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# AN AVERAGE OF SURFACES AS FUNCTIONS IN THE TWO-PARAMETER WIENER SPACE FOR A PROBABILISTIC 3D SHAPE MODEL

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Abstract. We define the average of a set of continuous functions of two variables (surfaces) using the structure of the two-parameter Wiener space that constitutes a probability space. The average of a sample set in the two-parameter Wiener space is defined employing the two-parameter Wiener process, which provides the concept of distribution over the two-parameter Wiener space. The average defined in our work, called an average function, also turns out to be a continuous function which is very desirable. It is proved that the average function also lies within the range of the sample set. The average function can be applied to model 3D shapes, which are regarded as their boundaries (surfaces), and serve as the average shape of them.

## 1. Introduction

The work proposed in this paper contributes to probability theory on a function space. We specifically develop a new concept of the average of a two-variable function set. There has been a widely used concept of the average of a function; often it is called the average value of the function. This is known to be the Mean Value Theorem for integrals or similar statements found in literature at the undergraduate level. On the other hand, the concept of an average of a set of functions, which we also expect to be a function, is not common. For a set of one-variable functions, their average has been defined in the author's previous work [9]. However, to the best of the author's knowledge, there is no concept of an average of a set of two-variable functions.

We aim to define such an average in this paper. We use the structure of the two-parameter Wiener space, the set of continuous functions of two variables. The two-parameter Wiener space was first introduced by Yeh in [18] and further work [19], originally called "the Wiener space of functions

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of two variables" therein. In these papers Yeh developed the structure of a Gaussian measure analogous to the Wiener measure but defined on the space of continuous functions of two variables.

Like the Wiener space, the two-parameter Wiener space is a probability space. For a given sample set in the two-parameter Wiener space, we mathematically define an average of the sample set using the two-parameter Wiener process, which provides the concept of distribution over the two-parameter Wiener space, as varying Gaussians. For each parameter of the process, the average of the random variable corresponding to the parameter is defined in terms of the two-parameter Wiener integral. The two-parameter Wiener integral involved in estimating the average can be converted to a Lebesgue integral and thus calculable. Then we derive the formula for the average. It is proved that the resulting average is also a continuous function which is very desirable. Moreover, the resulting average lies within the range of the sample set. We call the resulting average the average function onward.

Our average function can be applied to a mathematical model of three-dimensional (3D) shapes. When shapes are regarded as continuous entities, estimating an average of those shapes has been a very difficult problem in shape modelling. This problem has strongly motivated our work.

An initial result at an early stage of our work was presented at the International Congress of Mathematicians (ICM) in 2014 and only an abstract has appeared in its proceedings [8].

In the next section, we briefly introduce preliminaries necessary for the construction of an average: the background of the two-parameter Wiener space. The procedure of how our average function is formulated on the two-parameter Wiener space is given in Section 3. A potential application of our average function to 3D shapes is outlined in Section 4.

## 2. Preliminaries: the two-parameter Wiener space

This section introduces useful properties of the two-parameter Wiener space, mostly based on [18, 19], and [11, 17].

Given a rectangle $Q = [a, b] \times [c, d]$ in $\mathbb{R}^2$, $C(Q)$ will denote the space of all real-valued continuous functions on $Q$. We are interested in a subset of $C(Q)$ defined below.

**Definition 2.1.** Let $Q = [a, b] \times [c, d]$ in $\mathbb{R}^2$. Then

$$C_2(Q) := \{x \in C(Q) \mid x(a, \cdot) = x(\cdot, c) = 0\}.$$

**Definition 2.2** (Yeh)**.** Let $Q = [a, b] \times [c, d]$, and $a = s_0 < s_1 < \cdots < s_m = b$, $c = t_0 < t_1 < \cdots < t_m = d$,

$$I := \{x \in C_2(Q) \mid (x(s_1, t_1), \dots, x(s_m, t_n)) \in D\}, \tag{2.1}$$

where $D = (a_{11}, b_{11}] \times \cdots \times (a_{mn}, b_{mn}] \subset \mathbb{R}^{mn}$, $-\infty \leq a_{jk} \leq b_{jk} \leq \infty$, and $j = 1, \dots, m$, $k = 1, \dots, n$.

