU_B, -) : \mathrm{App}(U) \to \mathrm{Bild}$$

induces an equivalence, its inverse is $U \otimes_B -$.

Remark 2.6. Following [18], we call M (resp., N) U-static (resp., U-adstatic) if θ_M (resp., μ_N) is an isomorphism. We denote by $\mathrm{Stat}(U)$ and $\mathrm{Adst}(U)$ the class of all U-static modules and the class of all U-adstatic modules, respectively.

Lemma 2.7 ([18, Observation 2.4])**.** *For any A-mod ${}_AU$, the functor*

$$\mathrm{Hom}_A({}_AU_B, -) : \mathrm{Stat}(U) \to \mathrm{Adst}(U)$$

defines an equivalence with inverse is $U \otimes_B -$.

3. Subcategories induced by semi-Wakamatsu-tilting modules

In [11], the author introduced and investigated properties of the Auslander and Bass classes, C-flats, C-projectives, and C-injectives with respect to a semidualizing (S, R)-bimodule $C = {}_SC_R$. In this section, we generalize these concepts by replacing the semidualizing module C with the semi-Wakamatsu-tilting module ${}_AT_B$. Furthermore, we investigate many homological properties of them.

Definition 3.1. (a) The T-Auslander class $\mathcal{A}_T(B)$ with respect to T consists of all B-mod N satisfying

$(A1)$ $\mathrm{Tor}^B_{i\geqslant 1}(T, N) = 0$;

$(A2)$ $\mathrm{Ext}_A^{i\geqslant 1}(T, T \otimes_B N) = 0$;

$(A3)$ μ_N is an isomorphism.

(b) The T-Bass class $\mathcal{B}_T(A)$ with respect to T consists of all A-mod M satisfying
($B1$) $\mathrm{Ext}_A^{i\geqslant 1}(T, M) = 0$;
($B2$) $\mathrm{Tor}_{i\geqslant 1}^B(T, \mathrm{Hom}_A(T, M)) = 0$;
($B3$) θ_M is an isomorphism.

It is easy to check that the following results are hold by routine verification.

Remark 3.2. (1) $\mathcal{A}_T(B)$ (resp., $\mathcal{B}_T(A)$) contains Auslander classes with respect to a semidualizing bimodule T (resp., Bass classes with respect to a semidualizing bimodule T).

(2) One can directly verify that $\mathcal{A}_T(B)$ contains all projective B-mod and $\mathcal{B}_T(A)$ contains all injective A-mod.

(3) Given modules $N \in B$-mod, $M \in A$-mod. The morphisms

$$T \otimes_B \mathrm{Hom}_A(T, T \otimes_B N) \underset{T \otimes_B \mu_N}{\overset{\theta_{(T\otimes_B N)}}{\rightleftarrows}} T \otimes_B N$$

and

$$\mathrm{Hom}_A(T, M) \underset{\mathrm{Hom}_A(T, \theta_M)}{\overset{\mu_{\mathrm{Hom}_A(T,M)}}{\rightleftarrows}} \mathrm{Hom}_A(T, T \otimes_B \mathrm{Hom}_A(T, M))$$

from Definition 3.1 yield

$$\theta_{T\otimes_B N} \cdot (T \otimes_B \mu_N) = \mathrm{Id}_{T\otimes_B N}$$

and

$$\mathrm{Hom}_A(T, \theta_M) \cdot \mu_{\mathrm{Hom}_A(T,M)} = \mathrm{Id}_{\mathrm{Hom}_A(T,M)}.$$

In fact, there is an equivalence between $\mathcal{A}_T(B)$ and $\mathcal{B}_T(A)$.

Proposition 3.3. *Let ${}_AT_B$ be a semi-Wakamatsu-tilting module. Then there is an equivalence of categories*

$$T \otimes_B - : \mathcal{A}_T(B) \leftrightarrows \mathcal{B}_T(A) : \mathrm{Hom}_A(T, -).$$

Proof. For any $N \in \mathcal{A}_T(B)$, we have $\mathrm{Ext}_A^{i\geqslant 1}(T, T \otimes_B N) = 0$. Also, one can get $0 = \mathrm{Tor}_{i\geqslant 1}^A(T, N) \cong \mathrm{Tor}_{i\geqslant 1}^B(T, \mathrm{Hom}_A(T, T\otimes_B, N))$ since μ_N is an isomorphism. Moreover, the morphism $T \otimes_B \mu_N$ is an isomorphism, hence $\theta_{T\otimes_B N}$ is an isomorphism. Thus, $T \otimes_B N \in \mathcal{B}_T(A)$.

For any $M \in \mathcal{B}_T(A)$, by a routine verification similar to the above arguments, we can imply $\mathrm{Hom}_A(T, M) \in \mathcal{A}_T(B)$.

Furthermore, if $N \in \mathcal{A}_T(B)$ and $M \in \mathcal{B}_T(A)$, there are natural isomorphisms

$$\mu_N : N \longrightarrow \mathrm{Hom}_A(T, T \otimes_B N)$$

and

$$\theta_M : T \otimes_B \mathrm{Hom}(T, M) \longrightarrow M.$$

Then it follows that the equivalence holds. $\square$

Proposition 3.4. *$\mathcal{A}_T(B)$ and $\mathcal{B}_T(A)$ are closed under finite direct sums and direct summands.*

Proof. Note that the functors Hom, Ext and Tor are additive, then we can verify the results directly. □

We also introduce the following concepts as generalizations of C-flats, C-projectives, and C-injectives with respect to a semidualizing bimodule C.

Definition 3.5. An A-mod is T-flat (resp., T-projective) if it has the form $T \otimes_B F$ for some flat (resp., projective) module ${}_BF$. A B-mod is T-injective if it has the form $\mathrm{Hom}_A(T, I)$ for some injective module ${}_AI$. Set the notation:

$$\begin{aligned}
\mathcal{F}_T(A) &= \{T \otimes_B F \mid_B F \text{ is flat}\},\\
\mathcal{P}_T(A) &= \{T \otimes_B P \mid_B P \text{ is projective}\},\\
\mathcal{I}_T(B) &= \{\mathrm{Hom}_A(T, I) \mid_A I \text{ is injective}\}.
\end{aligned}$$

The next lemma is key to describe the relationship between these mentioned classes.

Lemma 3.6. *For modules ${}_BU$, ${}_AV$, the following are true*:

(a) *$V \in \mathcal{F}_T(A)$ if and only if $V \in \mathcal{B}_T(A)$ and $\mathrm{Hom}_A(T, V)$ is flat over B.*
(b) *$U \in \mathcal{P}_T(A)$ if and only if $U \in \mathcal{B}_T(A)$ and $\mathrm{Hom}_A(T, U)$ is projective over B.*
(c) *$Y \in \mathcal{I}_T(B)$ if and only if $Y \in \mathcal{A}_T(B)$ and $T \otimes_B Y$ is injective over A.*

Proof. (a) For the sufficiency, by the definition of $\mathcal{B}_T(A)$, one can get θ_V is an isomorphism. Thus, $V \in \mathcal{F}_T(A)$ by the assumption.

