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A NOTE ON GENERALIZED DERIVATIONS AS A JORDAN HOMOMORPHISMS

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Abstract. Let R be a prime ring of characteristic different from 2. Suppose that F, G, H and T are generalized derivations of R. Let U be the Utumi quotient ring of R and C be the center of U, called the extended centroid of R and let $f(x_1,\dots,x_n)$ be a non central multilinear polynomial over C. If

$$F(f(r_1,\dots,r_n))G(f(r_1,\dots,r_n)) - f(r_1,\dots,r_n)T(f(r_1,\dots,r_n)) = H(f(r_1,\dots,r_n)^2)$$

for all $r_1,\dots,r_n \in R$, then we describe all possible forms of F, G, H and T.

1. Introduction

Throughout the article, R always denotes an associative prime ring with center $Z(R)$. The Utumi quotient ring of R is denoted by U. The center of U is called the extended centroid of R and it is denoted by C. Note that the extended centroid C, of a prime ring R, is always a field. The definition and axiomatic formulation of Utumi quotient ring U can be found in [3] and [8]. The Lie product of $x,y \in R$ is denoted by $[x,y]$ and $[x,y] = xy - yx$. A ring R is said to be a prime ring if for any $a,b \in R$, $aRb = 0$ implies either $a = 0$ or $b = 0$. Suppose S is a non empty subset of R and f is a mapping on R. A mapping f is called centralizing function (commuting function) on S if $[f(s),s] \in Z(R)([f(s),s] = 0)$ for all $s \in S$.

The study of commuting and centralizing mappings goes long back. In 1955 Divinsky [13] studied the commuting automorphism on rings. More precisely, Divinsky proved that a simple artinian ring is commutative if it has a commuting automorphism different from the identity mapping. It is natural to ask what happens if derivations behave like a centralizing functions on R? By derivation, we mean an additive mapping d on R such that $d(xy) = d(x)y + xd(y)$ for all $x,y \in R$. Define a mapping d_a on R by $d_a(x) = [a,x]$ for all $x \in R$, where

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$a \in R$ is fixed. Notice that d_a is a derivation on R, called an inner derivation induced by an element $a \in R$. The derivation is called an outer derivation if it is not an inner derivation. The answer to the above question on centralizing derivations on a prime ring R was given by Posner in [26]. More precisely, Posner proved that a prime ring must be commutative if it has a non zero centralizing derivation on R. In 1993, Brešar [6] extended the Posner's [26] result by taking two derivations. Brešar proved that if d and δ are two derivations of R such that $d(x)x - x\delta(x) \in Z(R)$ for all $x \in R$, then either $d = \delta = 0$ or R is commutative. Later on, many mathematicians extended these results on some appropriate subsets of a prime rings.

Another question that arises is what happens if x is replaced with multilinear polynomials in Posner's and Brešar's results in [26] and [6], respectively. The definition of a multilinear polynomial is given below.

Let $X = \{x_1, x_2, \ldots\}$ be a countable set with non commuting variables $x_1, x_2, \ldots$. Let $\mathbb{Z}\langle X\rangle$ be the free algebra on X over $\mathbb{Z}$. Let $f = f(x_1, \ldots, x_n) \in \mathbb{Z}\langle X\rangle$ be a polynomial such that at least one of its monomials of highest degree has coefficient 1. Let R be a nonempty subset of a ring A. We say that f is a polynomial identity on R if $f(r_1, \ldots, r_n) = 0$ for all $r_1, \ldots, r_n \in R$.

Definition. A polynomial $f = f(x_1, \ldots, x_n) \in \mathbb{Z}\langle X\rangle$ is said to be a multilinear if every $x_i, 1 \leq i \leq n$, appears exactly once in each of the monomials of f.