(1) $I$ is called a strict interval in $C_2(Q)$.
(2) If $D$ is a measurable set in $\mathbb{R}^{mn}$, then $I$ is called a cylinder.
(3) $(s_1, t_1), \dots, (s_m, t_n)$ are called restriction points of $I$.
(4) $\mathcal{I} =$ the collection of all strict intervals $I$. (Then $\mathcal{I}$ is a semi-algebra.)

**Note.** For (2), some researchers call $I$ an interval. In this paper however we prefer the terminology "cylinder" to an interval in order to avoid confusions.

**Definition 2.3** (Yeh)**.** For $I \in \mathcal{I}$, the two-parameter Wiener measure $m_2$ of $I$ is defined by

$$m_2(I) = \int_E \dots (mn) \cdots \int W(\vec{u}, \vec{s}, \vec{t})\, d\vec{u},$$

where

$$\begin{aligned} W(\vec{u}; \vec{s}, \vec{t}) &= W(u_{11}, u_{12}, \dots, u_{mn}; s_1, s_2, \dots, s_m, t_1, t_2, \dots, t_n) \\ &= \prod_{k=1}^{n} \prod_{j=1}^{m} \Big( [\pi(s_j - s_{j-1})(t_k - t_{k-1})]^{\frac{-1}{2}} \\ &\quad \cdot \exp\{ \frac{-(u_{j,k} - u_{j-1,k} - u_{j,k-1} + u_{j-1,k-1})^2}{(s_j - s_{j-1})(t_k - t_{k-1})} \} \Big) \\ &= \pi^{\frac{-mn}{2}} [(s_1 - a)(s_2 - s_1) \cdots (b - s_{m-1})]^{\frac{-m}{2}} \\ &\quad [(t_1 - c)(t_2 - t_1) \cdots (d - t_{n-1})]^{\frac{-n}{2}} \\ &\quad \cdot \exp\left\{ - \sum_{k=1}^{n} \sum_{j=1}^{m} \left( \frac{(u_{j,k} - u_{j-1,k} - u_{j,k-1} + u_{j-1,k-1})^2}{(s_j - s_{j-1})(t_k - t_{k-1})} \right) \right\} \end{aligned} \tag{2.2}$$

Here, $u_{j,0} = u_{0,k} = 0$ for $0 \le j \le m$, $0 \le k \le n$.

**Proposition 2.1** (Yeh)**.**
1. *The measure $m_2$ is well defined and countably additive on $\mathcal{I}$.*
2. $m_2(C_2(Q)) = 1$.
3. *If $I \in \mathcal{I}$, $I \neq \emptyset$, then $m_2(I) > 0$.*

Since the measure $m_2$ is countably additive, it can be extended in the usual way to the $\sigma$-algebra $\sigma(\mathcal{I})$ of sets generated by $\mathcal{I}$ and then can be further extended so as to be a complete measure. We will denote this completed measure space by $(C_2(Q), \mathcal{M}_2, m_2)$, and call the *two-parameter Wiener space.* We can see it is a probability space since $m_2(C_2(Q)) = 1$.

**Proposition 2.2** (Yeh)**.** *Let $F : C_2(Q) \longrightarrow \mathbb{R}$. Let $f(u_{11}, u_{12}, \dots, u_{mn})$ be a Lebesgue measurable function on $\mathbb{R}^{mn}$ and $F(x) = f(x(s_1, t_1), \dots, x(s_m, t_n))$, then $F$ is measurable in the two-parameter Wiener space and*

$$\int_{C_2(Q)} F(x)\, dm_2(x) = \int_{\mathbb{R}^{mn}} f(\vec{u}) W(\vec{u}; \vec{s}, \vec{t})\, d\vec{u}.$$

The integral over the infinite dimensional function space $C_2(Q)$ on the left is reduced to an ordinary Lebesgue integral over the finite dimensional space $\mathbb{R}^{mn}$ on the right [7, 18].

Some well-known properties are listed in the example below.

**Example 2.3.** Let $E[\cdot]$ denote the expectation of a random variable. Then

(a) $E[x(s,t)] = \int_{C_2(Q)} x(s,t)dm_2(x) = 0$, since $\frac{1}{\sqrt{\pi(s-a)(t-c)}} \int_{-\infty}^{\infty} u_{11} \ \exp\{\frac{-u_{11}^2}{(s-a)(t-c)}\} \ du_{11} = 0$ (note that its integrand is an odd function).

(b) $E[x(s,t)^2] = \int_{C_2(Q)} x(s,t)^2 \ dm_2(x) = \frac{(s-a)(t-c)}{2}$.