For the necessity, firstly, by Lemma 2.2, we have:

$$\mathrm{Ext}_A^{i\geqslant 1}(T, T \otimes_B F) \cong \mathrm{Ext}_A^{i\geqslant 1}(T, T) \otimes_B F = 0,$$

where F is finitely generated flat left B-mod. Thus, the module $V = T \otimes_B F$ satisfies condition $(B1)$ in Definition 3.1(2). Next, one can get that $\mathrm{Hom}_A(T, V)$ is flat B-mod since

$$\begin{aligned}
\mathrm{Hom}_A(T, V) &= \mathrm{Hom}_A(T, T \otimes_B F)\\
&\cong \mathrm{Hom}_A(T, T \otimes_B (\varinjlim P_i))\\
&\cong \varinjlim \mathrm{Hom}_A(T, T \otimes_B P_i))\\
&\cong \varinjlim(\mathrm{Hom}_A(T, T_i))\\
&\cong \varinjlim P_i\\
&\cong \ {}_BF,
\end{aligned} \tag{3.1}$$

where P_i is finitely generated projective left B-mod and ${}_BF$ is flat. Hence, the condition $(B2)$ is automatically meet. Moreover, the above arguments imply the following result hold:

$$T \otimes_B \mathrm{Hom}_A(T, T \otimes_B F) \cong T \otimes_B F. \tag{3.2}$$

Thus, the module V also satisfies condition $(B3)$. Consequently, the equivalence holds.

(b) Compared to (a), the proof of (b) requires only minor adjustments. We omit it.

(c) For the sufficiency, by the definition of $\mathcal{A}_T(B)$, one can get μ_Y is an isomorphism. Thus, $V \in \mathcal{I}_T(B)$ by the assumption.

For the necessity, one may assume $Y = \mathrm{Hom}_A(T, I)$. By Remark 3.2(1), we know that for any injective module I, θ_I is an isomorphism. That is , $T \otimes_B Y$ is injective over A. So one can easily observe that $\mathrm{Ext}_A^{i\geqslant 1}(T, T \otimes_B Y) = 0$. Hence μ_Y is an isomorphism. Moreover, $Y \cong \mathrm{Hom}_A(T, T \otimes_B Y)$. By Lemma 2.3, ${\mathrm{Tor}^{\mathrm{B}}}_{i\geqslant 1}(T, Y) = {\mathrm{Tor}^{\mathrm{B}}}_{i\geqslant 1}(T, \mathrm{Hom}_A(T, I)) \cong \mathrm{Hom}_A(\mathrm{Ext}_A^{i\geqslant 1}(T, T), I) = 0$, as desired. $\square$

The next two propositions show basic properties on $\mathcal{F}_T(A)$, $\mathcal{P}_T(A)$ and $\mathcal{I}_T(B)$.

Proposition 3.7. (1) $\mathcal{P}_T(A) =$ *add* ${}_AT$.

(2) $\mathcal{I}_T(B) =$ *Prod* T^+, *where* $T^+ = \mathrm{Hom}_A(T, Q)$ *with* ${}_AQ$ *an injective cogenerator.*

Proof. (1) It is clear that $\mathcal{P}_T(A) \subseteq \mathrm{add}T$. On the contrary, for the module $T^{(k)}$, we can obtain that

$$\begin{aligned} T \otimes_B \mathrm{Hom}_A(T, T^{(k)}) &\cong T \otimes_B \mathrm{Hom}_A(T, T)^{(k)} \\ &\cong (T \otimes_B \mathrm{Hom}_A(T, T))^{(k)} \\ &\cong T^{(k)}. \end{aligned} \tag{3.3}$$

That is, $\theta_{T^{(k)}}$ is an isomorphism (or, $T^{(k)} \in \mathrm{App}(T)$, by Lemma 2.4, $\theta_{T^{(k)}}$ is an isomorphism). Now, suppose that ${}_AM \in \mathrm{add}T$ and $M \oplus N = T^{(k)}$ for some $N \in A$-mod. Then there is a split exact sequence $0 \to M \to T^{(k)} \to N \to 0$, which induces the following commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & T \otimes_B M_* \arrow[r] \arrow[d, "\theta_M"] & T \otimes_B T_*^{(k)} \arrow[r] \arrow[d, "\theta_{T^{(k)}}"] & T \otimes_B N_* \arrow[r] \arrow[d, "\theta_N"] & 0 \\ 0 \arrow[r] & M \arrow[r] & T^{(k)} \arrow[r] & N \arrow[r] & 0. \end{tikzcd}$$

By the five lemma, θ_M is monic, hence θ_N is monic, and so θ_M is an isomorphism by the five lemma again. Notice that $\mathrm{Hom}_A(T, M)$ is a projective left A-mod since $\mathrm{Hom}_A(T, M) \oplus \mathrm{Hom}_A(T, N) \cong \mathrm{Hom}_A(T, T^{(k)}) \cong B^{(k)}$. So ${}_AM \cong T \otimes_B M_* \in \mathrm{add}T$.

(2) The proof is dual to that for (1). For the convenience of readers, we give a complete proof. Firstly, It is clear that $\mathcal{I}_T(B) \subseteq \mathrm{Prod}\ T^+$. Conversely, for

the module $(T^+)^J$, we can obtain that

$$\begin{aligned}\mathrm{Hom}_A(T, T\otimes_B (T^+)^J) &\cong \mathrm{Hom}_A(T, (T\otimes_B T^+)^J)\\ &\cong (\mathrm{Hom}_A(T, T\otimes_B T^+))^J\\ &\cong (T^+)^J.\end{aligned} \tag{3.4}$$

That is, $\mu_{(T^+)^J}$ is an isomorphism. Now, suppose that ${}_BM \in \mathrm{Prod}\,(T^+)^J$ and $M \oplus N = (T^+)^J$ for some $N \in B$-mod. Then there is a split exact sequence $0 \to M \to (T^+)^J \to N \to 0$, which induces the following commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "\mu_M"] & (T^+)^J \arrow[r] \arrow[d, "\mu_{(T^+)^J}"] & N \arrow[r] \arrow[d, "\mu_N"] & 0 \\ 0 \arrow[r] & (T\otimes_B M)_* \arrow[r] & (T\otimes_B (T^+)^J)_* \arrow[r] & (T\otimes_B N)_* \arrow[r] & 0. \end{tikzcd}$$

Using the same technique in (1), we know that μ_M is an isomorphism. Notice that $T\otimes_B M$ is an injective A-mod since $(T\otimes_B M)\oplus(T\otimes_B N) \cong T\otimes_B (T^+)^J \cong (T\otimes_B (T^+))^J \cong Q^J$. Thus, ${}_BM \cong \mathrm{Hom}_A(T, T\otimes_B M) \in \mathrm{Prod}\, T^+$. □

Proposition 3.8. *The classes $\mathcal{F}_T(A)$, $\mathcal{P}_T(A)$ and $\mathcal{I}_T(B)$ are closed under extensions.*

Proof. We only to show the proof for $\mathcal{P}_T(A)$, since the proof of $\mathcal{F}_T(A)$ is similar and the proof of $\mathcal{I}_T(B)$ is dual. Assume that $0 \to L \to M \to N \to 0$ is an exact sequence of A-mod, and $L, N \in \mathcal{P}_T(A)$. Let $L = T\otimes_B P'$, where P' is a projective B-mod. By Lemma 2.2, we have $\mathrm{Ext}^1_A(T, L) = \mathrm{Ext}^1_A(T, T\otimes_B P') = 0$. Then we can obtain the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & T\otimes_B L_* \arrow[r] \arrow[d, "\theta_L"] & T\otimes_B M_* \arrow[r] \arrow[d, "\theta_M"] & T\otimes_B N_* \arrow[r] \arrow[d, "\theta_N"] & 0 \\ 0 \arrow[r] & L \arrow[r] & M \arrow[r] & N \arrow[r] & 0. \end{tikzcd}$$

Lemma 3.6 implies θ_L, θ_N are isomorphisms and the five lemma forces θ_M to be an isomorphism as well. Then by the definition of the functors Ext, Tor, we obtain $M \in \mathcal{B}_T(A)$. Moreover, L_*, N_* are projective modules by Lemma 3.6. Thus, M_* is a projective module. Consequently, $M \in \mathcal{P}_T(A)$ by Lemma 3.6 again. □

We conclude this section by giving a beautiful characterization on $\mathcal{A}_T(B)$, $\mathcal{B}_T(A)$. In order to give a concise proof, we show the next result.