The answer to the above question was given by Lee and Shiue [22]. They proved that if R is a prime ring, $f(x_1, \ldots, x_n)$ a multilinear polynomial over C which is not central valued on R and d, g are derivations of R such that

$$d(f(x_1, \ldots, x_n))f(x_1 \ldots, x_n) - f(x_1, \ldots, x_n)g(f(x_1 \ldots, x_n)) \in C$$

for all $x_1, \ldots, x_n \in R$, then either $d = 0 = g$ or $d = -g$ and $f(x_1, \ldots, x_n)^2$ is central valued on R, except when $\mathrm{char}(R) = 2$ and $\dim_C (RC) = 4$.

An additive mapping f on ring R is said to be a homomorphism (or an anti-homomorphism) if $f(ab) = f(a)f(b)$ (or $f(ab) = f(b)f(a)$) for all $a, b \in R$. Bell and Kappe [4], proved that if d is a derivation of a prime ring R such that d acts as a homomorphism or as an anti-homomorphism on a non zero right ideal of R, then $d = 0$. The one extension of derivation is a generalized derivation. The notion of generalized derivation is given first by Brešar [5]. The definition of generalized derivation is given below.

Definition. Let R be a ring. A mapping F on R is called a generalized derivation on R if there exists a derivation d on R such that

$$F(x + y) = F(x) + F(y) \text{ and } F(xy) = F(x)y + xd(y)$$

for all $x, y \in R$. If R is a prime or a semiprime ring, then the derivation d is uniquely determined by F and is called the associated derivation of F.

Here, we notice that every derivation is a generalized derivation but the converse need not be true in general. An example of a generalized derivation which is not a derivation is given below.

Example 1.1. Let $\mathbb{Z}$ be the set of integers. Suppose $R = \{\left(\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right) \mid x, y, z \in \mathbb{Z}\}$. Define $d : R \to R$ as $d\left(\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & y \\ 0 & 0 \end{smallmatrix}\right)$. Then d is a derivation on R. Define a mapping F on R such that $F\left(\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & y \\ 0 & z \end{smallmatrix}\right)$. Then F is a generalized derivation associated with a non zero derivation d on R. Here, we see that F is not a derivation on R.

Another example of a generalized derivation is a mapping of the form $F(x) = ax + xb$ for all $x \in R$, where $a, b \in R$ is fixed. Such generalized derivations are called a generalized inner derivation. Generalized inner derivations and left multipliers are primarily studied on operator algebras. Therefore, any study from algebraic point of view might be interesting (see for example [2, 10, 17, 21, 23, 28]).

In 2016, Tiwari et al. [29] studied the results of Bell and Kappe [4] by replacing derivation with generalized derivation on ideals of prime rings. Further, Tiwari et al. [30] extended [29] and [4] results to the case of multiplicative (generalized)-derivation on ideals of semiprime rings.

Definition. An additive mapping f a ring R is said to be a Jordan homomorphism if $f(a^2) = (f(a))^2$ for all $a \in R$.

It is easily seen that every homomorphism is a Jordan homomorphism but the converse need not true in general. An example of a Jordan homomorphism, which is not a homomorphism is given below.

Example 1.2. Let R be a ring with involution $*$. Let $S = R \bigoplus R$ and $a \in Z(R)$ such that $r_1 a r_2 = 0$ for all $r_1, r_2 \in R$. Define a function f on S by $f(r, t) = (ar, t^*)$ for all $r, t \in R$. Then f is a Jordan homomorphism but not a homomorphism.

If R satisfies $f(a^2) = (f(a))^2$ for all $a \in R$, then by linearizing this we get $f(ab + ba) = f(a)f(b) + f(b)f(a)$ for all $a, b \in R$. It implies that $f(a \circ b) = f(a) \circ f(b)$ for all $a, b \in R$. If R is a 2-torsion free ring, then both properties are equivalent.

In 1956, Herstein [16] proved that if $f : R \to R'$ is a Jordan homomorphism, where R' is a prime ring and R is a ring with characteristic of R' is different from 2 and 3, then either f is a homomorphism or an anti-homomorphism. Further, Smiley [27] extended the above result and removed the restriction on characteristic not equal to 3 in the hypothesis of the Herstein's [16] theorem and proved that every Jordan homomorphism from ring R onto prime ring R' of characteristic different from 2 is either homomorphism or anti-homomorphism.