The two-parameter Wiener space has been studied in literature such as [2, 4, 12, 13, 17] and further properties of it can be found. The space is also called the Yeh-Wiener space in some of the literature.

We introduce definitions of equicontinuity and the concept of an average of a measurable function which are necessary for our formulation in the next section.

**Definition 2.4** (p. 245 in [16])**.** Let $\mathcal{F}$ be a collection of functions on a metric space $A$ to $\mathbb{R}$ with metric $\rho$. We say that $\mathcal{F}$ is equicontinuous if to every $\varepsilon > 0$ corresponds a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for every $f \in \mathcal{F}$ and for all pairs of points $x$, $y \in A$ with $\rho(x,y) < \delta$. (In particular, every $f \in \mathcal{F}$ is then uniformly equicontinuous.)

*Remark* 2.4. It has been seen that a finite set of continuous functions on $A$ is equicontinuous.

**Definition 2.5** (p. 30 in [16])**.** Let $(M, \mathcal{M}, \mu)$ be a measure space and $\mu(M) < \infty$. By an average of a measurable function $f$ on a measurable set $E$ in $\mathcal{M}$ we mean

$$\bar{f}_E := \frac{\int_E f \ d\mu}{\mu(E)}, \tag{2.3}$$

where $\mu(E) > 0$.

## 3. Formulation of the average of a function set

As introduced in the previous section, $C_2(Q)$ is the set of continuous functions of two variables defined on the rectangle $Q = [a,b] \times [c,d]$. In this section, we attempt to build an average of a set of functions in $C_2(Q)$.

**Definition 3.1.** Let $\mathcal{C}$ be a subset of $C_2(Q)$ with a positive measure and be a bounded and uniformly equicontinuous collection. Define the range of a cylinder at $(s,t) \in (a,b] \times (c,d]$ by

$$\alpha(s,t) := \inf_{x \in \mathcal{C}} \{x(s,t)\}, \qquad \beta(s,t) := \sup_{x \in \mathcal{C}} \{x(s,t)\}.$$

**Definition 3.2.** A subset $\mathcal{C}$ of $C_2(Q)$ is said to be nowhere compressible in $(a,b]\times(c,d]$ if for $(s,t)\in(a,b]\times(c,d]$ there are two functions (surfaces) $x,\ y\in\mathcal{C}$ with $x(s,t)\neq y(s,t)$.

*Remark* 3.1. 1) The values $\alpha$ and $\beta$ obviously depend on $(s,t)$. Note that both $\inf_{x\in\mathcal{C}}\{x(s,t)\}$ and $\sup_{x\in\mathcal{C}}\{x(s,t)\}$ are continuous functions of $(s,t)$ since the set $\mathcal{C}$ is a bounded and uniformly equicontinuous collection [1].

2) At each $(s,t)\in Q$, we can define a cylinder (see Definition 2.2) of the form (one dimensional, with a single restriction point):

$$E_{(s,t)}=\{x\in C_2(Q):\alpha(s,t)<x(s,t)\leq\beta(s,t)\}. \tag{3.1}$$

3) If $\mathcal{C}$ is nowhere compressible, then $\alpha(s,t)<\beta(s,t)$, $(s,t)\in(a,b]\times(c,d]$.

4) Throughout this paper, $\mathcal{C}$ be a subset of $C_2(Q)$ that is equicontinuous and nowhere compressible in $(a,b]\times(c,d]$.

Now we introduce the two-parameter Wiener process that will provide the concept of distribution over the two-parameter Wiener space.

**Definition 3.3** (Yeh)**.** Let $(s,t)\in Q$ and $X_{(s,t)}$ is defined on the two-parameter Wiener space by

$$\begin{aligned} X_{(s,t)}:C_2(Q)&\longrightarrow\mathbb{R}\\ x&\longmapsto X_{(s,t)}(x):=x(s,t). \end{aligned} \tag{3.2}$$

Then the function $X_{(s,t)}$ is measurable on the two-parameter Wiener space $C_2(Q)$ [18] and is a random variable. The system of random variables $\{X_{(s,t)}: s,t\geq 0\}$ is called the *two-parameter Wiener process*, and also called the Yeh-Wiener process in literature such as [3,10,11,13]. Obviously, for all $x\in C_2(Q)$, $X_{(a,t)}(x)=0$ for all $t\in[c,d]$, and $X_{(s,c)}(x)=0$ for all $s\in[a,b]$ as $x(a,t)=x(s,c)=0$ (see Definition 2.1).