Lemma 3.9. *For any module $M \in B$-mod, if $\mathrm{Ext}^{i\geqslant 1}_A(T,\ T\otimes_B M) = 0$, then*

$$\mathrm{Ext}^{i\geqslant 1}_A(T\otimes_B P,\ T\otimes_B M) = 0$$

for all projective B-mod P.

Proof. Let $I^\bullet$ be an injective resolution of the A-mod $T \otimes_B M$ and P be an arbitrary projective B-mod. Hence, there are isomorphisms:

$$\begin{aligned}\mathrm{Ext}_A^{i\geqslant 1}(T\otimes_B P,\ T\otimes_B M) &\cong H_{-i}\mathrm{Hom}_B(T\otimes_B P,\ I^\bullet)\\ &\cong H_{-i}\mathrm{Hom}_B(P,\ \mathrm{Hom}_A(T,\ I^\bullet))\\ &\cong \mathrm{Hom}_B(P,\ H_{-i}\mathrm{Hom}_A(T,\ I^\bullet))\\ &\cong \mathrm{Hom}_B(P,\ \mathrm{Ext}_A^{i\geqslant 1}(T,\ T\otimes_B M))\\ &= 0.\end{aligned} \tag{3.5}$$

By the assumption, the desired conclusion follows. $\square$

Theorem 3.10. *Let $U, V \in B$-mod, $S, W \in A$-mod, and $S' \in$ mod-A.*

(1) *If $U \in \mathcal{A}_T(B)$ and $\mathrm{Tor}^B_{i\geqslant 1}(T,\ V) = 0$, then*

$$\mathrm{Ext}_B^{i\geqslant 1}(V,\ U) \cong \mathrm{Ext}_A^{i\geqslant 1}(T\otimes_B V,\ T\otimes_B U).$$

(2) *If $S \in \mathcal{B}_T(A)$ and $\mathrm{Ext}_A^{i\geqslant 1}(T,\ W) = 0$, then*

$$\mathrm{Ext}_A^{i\geqslant 1}(S,\ W) \cong \mathrm{Ext}_B^{i\geqslant 1}(\mathrm{Hom}_A(T,\ S),\ \mathrm{Hom}_A(T,\ W)).$$

(3) *If $S \in \mathcal{A}_T(B)$ and $\mathrm{Tor}^A_{i\geqslant 1}(S',\ T) = 0$, then*

$$\mathrm{Tor}^A_{i\geqslant 1}(S',\ S) \cong \mathrm{Tor}^A_{i\geqslant 1}(S'\otimes_A T,\ \mathrm{Hom}_A(T,\ S)).$$

Moreover, these isomorphisms are natural isomorphisms of abelian groups.

Proof. (1) We show the proof by induction on i. For the case $i = 0$, the fact that $U \in \mathcal{A}_T(B)$ and the Hom-tensor adjointness follows that

$$\begin{aligned}\mathrm{Hom}_B(V,\ U) &\cong \mathrm{Hom}_B(V,\ \mathrm{Hom}_A(T,\ T\otimes_B U))\\ &\cong \mathrm{Hom}_A(T\otimes_B V,\ T\otimes_B U).\end{aligned} \tag{3.6}$$

Next, we suppose that $i > 0$ and the conclusion hold for $j < i$. That is, there are isomorphisms

$$\mathrm{Ext}_B^j(L',\ L) \cong \mathrm{Ext}_A^j(T\otimes_B L',\ T\otimes_B L),$$

where $L \in \mathcal{A}_T(B)$ and $\mathrm{Tor}^B_{i\geqslant 1}(T, L') = 0$. Now, we consider $U \in \mathcal{A}_T(B)$ and $\mathrm{Tor}^B_{i\geqslant 1}(T, V) = 0$. Hence, there is an exact sequence

$$0 \to V' \to P' \to V \to 0, \tag{η_{21}}$$

where P' is a projective B-mod. Obviously, $\mathrm{Tor}^B_{i\geqslant 1}(T, V') = 0$ and the sequence

$$0 \to T\otimes_B V' \to T\otimes_B P' \to T\otimes_B V \to 0 \tag{$T\otimes_B \eta_{21}$}$$

is exact. Now, applying the functor $\mathrm{Hom}_B(-, U)$ to η_{21} (resp., $\mathrm{Hom}_B(-, T\otimes_B U)$ to $(T\otimes_B \eta_{21})$), we can obtain the following commutative diagram with exact columns:

$$
\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
\mathrm{Ext}_B^{i-1}(P',U) \arrow[r, "\cong"] \arrow[d] & \mathrm{Ext}_A^{i-1}(T\otimes_B P', T\otimes_B U) \arrow[d] \\
\mathrm{Ext}_B^{i-1}(V',U) \arrow[r, "\cong"] \arrow[d] & \mathrm{Ext}_A^{i-1}(T\otimes_B V', T\otimes_B U) \arrow[d] \\
\mathrm{Ext}_B^{i}(V,U) \arrow[r, dotted] \arrow[d] & \mathrm{Ext}_A^{i}(T\otimes_B V, T\otimes_B U) \arrow[d] \\
\mathrm{Ext}_B^{i}(P',U) \arrow[d, equal] & \mathrm{Ext}_A^{i}(T\otimes_B P', T\otimes_B U) \arrow[d, equal] \\
0 & 0.
\end{tikzcd} \tag{3.7}
$$

The two isomorphisms can be obtained by the induction hypothesis. On the other hand, $U \in \mathcal{A}_T(B)$, and hence $\mathrm{Ext}_A^{i\geqslant 1}(T, T\otimes_B M) = 0$. Then the right zero follows from Lemma 3.9. Hence, there exists a unique isomorphism

$$\mathrm{Ext}_B^i(V,U) \cong \mathrm{Ext}_A^i(T\otimes_B V, T\otimes_B U),$$

which makes the induced diagram commutative. Also, it is not difficult to verify the isomorphism is natural in U and V. The similar statements (2), (3) are proved accordingly. $\square$

4. Equivalence and relative homological dimensions

We firstly recall the concept of n-T-cotorsionfree modules [12]. Then, we prove that there is an equivalence between ∞-T-cotorsionfree modules and a subclass of the class of T-adstatic modules. Some important results on relative homological dimension are obtained.

Definition 4.1. Let M be a left A-module in $\mathrm{Copre}({}_AT)$, that is, there is an exact sequence

$$0 \longrightarrow M \xrightarrow{f^0} T^0 \xrightarrow{f^1} T^1. \tag{ρ_{22}}$$

Applying $\mathrm{Hom}({}_AT,-)$ to (ρ_{22}), we call $c\Sigma_T(M) := \mathrm{Coker}\ f^1_*$ the relative cotranspose of M with respect to T, or T-cotranspose of M. Moreover, there exists an exact sequence

$$0 \longrightarrow (co\Omega_T^i(M))_* \longrightarrow T^i_* \longrightarrow (co\Omega_T^{i+1}(M))_* \longrightarrow 0,$$

where $M := co\Omega_T^0(M)$, $co\Omega_T^{i+1}(M) = \mathrm{Coker} f^i$, and $i = 0, 1$.