On the other hand Filippis et al. in [7], proved the following. Let R be a non commutative prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C, $f(x_1, \ldots, x_n)$ a multilinear polynomial over C which is not an identity for R, F and G non zero generalized derivations of R. If $F(u)G(u) = 0$ for all $u \in f(R) = \{f(r_1, \ldots, r_n) \mid r_i \in R\}$, then one of the following holds:

(i) There exist $a, c \in U$ such that $ac = 0$ and $F(x) = xa$, $G(x) = cx$ for all $x \in R$;
(ii) $f(x_1, \ldots, x_n)^2$ is central valued on R and there exist $a, c \in U$ such that $ac = 0$ and $F(x) = ax$, $G(x) = xc$ for all $x \in R$;
(iii) $f(x_1, \ldots, x_n)$ is central valued on R and there exist $a, b, c, q \in U$ such that $(a + b)(c + q) = 0$ and $F(x) = ax + xb$, $G(x) = cx + xq$ for all $x \in R$.

More recently, in 2018, Dhara [9] studied the following identities.

Let R be a non commutative prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C, $f(x_1, \ldots, x_n)$ a multilinear polynomial over C which is not central valued on R, F, G and H are generalized derivations of R. If $F(u)G(u) = H(u^2)$ for all $u = f(r_1, \ldots, r_n) \in f(R)$, then one of the following holds:

(i) there exist $a \in C$ and $b \in U$ such that $F(x) = ax$, $G(x) = xb$ and $H(x) = xab$ for all $x \in R$;
(ii) there exist $a, b \in U$ such that $F(x) = xa$, $G(x) = bx$ and $H(x) = abx$ for all $x \in R$ with $ab \in C$;
(iii) there exist $a \in U$ and $b \in C$ such that $F(x) = ax$, $G(x) = bx$ and $H(x) = abx$ for all $x \in R$;
(iv) $f(x_1, \ldots, x_n)^2$ is central valued on R and one of the following holds:
   (a) there exist $a, b, c, q \in U$ such that $F(x) = ax$, $G(x) = xb$ and $H(x) = cx + xq$ for all $x \in R$ with $ab = c + q$;
   (b) there exist $a, b, c, q \in U$ such that $F(x) = xa$, $G(x) = bx$ and $H(x) = cx + xq$ for all $x \in R$ with $c + q = ab \in C$.

2. Main results

One might wonder if it is possible that a generalized derivation acts as a Jordan homomorphism on some subset of a prime ring. Following this line of investigation, our main theorem gives a complete description of the forms of generalized derivations F, G, H and T of a prime ring R, in the case when generalized derivation F acts as a Jordan homomorphism. Further, our aim is to extend the results of De Filippis et al. [7], Dhara [9], Argac and De Filippis [1], Tiwari [28]. The statement of our main result is the following.

Theorem 2.1. *Let R be a prime ring of characteristic different from 2 and F, G, H and T generalized derivations on R. Let U be the Utumi ring of quotients of R with extended centroid C and $f(x_1, \ldots, x_n)$ be a non central multilinear polynomial over C. If*

$$F(f(r))G(f(r)) - f(r)H(f(r)) = T(f(r)^2)$$

for all $r = (r_1, \ldots, r_n) \in R^n$, then one of the following holds:

(i) *there exist $a \in C$, $b, b', c \in U$ such that $F(x) = ax$, $G(x) = bx + xb'$, $H(x) = abx - x(c - ab')$ and $T(x) = xc$ for all $x \in R$;*