*Remark* 3.2. By Example 2.3, we know the expectation and variance of the random variable $X_{(s,t)}$;

$$E[X_{(s,t)}]=0\ \text{ and }\ V[X_{(s,t)}]=\frac{(s-a)(t-c)}{2}.$$

We use the average formula $\bar{f}_E:=\frac{\int_E f\,d\mu}{\mu(E)}$ in Definition 2.5 for the random variable $X_{(s,t)}$ over the set $E_{(s,t)}$ in $(C_2(Q),\mathcal{M}_2,m_2)$ in the definition below.

**Definition 3.4.** Let $(s,t)\in(a,b]\times(c,d]$. The average value $\bar{X}_{(s,t)}$ of a measurable function $X_{(s,t)}$ over a two-parameter Wiener measurable set $E_{(s,t)}$ is defined as follows.

$$\bar{X}_{(s,t)}:=\frac{\int_{E_{(s,t)}}X_{(s,t)}(x)\,dm_2(x)}{m_2(E_{(s,t)})}. \tag{3.3}$$

We show this newly defined formula in Equation (3.3) is well-defined.

**Lemma 3.3.** *$\bar{X}_{(\cdot,\cdot)}$ is a function from $(a,b]\times(c,d]$ to $\mathbb{R}$ and well-defined.*

*Proof.* The two-parameter Wiener integrals on a cylinder $E_{(s,t)}$ in Equation (3.3) are calculable since both are represented by Lebesgue integrals and can be regarded as Riemann integrals [14]. In fact, using Definition 2.3 and Proposition 2.2, the value $\bar{X}_{(s,t)}$ in Equation (3.3) becomes

$$\bar{X}_{(s,t)} = \frac{\int_\alpha^\beta \frac{1}{\sqrt{\pi(s-a)(t-c)}}\, u\, e^{\frac{-u^2}{(s-a)(t-c)}}\, du}{\int_\alpha^\beta \frac{1}{\sqrt{\pi(s-a)(t-c)}}\, e^{\frac{-u^2}{(s-a)(t-c)}}\, du} = \frac{\int_\alpha^\beta u\, e^{\frac{-u^2}{(s-a)(t-c)}}\, du}{\int_\alpha^\beta e^{\frac{-u^2}{(s-a)(t-c)}}\, du}. \tag{3.4}$$

We can integrate the last numerator in Equation (3.4). For simplicity, we denote $\alpha = \alpha(s,t)$ and $\beta = \beta(s,t)$ in Equation (3.4). The integrands both in the numerator and denominator are continuous functions of $u$, so they are integrable. Since $\alpha(s,t) < \beta(s,t)$ and the integrand $e^{\frac{-u^2}{(s-a)(t-c)}}$ of the denominator is positive, the denominator is positive; it is nonzero as required in Definition 2.5. Hence, for each $(s,t)$, $\bar{X}_{(s,t)}$ is uniquely mapped to a real number; i.e., it is a real-valued function defined on $(a,b] \times (c,d]$. $\square$

We now give the formal definition of an average function over a subset of $C_2(Q)$ by extending the definition $\bar{X}_{(\cdot,\cdot)}$ further to the rectangle $Q$. For the formal definition, we need to add the boundary, $s = a$ or $t = c$, to the set $(a,b] \times (c,d]$ in Lemma 3.3, and to give sensible function values of $\bar{X}_{(a,\cdot)}$ and $\bar{X}_{(\cdot,c)}$ there. For $s = a$, as explained for Definition 3.3, $X_{(a,\cdot)} \equiv 0$. Therefore, we define $\bar{X}_{(a,\cdot)} = 0$. Similarly for $t = c$, we define $\bar{X}_{(\cdot,c)} = 0$. We denote the extension of $\bar{X}_{(\cdot,\cdot)}$ by $\bar{X}(\cdot,\cdot)$ to all $(s,t) \in Q$ and use the notation in the next definition.