Definition 4.2 ([12, Definition 2.4]). Let M be a finitely generated left A-module in $\mathrm{Copre}({}_AT)$. Then M is called n-T-cotorsionfree if $\mathrm{Tor}_i^B(T, c\Sigma_T(M)) = 0$ for all $1 \leqslant i \leqslant n$.

If $\mathrm{Tor}_i^B(T, c\Sigma_T(M)) = 0$ for all $i \geqslant 1$, then M is called ∞-T-cotorsionfree. The class of ∞-T-cotorsionfree modules is denoted by $\triangle(A)$. Particularly, every module in A-mod is 0-T-cotorsionfree.

Thought out this section, we always suppose that $\triangle(A) \subseteq \mathrm{Copre}({}_AT)$ (resp., ${}_AT^{\perp} \subseteq \mathrm{Copre}({}_AT)$).

Remark 4.3. By [12, Corollary 2.6], a module M is 2-T-cotorsionfree if and only if $M \in \mathrm{Stat}(T)$.

We denote that $\mathcal{S} = \{M \in B\text{-mod} \mid \mathrm{Ext}_B^{i \geqslant 1}(M, T^+) = 0\}$ and $\mathcal{T} = \mathrm{Adst}(T) \cap \mathcal{S}$, where $(-)^+ = \mathrm{Hom}_{\mathbf{Z}}(-, \mathbf{Q}/\mathbf{Z})$ with $\mathbf{Z}$ the additive group of integers and $\mathbf{Q}$ the additive group of rational numbers.

Proposition 4.4. (1) *If $pd_AT < \infty$, then $\triangle(A) \subseteq {}_AT^{\perp}$.*

(2) *If $pd_{B^{op}}T < \infty$, then ${}_AT^{\perp} \subseteq \triangle(A)$.*

Proof. (1) Let $M \in \triangle(A)$. Then by [12, Theorem 2.9], there exists a $\mathrm{Hom}_A({}_AT, -)$-exact exact sequence

$$\cdots \longrightarrow T^n \longrightarrow \cdots \longrightarrow T^1 \longrightarrow T^0 \longrightarrow M \longrightarrow 0,$$

with all $T^i \in \mathrm{add}T$. Set $K_i = \mathrm{Im}(T^i \to T^{i-1})$ for any $i \geqslant 1$. We may assume $pd_AT = n < \infty$ by assumption. Thus, one can obtain $\mathrm{Ext}_B^i(T, M) \cong \mathrm{Ext}_A^{i+n}(T, K_n) = 0$ for any $i \geqslant 1$ by dimension shifting. Hence, $M \in {}_AT^{\perp}$.

(2) Let $M \in {}_AT^{\perp}$ and $pd_{B^{op}}T = n < \infty$. Then for $i = 0, 1$, we consider the following B-mod exact sequence

$$0 \longrightarrow (co\Omega_T^i(M))_* \longrightarrow T_*^i \longrightarrow (co\Omega_T^{i+1}(M))_* \longrightarrow 0.$$

By [12, Lemma 2.2], we have $\mathrm{Tor}_n^B(T_B, \mathrm{Hom}(T, T^i)) = 0$, $n \geqslant 1$. Then there is an isomorphism

$$\mathrm{Tor}_j^B(T, co\Omega_T^i(M)_*) \cong \mathrm{Tor}_{j+n}^B(T, co\Omega_T^{i+n}(M)_*).$$

In particular, $\mathrm{Tor}_1^B(T, co\Omega_T^2(M)_*) = 0$. Hence we have the following diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & T \otimes_B (co\Omega_T^1(M)) \arrow[r] \arrow[d, "\theta_{(co\Omega_T^1(M))}"] & T \otimes_B T_*^1 \arrow[d, "\theta_{T^1}"] \\
0 \arrow[r] & co\Omega_T^1(M) \arrow[r] & T^1.
\end{tikzcd}$$

Because θ_{T^1} is an isomorphism by Lemma 2.4, $\theta_{co\Omega_T^1(M)}$ is a monomorphism. Thus, $co\Omega_T^1(M)$ is 2-T-cotorsionfree by [12, Corollary 2.6(2)]. On the other hand, $\mathrm{Tor}_1^B(T, co\Omega_T^1(M)_*) = 0$ by the argument, we also can get the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & T \otimes_B M_* \arrow[r] \arrow[d, "\theta_M"] & T \otimes_B T^0_* \arrow[r] \arrow[d, "\theta_{T^0}"] & T \otimes_B co\Omega_T^1(M)_* \arrow[r] \arrow[d, "\theta_{co\Omega_T^1(M)}"] & 0 \\
0 \arrow[r] & M \arrow[r] & T^0 \arrow[r] & co\Omega_T^1(M) \arrow[r] & 0.
\end{tikzcd}$$

Because θ_{T^0} is an isomorphism by Lemma 2.4, and by the snake lemma we know that θ_M is also an isomorphism. That is, M is 2-T-cotorsionfree. So by [12, Corollary 2.10], there exists an exact sequence $0 \to K_1 \to T^{0'} \to M \to 0$ with $T^{0'} \in \mathrm{add}T$ and $\mathrm{Ext}_A^1(T, K_1) = 0$. Observe that $K_1 \in {}_AT^{\perp}$ for $M \in {}_AT^{\perp}$, then repeat the same process for K_1, we obtain an A-mod exact sequence $0 \to K_2 \to T^{1'} \to K_1$ with $T^{1'} \in \mathrm{add}T$ and $\mathrm{Ext}_A^1(T, K_2) = 0$. Continue the discussion, we finally get a proper add$T$-resolution

$$\cdots \longrightarrow T^{n'} \longrightarrow \cdots \longrightarrow T^{1'} \longrightarrow T^{0'} \longrightarrow M \longrightarrow 0.$$

Consequently, $M \in \triangle(A)$. □

The next theorems are our main results in this section:

Theorem 4.5. *Let ${}_AT_B$ be a semi-Wakamatsu-tilting module. There is an equivalence of categories $(-)^* : \triangle(A) \leftrightarrows \mathcal{T} : T \otimes_B -$.*

Proof. According to Lemma 2.7, the category of all 2-T-cotorsionfree modules and Adst(T) can form an equivalence, which is induced by the functors $(-)_*$ and $T \otimes_B -$. Hence, we only need to show that $(-)_*$ (resp., $T \otimes_B -$) maps $\triangle(A)$ (resp., $\mathcal{T}$) to $\mathcal{T}$ (resp., $\triangle(A)$).