(ii) *there exist* $a, b, c \in U$ *such that* $F(x) = xa$, $G(x) = bx$, $H(x) = abx - xc$ *and* $T(x) = xc$ *for all* $x \in R$;
(iii) *there exist* $a, c \in U$, $b \in C$ *such that* $F(x) = ax$, $G(x) = bx$, $H(x) = xc$ *and* $T(x) = abx - xc$ *for all* $x \in R$;
(iv) *there exist* $b, c \in U$, $a \in C$ *such that* $F(x) = ax$, $G(x) = xb$, $H(x) = xc$ *and* $T(x) = x(ab - c)$ *for all* $x \in R$;
(v) *there exist* $a, b, p \in U$, $c, \lambda \in C$ *such that* $F(x) = ax + xb$, $G(x) = cx$, $H(x) = bcx - \lambda x - xp$ *and* $T(x) = \lambda x + acx + xp$ *for all* $x \in R$;
(vi) $f(r_1, \ldots, r_n)^2$ *is central valued on* R *and one of the following holds*;
  (a) *there exist* $a \in C$, $b, b', p, p' \in U$ *such that* $F(x) = ax$, $G(x) = bx + xb'$, $H(x) = abx + xab' - x(p + p')$ *and* $T(x) = px + xp'$ *for all* $x \in R$;
  (b) *there exist* $a, b, p, p' \in U$ *such that* $F(x) = xa$, $G(x) = bx$, $H(x) = abx - x(p + p')$ *and* $T(x) = px + xp'$ *for all* $x \in R$;
  (c) *there exist* $a, b, c, p \in U$ *such that* $F(x) = ax$, $G(x) = xb$, $H(x) = xc$ *and* $T(x) = [p, x] - xc + xab$ *for all* $x \in R$;
  (d) *there exist* $a, b, p, q \in U$, $c \in C$ *such that* $F(x) = ax + xb$, $G(x) = cx$, $H(x) = c(bx + xa) - x(p + q)$ *and* $T(x) = px + xq$ *for all* $x \in R$.

The following corollaries are immediate consequences of our Theorem 2.1.

Corollary 2.2 ([9, Main Theorem])**.** *If we take* $H = 0$ *in our Theorem 2.1, then we get the theorem of Dhara* [9].

Corollary 2.3 ([7, Main Theorem])**.** *If we take* $H = 0 = T$ *in our Theorem 2.1, then we get the Carini, Filippis and Gsudo* [7, Main Theorem] *result.*

In particular, when $F = G$ and $H = 0$ in our Theorem 2.1, we obtain a particular result of De Filippis and Scudo [11, Theorem 1].

Corollary 2.4. *Let* R *be a prime ring of characteristic different from* 2 *and* F, T *generalized derivations on* R. *Let* U *be the Utumi ring of quotients of* R *with extended centroid* C *and* $f(x_1, \ldots, x_n)$ *be a non central multilinear polynomial over* C. *If*

$$(F(f(r)))^2 = T(f(r)^2)$$

for all $r = (r_1, \ldots, r_n) \in R^n$, *then one of the following holds*:
(i) *there exists* $a \in C$ *such that* $F(x) = ax$, $T(x) = a^2x$ *for all* $x \in R$;
(ii) *there exist* $a \in C$, $b, c \in U$ *such that* $F(x) = ax$ *and* $T(x) = bx + xc$ *for all* $x \in R$ *with* $a^2 = b + c$ *and* $f(x_1, \ldots, x_n)^2$ *is central valued on* R.

If we take $F = T$ in Corollary 2.4, we get the following.

Corollary 2.5. *Let* R *be a prime ring with characteristic different from* 2 *and* F *a generalized derivation on* R. *Let* U *be the Utumi ring of quotients with extended centroid* $C = Z(U)$ *and let* $f(x_1, \ldots, x_n)$ *be a multilinear polynomial over* C, *which is not central valued on* R. *If* $F(u^2) = (F(u))^2$ (*i.e., if* F *acts as a Jordan homomorphism*) *for all* $u \in f(R)$, *then* $F(x) = x$ *for all* $x \in R$.

The following corollary is an immediate application of Corollary 2.5.