**Definition 3.5.** Let $Q = [a,b] \times [c,d]$ and $\mathcal{D} = \cap_{(s,t)\in Q} E_{(s,t)}$. Then the function $\bar{X} : Q \to \mathbb{R}$ defined by

$$\bar{X}(s,t) := \begin{cases} \bar{X}_{(s,t)} & \text{if } (s,t) \in (a,b] \times (c,d], \\ 0 & \text{if } s = a \text{ or } t = c, \end{cases} \tag{3.5}$$

is the average function of all $x$ in $\mathcal{D}$, where $\bar{X}_{(s,t)}$ is in the formula (3.3) (equivalently, (3.4)).

Here, $\mathcal{D}$ is bigger (denser) than $\mathcal{C}$ and the two function sets have cylinder boundaries $\alpha(s,t)$, $\beta(s,t)$ in common for all $(s,t) \in Q$. Note that we recycle the notation $\bar{X}$ in Equation (3.5).

**Theorem 3.4.** *Let $\{X_{(s,t)},\ (s,t) \in (a,b] \times (c,d]\}$ be a system of random variables in the formula* (3.2) *and $\bar{X}_{(s,t)}$ be the average value of $X_{(s,t)}$ over the set $E_{(s,t)}$ defined in Definition 3.5. Then for each $(s,t)$,*

$$\alpha(s,t) < \bar{X}_{(s,t)} \leq \beta(s,t). \tag{3.6}$$

*Proof.* Let $(s,t)$ be fixed in $(a,b] \times (c,d]$ and consider $E_{(s,t)}$ in the formula (3.1) and $\bar{X}_{(s,t)}$ in (3.4). We denote $\alpha = \alpha(s,t)$ and $\beta = \beta(s,t)$ for simplicity within the proof. We simplify the inequality (3.6) by multiplying the denominator in Equation (3.4), which is positive. The inequality $\alpha < \bar{X}_{(s,t)}$ is equivalent to

$$\alpha \int_{\alpha}^{\beta} e^{\frac{-u^2}{2(s-a)(t-c)}}\, du < \int_{\alpha}^{\beta} u\, e^{\frac{-u^2}{2(s-a)(t-c)}}\, du$$

and it suffices to show that

$$\int_{\alpha}^{\beta} (u-\alpha)\, e^{\frac{-u^2}{2(s-a)(t-c)}}\, du > 0.$$

Since the function $u$ is linear and $\alpha$ is constant, both are integrable with respect to $u$. The inequality follows as required since $\alpha < u$ and the exponential function in the integrand is positive for $(s,t) \in (a,b] \times (c,d]$ and $u \in (\alpha, \beta]$. The remaining part of the theorem, $\bar{X}_{(s,t)} \leq \beta$, follows in the same manner. These arguments are valid for all $(s,t) \in (a,b] \times (c,d]$ and so the inequalities (3.6) hold. □

**Theorem 3.5.** *If $f$ is continuous on $[v,w] \times Q$ to $\mathbb{R}$ and if $F$ is defined by $F(s,t) = \int_v^w f(x,s,t)dx$, then $F$ is continuous on $Q$.*

*Proof.* The uniform continuity theorem (such as Theorem 23.3 in [1]) implies that for $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that if $(s,t)$, $(s_0,t_0)$ belong to $Q$ and $|s-s_0| < \delta(\varepsilon)$ and $|t-t_0| < \delta(\varepsilon)$, then $|f(x,s,t) - f(x,s_0,t_0)| < \varepsilon$, for all $x$ in $[v,w]$. Then

$$\begin{aligned} |F(s,t) - F(s_0,t_0)| &= \left| \int_v^w f(x,s,t) - f(x,s_0,t_0)\ dx \right| \\ &\leq \int_v^w |f(x,s,t) - f(x,s_0,t_0)|\ dx \leq \varepsilon(w-v). \end{aligned}$$

Therefore, $F$ is continuous on $Q$. □

**Lemma 3.6.** *Let $f$ be a function from $[v,w] \times Q$ to $\mathbb{R}$ defined by*

$$f(u,s,t) = \begin{cases} e^{\frac{-u^2}{(s-a)(t-c)}} & \text{if } s > a,\ t > c, \\ 0 & \text{if } s = a \text{ or } t = c. \end{cases}$$

*Then $f$ is continuous on its domain.*

*Proof.* Since $f$ is an exponential function, it is continuous on $[v,w] \times (a,b] \times (c,d]$. For the points $(a,\cdot)$ and $(\cdot,c)$ of $(s,t)$, $\lim_{s\to a^+} e^{\frac{-u^2}{(s-a)(t-c)}} = 0$ and $\lim_{t\to c^+} e^{\frac{-u^2}{(s-a)(t-c)}} = 0$. Therefore, $f$ is continuous on $[v,w] \times Q$. □

## 3.1. Main theorems: continuity of the average function

It would be very desirable that the average function $\bar{X}(\cdot,\cdot)$ of the set $\mathcal{D}$ has continuity property on its domain $Q$. We examine the continuity in the next two theorems.