Let $M \in \triangle(A)$. Then by Lemma 2.7, we have $M_* \in \mathrm{Adst}(T)$. Moreover, by [12, Theorem 2.9], there exists a proper add_AT-resolution

$$\cdots \longrightarrow T^n \longrightarrow \cdots \longrightarrow T^1 \longrightarrow T^0 \longrightarrow M \longrightarrow 0$$

of $M \in A$-mod. Thus, we can also get an exact sequence

$$\cdots \longrightarrow (T^n)_* \longrightarrow \cdots \longrightarrow (T^1)_* \longrightarrow (T^0)_* \longrightarrow M_* \longrightarrow 0$$

in B-mod. Applying the functor $T \otimes_B -$ to this exact sequence, then we easily imply $\mathrm{Tor}^B_{i\geqslant 1}(T, M_*) = 0$ by dimension shifting and [12, Lemma 2.2]. Note that there is an isomorphism

$$\mathrm{Ext}_B^{i\geqslant 1}(M_*, T^+) \cong [\mathrm{Tor}^B_{i\geqslant 1}(T, M_*)]^+ = 0.$$

So $M_* \in \mathrm{KerExt}_B^{i\geqslant 1}(-, T^+)$ and $M_* \in \mathcal{T}$.

On the other hand, let $N \in \mathcal{T}$. Then μ_N is an isomorphism. Notice that there also exist isomorphisms

$$[\mathrm{Tor}^B_{i\geqslant 1}(T, (T \otimes_B N)_*)]^+ \cong [\mathrm{Tor}^B_{i\geqslant 1}(T, N)]^+ \cong \mathrm{Ext}_B^{i\geqslant 1}(N, T^+) = 0$$

and $\mathrm{Tor}^B_{i\geqslant 1}(T,(T\otimes_B N)_*)=0$. Additionally, $T\otimes_B N$ is 2-T-cotorsionfree by Lemma 2.7. Thus, we can obtain that $T\otimes_B N\in\triangle(A)$ by [12, Corollary 2.6(3)]. $\square$

For a subclass $\mathcal{C}\subseteq A$-mod, we denote $id_A\mathcal{C}:=\sup\{id_AC\,|\,C\in\mathcal{C}\}$. The following theorem establishes the relation between the relative homological dimensions of a module ${}_AM$ and the corresponding standard homological dimensions of M_*.

Theorem 4.6. (1) *$pd_BM_*\leqslant\mathcal{P}_T(A)$-$pd_AM$ for any $M\in A$-mod; the equality holds if $M\in\triangle(A)$.*

(2) *$id_AT\otimes_B N\leqslant\mathcal{I}_T(B)$-$id_BN$ for any $N\in B$-mod; the equality holds if $N\in\mathcal{A}_T(B)$.*

(3) *$\sup\{\mathcal{P}_T$-$pd_AM\mid M\in\triangle(A)$ with $\mathcal{P}_T$-$pd_AM<\infty\}\leqslant id_A\mathcal{P}_T(A)$.*

(4) *$\sup\{\mathcal{F}_T$-$pd_AM\mid M\in\triangle(A)$ with $\mathcal{F}_T$-$pd_AM<\infty\}\leqslant id_A\mathcal{F}_T(A)$.*

Proof. (1) Let $M\in A$-mod with $\mathcal{P}_T(A)$-$\mathrm{pd}_AM=n<\infty$. Then there exists an exact sequence

$$0\longrightarrow T^n\longrightarrow\cdots\longrightarrow T^1\longrightarrow T^0\longrightarrow M\longrightarrow 0$$

with all $T^i\in\mathrm{add}T$ by Proposition 3.7. Note that all the T^i_* are projective B-mod, and T is a semi-Wakamatsu-tilting module, we can get the following exact sequence

$$0\longrightarrow T^n_*\longrightarrow\cdots\longrightarrow T^1_*\longrightarrow T^0_*\longrightarrow M_*\longrightarrow 0$$

whenever applying the functor $(-)_*$. Thus, $\mathrm{pd}_BM_*\leqslant n$.

Conversely, assume that $M\in\triangle(A)$ and $pd_BM_*=n<\infty$. Then there is an exact sequence

$$0\longrightarrow P^n\longrightarrow\cdots\longrightarrow P^1\longrightarrow P^0\longrightarrow M_*\longrightarrow 0$$

with all P^i projective. By [12, Corollary 2.6(3)], we can obtain the following exact sequence

$$0\longrightarrow T\otimes_B P^n\longrightarrow\cdots\longrightarrow T\otimes_B P^1\longrightarrow T\otimes_B P^0\longrightarrow T\otimes_B M_*(\cong M)\longrightarrow 0.$$

Hence, $\mathcal{P}_T(A)$-$\mathrm{pd}_AM\leqslant n$.

(2) Let Q be an injective cogenerator. By the assumption, there exists an exact sequence

$$0\longrightarrow N\longrightarrow E^0\longrightarrow E^1\longrightarrow\cdots\longrightarrow E^n\longrightarrow 0$$

with all $E^i\in\mathrm{Prod}(\mathrm{Hom}_A(T,Q))$ by Proposition 3.7. It follows from Lemma 2.3 that $\mathrm{Tor}^A_{j\geqslant 1}(T,E^i)=0$ for any $0\leqslant i\leqslant n$. Applying the functor $T\otimes_B-$ to the above exact sequence, we can obtain the following exact sequence

$$0\longrightarrow T\otimes_B N\longrightarrow T\otimes_B E^0\longrightarrow T\otimes_B E^1\longrightarrow\cdots\longrightarrow T\otimes_B E^n\longrightarrow 0.$$

Then by Remark 3.2(1), $\mathcal{I}_T(B)$-id_BN.

Conversely, suppose $N \in \mathcal{A}_T(B)$, then we have $\mathrm{Tor}^B_{i\geqslant 1}(T, N) = 0$ and $\mathrm{Ext}^{i\geqslant 1}_A(T, T \otimes_B N) = 0$. If $\mathcal{I}_T(B)$-$\mathrm{id}_B N = n < \infty$, i.e., there is an exact sequence

$$0 \longrightarrow T \otimes_B N \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots \longrightarrow I^n \longrightarrow 0$$

with all E^i injective. Hence, it is not hard to get the following exact sequence

$$0 \longrightarrow (T \otimes_B N)_*(\cong N) \longrightarrow (I^0)_* \longrightarrow (I^1)_* \longrightarrow \cdots \longrightarrow (I^n)_* \longrightarrow 0$$

with all $(I^i)_* \in \mathcal{I}_T(B)$. Thus, $\mathcal{I}_T(B)$-$\mathrm{id}_B N \leqslant n$.

(3) We may assume that $id_A \mathcal{P}_T(A) = n < \infty$. Let $M \in \triangle(A)$ with $\mathcal{P}_T$-$\mathrm{pd}_A M = m < \infty$. By (1), $pd_B M_* = m$, and we can obtain the exact sequence

$$0 \longrightarrow T \otimes_B P^m \longrightarrow \cdots \longrightarrow T \otimes_B P^1 \longrightarrow T \otimes_B P^0 \longrightarrow T \otimes_B M_*(\cong M) \longrightarrow 0.$$

Notice that there is an isomorphism $\mathrm{Ext}^{i\geqslant 1}_A(T \otimes_B P^k, T \otimes_B P^j) = 0$. We suppose $m > n$. Then we can obtain $\mathrm{Ext}^1_A(K, T \otimes_B P^m) \cong \mathrm{Ext}^m_A(M, T \otimes_B P^m) = 0$ because $id_A \mathcal{P}_T(A) = n$, where $K = \mathrm{Coker}(T \otimes_B P^m \to T \otimes_B P^{m-1})$. Thus, the sequence $0 \to T \otimes_B P^m \to T \otimes_B P^{m-1} \to K \to$ splits and $K \in \mathcal{P}_T(A)$, which is a contradiction. Hence, $m \leqslant n$.

(4) It is similar to the proof of (3), we omit it. $\square$

To conclude this section, we give the following application, which as a criterion on $\mathcal{P}_T(A)$-$\mathrm{pd}_A M$.