Corollary 2.6. *Let R be a prime ring with characteristic different from 2 and F be a generalized derivation on R. Let U be the Utumi ring of quotients with extended centroid $C = Z(U)$ and let $f(x_1, \ldots, x_n)$ be a multilinear polynomial over C, which is not central valued on R. If $F(uv) = F(u)F(v)$ (i.e., F behaves as a homomorphism) or $F(uv) = F(v)F(u)$ (i.e., F behaves as an anti-homomorphism) for all $u, v \in f(R)$, then $F(x) = x$ for all $x \in R$.*

Proof. By our hypothesis, we have $F(uv) = F(u)F(v)$ for all $u, v \in f(R)$. This implies that $F(u^2) = (F(u))^2$ for all $u \in f(R)$. From Corollary 2.5, we get our conclusion. Similarly, we can show the case when F ia an anti-homomorphism. ☐

In particular if we take $f(r) = x$ in Corollary 2.5, we get the following.

Corollary 2.7. *Let R be a non commutative prime ring with characteristic different from 2 and F be a generalized derivation on R. Let U be the Utumi ring of quotients with extended centroid $C = Z(U)$. If $F(u^2) = (F(u))^2$ for all $u \in f(R)$, then $F(x) = x$ for all $x \in R$.*

If we take $T = 0$ and $G = I$, the identity mapping on R, then we get the result of Argac and De Filippis [1]. More precisely, we have:

Corollary 2.8 ([1, Main Theorem])**.** *Let R be a prime ring with characteristic different from 2 and U be its Utumi ring of quotients with extended centroid C. Suppose that F and G are two non zero generalized derivations of R such that $F(u)u - uG(u) = 0$ for all $u = f(x_1, \ldots, x_n) \in f(I)$, where $f(x_1, \ldots, x_n)$ is a non central multilinear polynomial over K and I is a non zero ideal of R. Then one of the following holds*:

(1) *there exists $a \in U$ such that $H(x) = xa$ and $G(x) = ax$ for all $x \in R$;*
(2) *$f(x_1, \ldots, x_n)^2$ is central valued on R and there exist $a, b \in U$ such that $H(x) = ax + xb$ and $G(x) = bx + xa$ for all $x \in R$.*

In particular for $G = I$, the identity mapping, we have the following corollary.

Corollary 2.9. *Let R be a prime ring with characteristic different from 2 and U be its Utumi ring of quotients with extended centroid $C = Z(U)$. Suppose that F, H and T are generalized derivations on R and $f(x_1, \ldots, x_n)$ is a non central multilinear polynomial over C such that $F(u)u - uH(u) = T(u^2)$ for all $u \in f(R)$, then one of the following holds.*

(i) *there exist $a, b, c \in U$, $\lambda \in C$ such that $F(x) = ax + xb$, $H(x) = (b - \lambda)x - xc$ and $T(x) = (a + \lambda)x + xc$ for all $x \in R$;*
(ii) *there exist $a, b, c, p \in U$ such that $F(x) = ax + xb$, $H(x) = bx + xa - x(c + p)$ and $T(x) = cx + xp$ for all $x \in R$ and $f(x_1, \ldots, x_n)^2$ is central valued on R.*

In particular for $F = H = T$ in Corollary 2.9, we obtain the following.

Corollary 2.10. *Let R be a prime ring with characteristic different from 2 and U be its Utumi ring of quotients with extended centroid $C = Z(U)$. Suppose that F is a generalized derivation on R and $f(x_1, \dots, x_n)$ is a non central multilinear polynomial over C such that $F(u)u - uF(u) = F(u^2)$ for all $u \in f(R)$, then $F = 0$.*

An immediate corollary is obtained by taking $G = I$, the identity mapping and $T = 0$ in our Theorem 2.1, which gives a particular case of Lee and Shiue [22], Brešar's [6]. Moreover by replacing $T = 0$, $G = I$, the identity mapping and $F = H = d$, a derivation, then corollary gives a famous result of Posner [26].