**Theorem 3.7.** *$\bar{X}(\cdot,\cdot)$ defined in Definition 3.5 is a continuous function of $(s,t)$ on $(a,b]\times(c,d]$.*

*Proof.* The function $\bar{X}_{(s,t)}$ is real-valued as proved in Lemma 3.3, so is $\bar{X}(\cdot,\cdot)$. To establish continuity, we need to show that the value $\bar{X}_{(s',t')}$ approaches to the value $\bar{X}_{(s,t)}$ whenever $(s',t')\to(s,t)$ for all $(s,t)\in(a,b]\times(c,d]$. Hence it suffices to show that

$$\lim_{(s',t')\to(s,t)} \bar{X}_{(s',t')} = \bar{X}_{(s,t)}.$$

As $\bar{X}_{(s',t')}$ and $\bar{X}_{(s,t)}$ are defined by integrals, the statement of limit above requires the interchange of a limit and an integral in its procedure. Then, we need to show

$$\begin{aligned}
\lim_{(s',t')\to(s,t)} \bar{X}_{(s',t')} &= \lim_{(s',t')\to(s,t)} \frac{\int_{\alpha(s',t')}^{\beta(s',t')} u\, e^{\frac{-u^2}{(s'-a)(t'-c)}}\, du}{\int_{\alpha(s',t')}^{\beta(s',t')} e^{\frac{-u^2}{(s'-a)(t'-c)}}\, du} \\
&= \frac{\lim_{(s',t')\to(s,t)} \int_{\alpha(s',t')}^{\beta(s',t')} u\, e^{\frac{-u^2}{(s'-a)(t'-c)}}\, du}{\lim_{(s',t')\to(s,t)} \int_{\alpha(s',t')}^{\beta(s',t')} e^{\frac{-u^2}{(s'-a)(t'-c)}}\, du} \qquad (3.7) \\
&= \frac{\int_{\alpha(s,t)}^{\beta(s,t)} u\, e^{\frac{-u^2}{(s-a)(t-c)}}\, du}{\int_{\alpha(s,t)}^{\beta(s,t)} e^{\frac{-u^2}{(s-a)(t-c)}}\, du} \qquad (3.8) \\
&= \bar{X}_{(s,t)}.
\end{aligned}$$

All of the integrals in these four equations are Riemann integrals. The first and last equations are trivial from Equation (3.4). We give the proof of Equations (3.7) and (3.8) in detail. The limit of a quotient can be evaluated by the quotient of limits since the limit of the denominator is non-zero, which has been discussed in Lemma 3.3. Then the basic limit laws are applied and Equation (3.7) follows. (The equation can also be obtained from the dominated convergence theorem.)

For Equation (3.8), let us denote the integral in its numerator by $\phi$:

$$\phi(s,t) = \int \alpha(s,t)^{\beta(s,t)} u\, e^{\frac{-u^2}{(s-a)(t-c)}}\, du.$$

Let $H$ be defined for $(y,z,s,t)$ by

$$H(y,z,s,t) = \int_y^z f(u,s,t)\, du,$$

when $f$ is a continuous function and $y,\ z$ belong to a bounded interval and $(s,t)$ belongs to the rectangle $Q$.

We introduce functions $H_1,\ H_2$ and $f$ to decompose $H$ into simpler expression;

$$
\begin{aligned}
f(u,s,t) &:= u\, e^{\frac{-u^2}{(s-a)(t-c)}};\\
H_1(y,z) &:= \int_y^z f(u,s,t)\,du;\\
H_2(s,t) &:= (\alpha(s,t),\beta(s,t));\\
H(y,z,s,t) &:= \int_y^z f(u,s,t)\,du.
\end{aligned}
$$

Here, the first variable $u$ of $f$ belongs to a bounded interval, and $(s,t)\in Q$. The exponential function in $f$ is continuous on $[y,z]\times Q$ by Lemma 3.6. Then the function $f$, which is an exponential function multiplied by a linear function, is continuous as well. $H_1$ is continuous at $(y,z)$ since if we regard $\int_y^v f(u,s,t)du$ (and also $\int_v^z f(u,s,t)du$) as the function of $v$ then it is continuous at $v\in[y,z]$ (p. 133 in [15]). $H_2$ is continuous at each $(s,t)$ as $\alpha$ and $\beta$ are continuous functions of $(s,t)$ as noted in Remark 3.1. Since $f$ is continuous, $H$ is continuous for $(s,t)$ in $Q$ by Theorem 3.5.