Proposition 4.7. *If $\mathcal{P}_T(A)$-$pd_A M < \infty$, then $\mathcal{P}_T(A)$-$pd_A M = \sup\{i \geqslant 0 \mid \mathrm{Ext}^i_A(M, T) \neq 0\}$.*

Proof. By the above argument in Theorem 4.6(1), we set $K^{n-1} = \mathrm{Coker}(T^n \to T^{n-1})$. It is clear that $\mathrm{Ext}^i_A(M, T) = 0$ for all $i \geqslant n + 1$. Assume that $\mathrm{Ext}^n_A(M, T) = 0$. Then $\mathrm{Ext}^1_A(K^{n-1}, T^n) \cong \mathrm{Ext}^n_A(M, T^n) = 0$. It follows that the exact sequence $0 \to T^n \to T^{n-1} \to K^{n-1} \to 0$ splits. Hence, it yields that $K^{n-1} \in \mathcal{P}_T(A)$ and $\mathcal{P}_T(A)$-$\mathrm{pd}_A M \leqslant n - 1$, which is a contradiction. As desired. $\square$

5. An analogous Auslander-Bridger approximation's theorem

In the following parts, we are committed to getting a similar version of the Auslander-Bridger approximation theorem (See [9, Theorem 3.8]) and devote ourself to give an application. In the end, we also obtain a nice property of relative cotranspose $c\Sigma_T(M)$ (See Definition 4.1).

For an integer $n \geqslant 0$, we have defined the concept of T-cograde of N with respect to T, refer to [12]. In order to better express the meaning of the first T in the notation 'T-cograde$_T N$', in this paper, we change the original notation 'T-cograde$_T N$' into 'Tor-cograde$_T N$'.

Definition 5.1. Let N be in B-mod and let $n \geqslant 0$. The Tor-cograde of N with respect to T, denoted by Tor-cograde$_T N$, is defined to be the integer $n = \inf\{i \mid \mathrm{Tor}^i(T, N) \neq 0\}$.

Dually, we give the following definition:

Definition 5.2. Let M be in A-mod and let $n \geqslant 0$. The Ext-cograde of M with respect to T, denoted by Ext-cograde$_T$M, is defined to be the integer $n = \inf\{i \mid \mathrm{Ext}^i(T, M) \neq 0\}$.

The next theorem is our main result in this section, which can be regard as a similar version of the Auslander-Bridger approximation theorem (See [9, Theorem 3.8]).

Theorem 5.3. *Let ${}_AT_B$ be a semi-Wakamatsu-tilting module, $M \in \mathrm{Copre}(T)$ and $n \geqslant 1$. If* Tor-*cograde*$_T$ $\mathrm{Ext}^i_A(T, M) \geqslant i$ *for any* $1 \leqslant i \leqslant n$*, then there exist a module* $U \in A$*-mod and a homomorphism* $f : U \to M$ *satisfying the following conditions*:
(1) $\mathcal{P}_T(A)$-$id_A U \leqslant n$;
(2) $\mathrm{Ext}^i_A(T, f)$ *is bijective for any* $1 \leqslant i \leqslant n$.

Proof. The proof is by induction on n. Firstly, let $n = 1$ and

$$P^1 \xrightarrow{f_1} P^0 \longrightarrow \mathrm{Ext}^1_A(T, M) \longrightarrow 0,$$

be a projective presentation of $\mathrm{Ext}^i_A(T, M)$. Hence, we can obtain the exact sequence

$$0 \longrightarrow U \longrightarrow T \otimes_B P^1 \xrightarrow{1 \otimes_B f_1} T \otimes_B P^0 \longrightarrow T \otimes_B \mathrm{Ext}^1_A(T, M) \longrightarrow 0$$

in B-mod, with $T \otimes_B P^1, T \otimes_B P^0 \in \mathcal{P}_T(A)$. Thus, the fact that $T \otimes_B \mathrm{Ext}^1_A(T, M) = 0$ by assumption follows that $\mathcal{P}_T(A)$-id$_A U \leqslant 1$.

On the other hand, note that $P^1, P^0 \in B$-mod are projective, we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
P^1 \arrow[r, "f_1"] \arrow[d, dashed, "g^1"] & P^0 \arrow[r, "\delta'"] \arrow[d, dashed, "g^0"] & \mathrm{Ext}^1_A(T, M) \arrow[r] \arrow[d, equal] & 0 \\
(T^0)_* \arrow[r] & co\Omega^1_T(M)_* \arrow[r, "\delta"] & \mathrm{Ext}^1_A(T, M) \arrow[r] & 0,
\end{tikzcd}$$

where g^1, g^0 are induced homomorphisms.

Then we can construct homomorphisms h^1, h^0 to ensure the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & U \arrow[r] \arrow[d, dashed, "f"] & T \otimes_B P^1 \arrow[r, "1_T \otimes_B f^1"] \arrow[d, "h^1"] & T \otimes_B P^0 \arrow[r] \arrow[d, "h^0"] & 0 \\
0 \arrow[r] & M \arrow[r] & T^0 \arrow[r, "\delta"] & co\Omega^1_T(M) \arrow[r] & 0,
\end{tikzcd}$$

where f is a induced homomorphism and $h^0 = \theta_{co\Omega^1_T(M)} \bullet (1_T \otimes_B g^0)$, $h^1 = \theta_{T^0} \bullet (1_T \otimes_B g_1)$.

Applying the functor $(-)_*$ to the above diagram, we can obtain the following commutative diagram with exact rows:

$$\begin{tikzcd}
(T\otimes_B P^1)_* \arrow[r, "(1_T\otimes_B f^1)_*"] \arrow[d, "h^1_*"] & (T\otimes_B P^0)_* \arrow[r, "\delta''"] \arrow[d, "h^0_*"] & \mathrm{Ext}^1_A(T,U) \arrow[r] \arrow[d, "\mathrm{Ext}^1_A(T,f)"] & 0 \\
(T^0)_* \arrow[r] & co\Omega^1_T(M)_* \arrow[r, "\delta"] & \mathrm{Ext}^1_A(T,M) \arrow[r] & 0.
\end{tikzcd}$$

Notice that μ_{P^0}, μ_{P^1} are isomorphisms since $P^1, P^0 \in B$-mod are projective. By the naturalness of the functor μ, we know that $(1_T\otimes_B g^0)_* \bullet \mu_{P^0} = \mu_{co\Omega^1_T(M)_*} \bullet g^0$. Hence, one can obtain that

$$\begin{aligned}
(h^0)_* \bullet \mu_{P^0} &= (\theta_{co\Omega^1_T(M)} \bullet (1_T\otimes_B g^0))_* \bullet \mu_{P^0} \\
&= (\theta_{co\Omega^1_T(M)})_* \bullet (1_T\otimes_B g^0)_* \bullet \mu_{P^0} \\
&= (\theta_{co\Omega^1_T(M)})_* \bullet \mu_{co\Omega^1_T(M)_*} \bullet g^0 \\
&= 1_{co\Omega^1_T(M)_*} \bullet g^0 \\
&= g^0.
\end{aligned} \tag{5.1}$$

Then, it is easy to obtain the fact that

$$\begin{aligned}
\delta' \bullet (\mu_{P^0})^{-1} &= \delta \bullet g^0 \bullet (\mu_{P^0})^{-1} \\
&= \delta \bullet (h^0)_* \bullet \mu_{P^0} \bullet (\mu_{P^0})^{-1} \\
&= \delta \bullet (h^0)_* \\
&= \mathrm{Ext}^1_A(T,f) \bullet \delta''.
\end{aligned} \tag{5.2}$$

In other words, we can get the following commutative diagram with exact rows:

$$\begin{tikzcd}
(T\otimes_B P^1)_* \arrow[r, "(1_T\otimes_B f^1)_*"] \arrow[d, "(\mu_{P^1})^{-1}"] & (T\otimes_B P^0)_* \arrow[r, "\delta''"] \arrow[d, "(\mu_{P^0})^{-1}"] & \mathrm{Ext}^1_A(T,U) \arrow[r] \arrow[d, "\mathrm{Ext}^1_A(T,f)"] & 0 \\
P^1 \arrow[r] & P^0 \arrow[r, "\delta"] & \mathrm{Ext}^1_A(T,M) \arrow[r] & 0.
\end{tikzcd}$$

Thus, $\mathrm{Ext}^1_A(T,f)$ is bijective by the five lemma.