Corollary 2.11. *Let R be a prime ring of characteristic different from 2 and I be a non zero ideal of R. Suppose that d is a non zero derivation on R such that $[d(x), x] = 0$ for all $x \in I$, then R is commutative.*

3. F, G, H and T are an inner generalized derivations

In this section we study the situation when F, G, H and T are generalized inner derivations of R. For some $a, a', b, b', c, c', p, p' \in U$, let $F(x) = ax + xa'$, $G(x) = bx + xb'$, $H(x) = cx + xc'$ and $T(x) = px + xp'$ for all $x \in R$. Then we prove the following proposition:

Proposition 3.1. *Let R be a prime ring of characteristic different from 2 and F, G, H and T generalized inner derivations on R. Let U be the Utumi ring of quotients of R with extended centroid C and $f(x_1, \dots, x_n)$ be a non central multilinear polynomial over C. If*

$$F(f(r))G(f(r)) - f(r)H(f(r)) = T(f(r)^2)$$

for all $r = (r_1, \dots, r_n) \in R^n$, then one of the following holds:

- (i) *there exist $a \in C$, $b, b', c \in U$ such that $F(x) = ax$, $G(x) = bx + xb'$, $H(x) = abx - x(c - ab')$ and $T(x) = xc$ for all $x \in R$;*
- (ii) *there exist $a, b, c \in U$ such that $F(x) = xa$, $G(x) = bx$, $H(x) = abx - xc$ and $T(x) = xc$ for all $x \in R$;*
- (iii) *there exist $a, c \in U$, $b \in C$ such that $F(x) = ax$, $G(x) = bx$, $H(x) = xc$ and $T(x) = abx - xc$ for all $x \in R$;*
- (iv) *there exist $b, c \in U$, $a \in C$ such that $F(x) = ax$, $G(x) = xb$, $H(x) = xc$ and $T(x) = x(ab - c)$ for all $x \in R$;*
- (v) *there exist $a, b, p \in U$, $c, \lambda \in C$ such that $F(x) = ax + xb$, $G(x) = cx$, $H(x) = bcx - \lambda x - xp$ and $T(x) = \lambda x + acx + xp$ for all $x \in R$;*
- (vi) *$f(r_1, \dots, r_n)^2$ is central valued on R and one of the following holds;*
  - (a) *there exist $a \in C$, $b, b', p, p' \in U$ such that $F(x) = ax$, $G(x) = bx + xb'$, $H(x) = abx + xab' - x(p + p')$ and $T(x) = px + xp'$ for all $x \in R$;*

(b) *there exist* $a, b, p, p' \in U$ *such that* $F(x) = xa$, $G(x) = bx$, $H(x) = abx - x(p + p')$ *and* $T(x) = px + xp'$ *for all* $x \in R$;
(c) *there exist* $a, b, c, p \in U$ *such that* $F(x) = ax$, $G(x) = xb$, $H(x) = xc$ *and* $T(x) = [p, x] - xc + xab$ *for all* $x \in R$;
(d) *there exist* $a, b, p, q \in U$, $c \in C$ *such that* $F(x) = ax + xb$, $G(x) = cx$, $H(x) = c(bx + xa) - x(p + q)$ *and* $T(x) = px + xq$ *for all* $x \in R$.

To prove Proposition 3.1, we need the following results.

Lemma 3.2 ([12, Lemma 1])**.** *Let C be an infinite field and $m \geq 2$. If $A_1, \dots, A_k$ are not scalar matrices in $M_m(C)$, then there exists some invertible matrix $P \in M_m(C)$ such that each matrix $PA_1P^{-1}, \dots, PA_kP^{-1}$ has all non zero entries.*

Proposition 3.3. *Let $R = M_m(C)$, $m \geq 2$, be the ring of all $m \times m$ matrices over the infinite field C. Suppose that $f(x_1, \dots, x_n)$ is a non central multilinear polynomial over C and $a, b, a', b', c, p, w \in R$ such that $af(r)^2b' + af(r)bf(r) + f(r)wf(r) + f(r)a'f(r)b' - f(r)^2c - pf(r)^2 = 0$ for all $r = (r_1, \dots, r_n) \in f(R)$. Then one of the following holds*:
(1) $a, a' \in Z(R)$; (2) $a, b' \in Z(R)$; (3) $b, a' \in Z(R)$; (4) $b, b' \in Z(R)$.