Then by substituting $y=\alpha(s,t)$ and $z=\beta(s,t)$ in $H$ above, we get $H(\alpha(s,t),\beta(s,t),s,t)$. Therefore, the function $\phi$ is, in fact, the composition $\phi(s,t)=H(\alpha(s,t),\beta(s,t),s,t)$. So, $\phi(s,t)$ is a continuous function of $(s,t)$.

Now we apply this continuity to any $(s,t)$ in $(a,b]\times(c,d]$ and so to $(s',t')$. Hence, we have

$$\lim_{(s',t')\to(s,t)}\phi(s',t')=\phi(s,t). \tag{3.9}$$

The limit implies that the numerators on both sides of Equation (3.8) are equal. Similar arguments are applied to the denominators and Equation (3.8) has been justified. Therefore, we have the desired result. □

By the fact that for a uniformly equicontinuous collection $\mathcal{C}$ (so that bounded) $\inf_{x\in\mathcal{C}}\{x(s,t)\}$, and $\beta(s,t):=\ \sup_{x\in\mathcal{C}}\ \{x(s,t)\}$ are both continuous functions of $(s,t)$. That is, $\alpha$ and $\beta$ are continuous on $(a,b]\times(c,d]$ as already mentioned in Remark 3.1. For the continuity of $\bar{X}(s,t)$ at the points of $s=a$ or $t=c$, we need the limit of $\alpha$ and $\beta$ on the boundary of the rectangle $Q$, which is clarified in the next lemma.

**Lemma 3.8.** $\displaystyle\lim_{(s,t)\to(a^+,c^+)}\beta(s,t)=0;\qquad \lim_{(s,t)\to(a^+,c^+)}\alpha(s,t)=0.$

*Proof.* We first show for $\beta$, here $\beta(s,t):=\ \sup_{x\in\mathcal{C}}\ \{x(s,t)\}$ (Definition 3.1). Suppose to the contrary that $\displaystyle\lim_{(s,t)\to(a^+,c^+)}\beta(s,t)=0$; i.e., the limit is non-zero. Then there exists a positive $\varepsilon_0$ such that for all $\delta>0$, if $a<s<a+\delta$, then $|\beta(s,t)|\geq\varepsilon_0$ for some $t$.

(i) For the case $\beta(s,t)\geq\varepsilon_0$: By the definition of supremum, there are infinitely many $x$ satisfying (for all $\delta>0$) $\beta(s,t)\geq x(s,t)\geq\varepsilon_0$ for $a<s<a+\delta$.

This contradicts to the fact that $x(a,t)=0$ for all $t$ and $x$ is continuous at $(a,t)$. Since $x(a,\cdot)=0$, $x(s,t)$ should be sufficiently close to 0 for $a<s<a+\delta$ and all $t$.

(ii) For the case $\beta(s,t)\le -\varepsilon_0$: Since $x(s,t)\le\beta(s,t)$ for all $x\in\mathcal{C}$, the case implies $x(s,t)\le -\varepsilon_0$ for all $x$ (for all $\delta>0$) for $a<s<a+\delta$. This also contradicts to the fact that $x(a,t)=0$ for all $x$ and $t$.

In either case of (i) and (ii), we are led to a contradiction, then $\lim\beta(s,t)=0$. The proof for infimum $\alpha$ is very similar to the arguments for supremum above, and we do not describe the proof for the second equation here. □

**Theorem 3.9.** *The average function $\bar{X}$ in Definition 3.5 is continuous on $Q$.*

*Proof.* The continuity in the statement already holds for $(s,t)\in(a,b]\times(c,d]$ as shown in Theorem 3.7 since $\bar{X}(s,t)=\bar{X}_{(s,t)}$ for $(s,t)\in(a,b]\times(c,d]$. Then it suffices to show that $\bar{X}(\cdot,\cdot)$ is continuous at points $(s,t)$, where $s=a$ or $t=c$. This requires to show

(3.10)
$$\lim_{(s,t)\to(a^+,t)}\bar{X}(s,t)=\bar{X}(a,t)(=0)\quad\text{and}\quad\lim_{(s,t)\to(s,c^+)}\bar{X}(s,t)=\bar{X}(s,c)(=0)$$

Note that $s>a$ when $s$ approaches $a$ in the limit since $a$ is the left end point of the interval $[a,b]$. So we consider only right continuity at points where $s=a$ or $t=c$.