Now, we assume the result holds for $i = n-1 \geqslant 2$. That is, there exist a module $H \in A$-mod and a homomorphism $h' : H \to M$ satisfying the following conditions:

(1) $\mathcal{P}_T(A)$-$\mathrm{id}_A H \leqslant n-1$;

(2) $\mathrm{Ext}^i_A(T,h')$ is bijective for any $1 \leqslant i \leqslant n-1$. Then by dimension shifting, also note that T is a semi-Wakamatsu-tilting module, there exists a $\mathrm{Hom}_A(-,\mathcal{P}_T(A))$-exact exact sequence

$$0 \longrightarrow H \xrightarrow{g'} W \longrightarrow X \longrightarrow 0,$$

where $W \in \mathcal{P}_T(A)$. So we can construct the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & M \arrow[r, equal] \arrow[d] & M \arrow[d] & \\
0 \arrow[r] & H \arrow[r, "\binom{f'}{g'}"] \arrow[d, equal] & M \oplus W \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H \arrow[r, "g'"] & W \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0, &
\end{tikzcd}$$

where the middle column is split, the $\mathrm{Hom}_A(-, \mathcal{P}_T(A))$-exact middle row can implies the following exact sequence

$$0 \longrightarrow H_* \longrightarrow (M \oplus W)_* \longrightarrow L_* \longrightarrow 0.$$

Moreover, the induction of hypothesis follows that

$$(*_{23}) \qquad \mathrm{Ext}_A^{1 \leqslant i \leqslant n-1}(T, L) = 0, \ \mathrm{Ext}_A^n(T, M) \cong \mathrm{Ext}_A^n(T, L).$$

Now, we take a projective resolution of $\mathrm{Ext}_A^n(T, M)$:

$$P^n \longrightarrow \cdots \longrightarrow P^1 \longrightarrow P^0 \longrightarrow \mathrm{Ext}_A^n(T, M) \longrightarrow 0.$$

By assumption, Tor-cograde$_T$ $\mathrm{Ext}_A^n(T, M) \geqslant n$, so we get the exact sequence:

$$0 \longrightarrow N \longrightarrow T \otimes_B P^n \longrightarrow \cdots \longrightarrow T \otimes_B P^1 \longrightarrow T \otimes_B P^0 \longrightarrow 0.$$

Hence, $\mathcal{P}_T(A)$-id$_A N \leqslant n$. Moreover, we applying the functor $(-)_*$ to the above exact sequence, we get another exact sequence:

$$0 \longrightarrow N_* \longrightarrow (T \otimes_B P^n)_* \longrightarrow \cdots \longrightarrow (T \otimes_B P^1)_* \longrightarrow (T \otimes_B P^0)_* \longrightarrow 0.$$

Note that μ_{P^i} are isomorphisms for all $1 \leqslant i \leqslant n$, we can easily get that

$$(*_{24}) \qquad \mathrm{Ext}_A^{1 \leqslant i \leqslant n-1}(T, N) = 0, \quad \mathrm{Ext}_A^n(T, N) \cong \mathrm{Ext}_A^n(T, M).$$

By $(*_{23})$, $(*_{24})$ We give an observation:

$$(*_{25}) \qquad \mathrm{Ext}_A^n(T, N) \cong \mathrm{Ext}_A^n(T, L).$$

Indeed, let $0 \to L \to I^0 \to \cdots \to I^{n-2} \to I^{n-1} \to K \to 0$ be induced by an injective resolution of L. Since $\mathrm{Ext}_A^{1 \leqslant i \leqslant n-1}(T, L) = 0$, one can obtain the following exact sequence:

$$\begin{tikzcd}
P^n_* \arrow[r] \arrow[d] & \cdots \arrow[r] & P^1 \arrow[r] \arrow[d] & P^0 \arrow[r] \arrow[d] & \mathrm{Ext}^1_A(T,M) \arrow[r] \arrow[d, equal] & 0 \\
I^0_* \arrow[r] & \cdots \arrow[r] & I^{n-1}_* \arrow[r] & K_* \arrow[r] & \mathrm{Ext}^1_A(T,L) \arrow[r] & 0,
\end{tikzcd}$$

where the vertical morphisms are induced by the projective modules.

By assumption, we applying the functor $T \otimes_B -$:

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, dashed] & T\otimes_B P^n \arrow[r] \arrow[d] & \cdots \arrow[r] & T\otimes_B P^1 \arrow[r] \arrow[d] & T\otimes_B P^0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L \arrow[r] & I^0 \arrow[r] & \cdots \arrow[r] & I^{n-1} \arrow[r] & K \arrow[r] & 0,
\end{tikzcd}$$

comparing the above two diagrams, we can easily obtain the observation $(*_{25})$.

By the above arguments, there exist two exact sequence: $0 \to N \to L \oplus W' \to N' \to 0$ and $\mathrm{Hom}_A(-,\mathcal{P}_T(A))$-exact exact sequence $0 \to H \to M \oplus W \oplus W' \to L \oplus W' \to 0$, where $W' = T \otimes_B P^n$. Notice the following pullback diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & H \arrow[r] \arrow[d, equal] & U \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H \arrow[r] & M\oplus L\oplus W \arrow[r] \arrow[d] & L\oplus W' \arrow[r] \arrow[d] & 0 \\
 & & N' \arrow[r, equal] \arrow[d] & N' \arrow[d] & \\
 & & 0 & 0. &
\end{tikzcd}$$

Then by the Horse Lemma, we finally get $\mathcal{P}_T(A)$-$\mathrm{id}_A U \leqslant n$ because $\mathcal{P}_T(A)$-$\mathrm{id}_A H \leqslant n-1$, $\mathcal{P}_T(A)$-$\mathrm{id}_A N \leqslant n$ and the first row in the above diagram is a $\mathrm{Hom}_A(-,\mathcal{P}_T(A))$-exact exact sequence.