Proof. By our hypothesis, R satisfies the generalized polynomial identity

$$\begin{aligned} &af(r_1, \dots, r_n)^2b' + af(r_1, \dots, r_n)bf(r_1, \dots, r_n) \\ &+ f(r_1, \dots, r_n)wf(r_1, \dots, r_n) + f(r_1, \dots, r_n)a'f(r_1, \dots, r_n)b' \\ &- f(r_1, \dots, r_n)^2c - pf(r_1, \dots, r_n)^2 \end{aligned} \tag{1}$$

for all $r_1, \dots, r_n \in R$. We shall prove this by contradiction. Suppose that $a \notin Z(R)$ and $b \notin Z(R)$.

Since $a \notin Z(R)$ and $b \notin Z(R)$ by Lemma 3.2 there exists a C-automorphism ϕ of $M_m(C)$ such that $\phi(a)$ and $\phi(b)$ have all non zero entries. Clearly $\phi(a)$, $\phi(b)$, $\phi(a')$, $\phi(b')$, $\phi(w)$, $\phi(c)$ and $\phi(p)$ must satisfy the condition (1).

Here e_{ij} denotes the matrix whose (i, j)-entry is 1 and rest of the entries are zero. Since $f(x_1, \dots, x_n)$ is not central, by [20] (see also [24]), there exist $s_1, \dots, s_n \in M_m(C)$ and $0 \neq \gamma \in C$ such that $f(s_1, \dots, s_n) = \gamma e_{ij}$, with $i \neq j$. Moreover, since the set $\{f(r_1, \dots, r_n) : r_1, \dots, r_n \in M_m(C)\}$ is invariant under the action of all C-automorphisms of $M_m(C)$, then for any $i \neq j$ there exist $r_1, \dots, r_n \in M_m(C)$ such that $f(r_1, \dots, r_n) = e_{ij}$. Hence by (1) we have

$$\phi(a)e_{ij}^2\phi(b') + \phi(a)e_{ij}\phi(b)e_{ij} + e_{ij}\phi(w)e_{ij} + e_{ij}\phi(a')e_{ij}\phi(b') - e_{ij}^2\phi(c) - \phi(p)e_{ij}^2 = 0.$$

That is

$$\phi(a)e_{ij}\phi(b)e_{ij} + e_{ij}\phi(w)e_{ij} + e_{ij}\phi(a')e_{ij}\phi(b') = 0. \tag{2}$$

Left multiplying by e_{ij}, we obtain $e_{ij}\phi(a)e_{ij}\phi(b)e_{ij} = 0$. Thus we have

$$\phi(a)_{ji}\phi(b)_{ji}e_{ij} = 0.$$

This gives a contradiction, since $\phi(a)$ and $\phi(b)$ have all non zero entries. Thus we conclude that either $\phi(a)$ or $\phi(b)$ is central. This gives either $a \in C$ or $b \in C$.

Next, we assume that $a' \notin Z(R)$ and $b' \notin Z(R)$. Using similar arguments as above we have used, we get the equation (2). Now, right multiplying by e_{ij} in the equation (2), we get

$$e_{ij}\phi(a')e_{ij}\phi(b')e_{ij} = 0$$

a contradiction, since $\phi(a')$ and $\phi(b')$ have all non zero entries. Combining these two we get the required results. □

Proposition 3.4. *Let $R = M_m(C)$, $m \geq 2$, be the ring of all matrices over the field C, with characteristic different from 2. Suppose that $f(x_1, \dots, x_n)$ is a non central multilinear polynomial over C and $a, b, a', b', c, p, w \in R$ such that $af(r)^2b' + af(r)bf(r) + f(r)wf(r) + f(r)a'f(r)b' - f(r)^2c - pf(r)^2 = 0$ for all $r = (r_1, \dots, r_n) \in f(R)$. Then one of the following holds*:
(1) $a, a' \in Z(R)$; (2) $a, b' \in Z(R)$; (3) $b, a' \in Z(R)$; (4) $b, b' \in Z(R)$.

Proof. If C is an infinite field, then conclusions follow from Proposition 3.3.