Since $\alpha<u<\beta$ in the integral, $\left|\int_\alpha^\beta u\, e^{\frac{-u^2}{(s-a)(t-c)}}\,du\right|\le\left|\int_\alpha^\beta\beta\, e^{\frac{-u^2}{(s-a)(t-c)}}\,du\right|$. Then

$$\begin{aligned}\lim_{(s,t)\to(a^+,t)}\left|\bar{X}_{(s,t)}\right| &\le \lim_{(s,t)\to(a^+,t)}\left|\frac{\beta(s,t)\int_{\alpha(s,t)}^{\beta(s,t)} e^{\frac{-u^2}{(s-a)(t-c)}}\,du}{\int_{\alpha(s,t)}^{\beta(s,t)} e^{\frac{-u^2}{(s-a)(t-c)}}\,du}\right|\\ &=\lim_{(s,t)\to(a^+,t)}\left|\beta(s,t)\right|=0.\end{aligned}$$

The last equality being zero comes from Lemma 3.8. For this right continuity, we can alternatively use $\alpha(s,t)<\bar{X}_{(s,t)}\le\beta(s,t)$ and the squeeze theorem.

The second equality in Equation (3.10) can be justified in the same manner. One can also prove Equation (3.10) indirectly using contradiction on the behaviour of $\bar{X}(s,t)$ in $\delta$-neighbourhoods of $(a,t)$. □

From Theorems 3.4, 3.7 and 3.9 for $\bar{X}(\cdot,\cdot)$, we have the next corollary.

**Corollary 3.10.** *For any $(s,t)\in Q$, $\bar{X}(\cdot,\cdot)$ is bounded by $\alpha(\cdot,\cdot)$ and $\beta(\cdot,\cdot)$, and continuous on $Q$; in fact, $\bar{X}\in E_{(s,t)}$ for $(s,t)\in Q$. Therefore, $\bar{X}$ belongs to the sample set $\mathcal{D}$ from which $\bar{X}$ has been derived.*

We have formulated the average function of the sample set of $\mathcal{D}$ in the two-parameter Wiener space $(C_2(Q),\mathcal{M}_2,m_2)$. It has been proved that our average function is uniquely defined and a continuous function; moreover, it belongs the range of the sample set $\mathcal{D}$, both of which are very desirable.

## 4. A potential application to a probabilistic model of 3D shapes

The average function defined in Section 3 can be applied to a 3D shape model. Shape models play a crucial role in many fields of computer vision such as image analysis. In particular, the standard representation, in terms of an average, of shapes is essential to explain shape variations.

A 3D shape can be modelled on the two-parameter Wiener space, as its boundary (a surface) is mathematically represented by a continuous function of two variables. Our average function serves as an average shape in shape modelling. The idea is focused on biological or anatomical shapes that are not well characterised by a small number of discrete points. Such shapes are better identified by continuous functions of two variables while existing 3D shape models assume shapes as sets of discrete points.

As mentioned in Section 1, the known difficulties in dealing with shapes as continuous entities has strongly motivated our work. For 2D shapes accounting for continuity, shape models [9] have been developed by the author. There are few models for 3D shapes. Gutman et al. pointed out in [6] that "Comparison of simple 3D shapes remains one of the staples of medical image analysis". Recently, 3D shapes have been compared on a Riemannian manifold [5], [6]. All they are modelled on discrete sets for shape representation.

If one uses a relevant parameterisation technique for a set of 3D shapes, then our average function can be implemented and visualised.

## 5. Conclusions and discussions

We have formulated a new definition of the average of a set of continuous two-variable functions. Two-variable functions are regarded as the members of the two-parameter Wiener space. Employing the two-parameter Wiener process for the concept of distribution over the two-parameter Wiener space, the average of a sample set is mathematically defined. We have proved that the resulting average is a continuous function and lies within the range of the sample set, both of which are very desirable.

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