In addition, by $(*_{23})$, $(*_{24})$ and $\mathrm{Ext}^n_A(T,\mathcal{P}_T(A)) = 0$, it is not hard for us to get the following commutative diagram:

$$\begin{tikzcd}
 & \mathrm{Ext}^n_A(T,H) \arrow[r] \arrow[d, equal] & \mathrm{Ext}^n_A(T,U) \arrow[r] \arrow[d] & \mathrm{Ext}^n_A(T,N) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Ext}^n_A(T,H) \arrow[r] & \mathrm{Ext}^n_A(T,M\oplus W\oplus W') \cong \mathrm{Ext}^n_A(T,M) \arrow[r] & \mathrm{Ext}^n_A(T,L\oplus W) \arrow[r] & 0.
\end{tikzcd}$$

Consequently, we obtain $\mathrm{Ext}^n_A(T,M) \cong \mathrm{Ext}^n_A(T,U)$. □

The following result is an application of Theorem 5.3:

Corollary 5.4. *Let $M \in \mathrm{Copre}({}_AT)$, and let $n \geqslant 1$. If Tor-cograde$_T$ $\mathrm{Ext}^i_A(T, M) \geqslant i+1$ for any $0 \leqslant i \leqslant n$, then* Ext*-cograde$_T$ $M \geqslant n+1$.*

Proof. We proceed by induction on n. If $n = 0$, and $(T \otimes_B M)_* = 0$. By Remark 3.2(2), μ_{M_*} is a split monomorphism and $M_* = 0$.

Secondly, we suppose $n \geqslant 1$. By the assumption, we have that Ext-cograde$_T$ $M \geqslant n$ and $\mathrm{Ext}^{0\leqslant i \leqslant n-1}_A(T, M) = 0$. By Theorem 5.3, there exist a module $U \in A$-mod and a homomorphism $f : U \to M \in A$-mod satisfying the following conditions:

(1) $\mathcal{P}_T(A)$-$\mathrm{id}_A U \leqslant n$;

(2) $\mathrm{Ext}^i_A(T, f)$ is bijective for any $1 \leqslant i \leqslant n$. Then there exists an exact sequence:

$$0 \longrightarrow U_* \longrightarrow P^0_* \cdots \longrightarrow P^{n-1}_* \longrightarrow P^n_* \longrightarrow \mathrm{Ext}^n_A(T, U) \longrightarrow 0,$$

where $P^i \in \mathcal{P}_T(A)$ for all $0 \leqslant i \leqslant n$. Obviously, $\mathrm{Ext}^n_A(T, M) \cong \mathrm{Ext}^n_A(T, U)$. Hence, by the assumption again, we obtain Tor-cograde$_T$ $\mathrm{Ext}^i_A(T, U) \geqslant n+1$.

Thus, there exists the following commutative diagram with exact rows:

$$\begin{tikzcd}
 & T \otimes_B U_* \arrow[r] \arrow[d, "\theta_U"] & T \otimes_B P^0_* \arrow[r] \arrow[d, "\theta_{P^0}"] & \cdots \arrow[r] & T \otimes_B P^0_* \arrow[r] \arrow[d, "\theta_{P^{n-1}}"] & T \otimes_B P^0_* \arrow[r] \arrow[d, "\theta_{P^n}"] & 0 \\
0 \arrow[r] & U \arrow[r] & P^0 \arrow[r] & \cdots \arrow[r] & P^{n-1} \arrow[r] & P^n \arrow[r] & 0.
\end{tikzcd}$$

By Remark 3.2(2), θ_{P^i} are isomorphisms for any $1 \leqslant i \leqslant n$. Hence, θ_U is epic. By the naturality of θ, we have the following commutative diagram:

$$\begin{tikzcd}
(T \otimes_B U)_* \arrow[r] \arrow[d, "\theta_U"] & (T \otimes_B M)_* \arrow[d, "\theta_M"] \\
U \arrow[r, "f"] & M.
\end{tikzcd}$$

The fact $(T \otimes_B M)_* = 0$ follows that $f \bullet \theta_U = 0$. Thus, $f = 0$, and $\mathrm{Ext}^i_A(T, f) = 0$. Consequently, $\mathrm{Ext}^i_A(T, M) = 0$ and Ext-cograde$_T$ $M \geqslant n+1$. □

The ending section of this paper presents the following interesting results, which can obtain a nice property of $c\Sigma_T(M)$.

Proposition 5.5. *If there exists an exact sequence*

$$(*_{26}) \qquad V_1 \xrightarrow{g} V_0 \longrightarrow N \longrightarrow 0$$

which satisfy the following conditions:

(1) μ_{V_0}, μ_{V_1} *are isomorphisms.*

(2) $\mathrm{Ext}_A^1(T, T \otimes_B V_0) = 0 = \mathrm{Ext}_A^1(T, T \otimes_B V_1) = 0 = \mathrm{Ext}_A^2(T, T \otimes_B V_1)$. *Then there exists an exact sequence*

$$0 \longrightarrow \mathrm{Ext}_A^1(T, L) \longrightarrow N \xrightarrow{\mu_N} (T \otimes_B N)_* \longrightarrow \mathrm{Ext}_A^2(T, L) \longrightarrow 0,$$

where $L = \mathrm{Ker}(1_T \otimes_B g)$.

Proof. By applying the functor $T \otimes_B -$ to $(*_{26})$, one can get an exact sequence:

$$0 \longrightarrow L \longrightarrow T \otimes_B V_1 \xrightarrow{1_T \otimes_B g} T \otimes_B V_0 \longrightarrow T \otimes_B N \longrightarrow 0.$$

Then it is easy to obtain the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Im}g \arrow[r] \arrow[d, dashed, "h"] & V_0 \arrow[r, "1_T \otimes \pi'"] \arrow[d, "\mu_{V_0}"] & N \arrow[r] \arrow[d, "\mu_N"] & 0 \\
0 \arrow[r] & (\mathrm{Im}(1_T \otimes_B g))_* \arrow[r] & (T \otimes_B V_0)_* \arrow[r, "\pi"] & (T \otimes_B N)_* \arrow[r] & \mathrm{Ext}_A^1(T, \mathrm{Im}(1_T \otimes_B g) \arrow[r] & 0,
\end{tikzcd}$$

where h is a induced morphism.

Hence, by the snake lemma, we have $\mathrm{Coker}\mu_N \cong \mathrm{Ext}_A^1(T, \mathrm{Im}(1_T \otimes_B g)$ and $\mathrm{Ker}\mu_N \cong \mathrm{Coker}h$. The rest proof can be obtained by the dually proof in [12, Theorem 2.3], so we omit it. □

Corollary 5.6. *Let* $M \in \mathrm{Copre}(T)$ *be in* A*-mod. Then there exists an exact sequence*

$$0 \to \mathrm{Ext}_A^1(T, M) \longrightarrow c\Sigma_T(M) \xrightarrow{\mu_{c\Sigma_T(M)}} (T \otimes_B c\Sigma_T(M))_* \longrightarrow \mathrm{Ext}_A^2(T, M) \to 0.$$

Proof. By the definition of $c\Sigma_T(M)$, there is an exact sequence

$$0 \longrightarrow M_* \longrightarrow T_*^0 \longrightarrow T_*^1 \longrightarrow c\Sigma_T(M) \longrightarrow 0.$$

Note that $\mu_{T_*^0}, \mu_{T_*^1}$ are isomorphisms, and $\mathrm{Ext}_A^{i \geqslant 1}(T, T \otimes_B T_*^1) = 0 = \mathrm{Ext}_A^{i \geqslant 1}(T, T \otimes_B T_*^0)$.

Then the result follows from Corollary 5.4. □

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Dajun Liu
Institute of Mathematics
School of Mathematics Sciences
Nanjing Normal University
Nanjing 210023, P. R. China
Email address: ldjnnu2017004@163.com

Jiaqun Wei
Institute of Mathematics
School of Mathematics Sciences
Nanjing Normal University
Nanjing 210023, P. R. China
Email address: weijiaqun@njnu.enu.cn