Now assume C is a finite field and let K be an infinite extension field of C. Let $\overline{R} = M_m(K) \cong R \otimes_C K$. Notice that the multilinear polynomial $f(x_1, \dots, x_n)$ is central valued on R if and only if it is central valued on $\overline{R}$. Suppose that the generalized polynomial $Q(r_1, \dots, r_n)$ such that

$$\begin{aligned} Q(r_1, \dots, r_n) = {} & af(r_1, \dots, r_n)^2b' + af(r_1, \dots, r_n)bf(r_1, \dots, r_n) \\ & + f(r_1, \dots, r_n)wf(r_1, \dots, r_n) + f(r_1, \dots, r_n)a'f(r_1, \dots, r_n)b' \\ & - f(r_1, \dots, r_n)^2c - pf(r_1, \dots, r_n)^2 \end{aligned}$$

is a generalized polynomial identity for R.

Moreover, it is a multihomogeneous of multidegree $(2, \dots, 2)$ in the indeterminates $r_1, \dots, r_n$. Hence the complete linearization of $Q(r_1, \dots, r_n)$ is a multilinear generalized polynomial $\Theta(r_1, \dots, r_n, x_1, \dots, x_n)$ in $2n$ indeterminates, moreover

$$\Theta(r_1, \dots, r_n, r_1, \dots, r_n) = 2^nQ(r_1, \dots, r_n).$$

It is clear that the multilinear polynomial $\Theta(r_1, \dots, r_n, x_1, \dots, x_n)$ is a generalized polynomial identity for both R and $\overline{R}$. Since characteristic of R is not two, we obtain $Q(r_1, \dots, r_n) = 0$ for all $r_1, \dots, r_n \in \overline{R}$ and then conclusion follows from Proposition 3.3. □

In view of above, we can write the following corollary.

Corollary 3.5. *Let $R = M_m(C)$ be the ring of all $m \times m$ matrices over the field C, where $m \geq 2$, with characteristic different from 2. If a_1, a_2, a_3, a_4, a_5 a_6, $a_7 \in R$ such that $a_1r^2a_2 + a_1ra_3r + ra_4r + ra_5ra_2 - r^2a_6 - a_7r^2 = 0$ for all $r \in R$, then either $a_1 \in C$ or $a_3 \in C$ and either $a_5 \in C$ or $a_2 \in C$.*

Proposition 3.6. *Let R be a primitive ring of characteristic different from 2 with a non zero socle which is isomorphic to a dense ring of linear transformations of an infinite dimensional vector space V over C. If a_1, a_2, a_3, a_4, a_5 a_6, $a_7 \in R$ such that $a_1r^2a_2 + a_1ra_3r + ra_4r + ra_5ra_2 - r^2a_6 - a_7r^2 = 0$ for all $r \in R$, then either $a_1 \in C$ or $a_3 \in C$ and either $a_5 \in C$ or $a_2 \in C$.*

Proof. We shall prove this proposition by contradiction. Suppose that neither a_1 nor a_3 and neither a_2 nor a_5 are in C. Since $\dim_C(V)$ is infinite. By Martindale's theorem [25, Theorem 3], for any $e^2 = e \in soc(R)$ we have $eRe \cong M_t(C)$ with $t = \dim_C Ve$. Since, none of a_1, a_3 in C and none of a_2 and a_5 in C, there exist $h_1, h_2, h_3, h_4 \in soc(R)$ such that $[a_1, h_1] \neq 0$, $[a_3, h_2] \neq 0$, $[a_5, h_3] \neq 0$ and $[a_2, h_4] \neq 0$. By Litoff's Theorem [15], there exists idempotent $e \in soc(R)$ such that $a_1h_1, h_1a_1, a_3h_2, h_2a_3, a_5h_3, h_3a_5, a_2h_4, h_4a_2, h_1, h_2, h_3, h_4 \in eRe$. Since R satisfies generalized identity

$$
\begin{aligned}
&e\Big\{a_1f(ex_1e,\dots,ex_ne)^2a_2 + a_1f(ex_1e,\dots,ex_ne)a_3f(ex_1e,\dots,ex_ne)\\
&\quad + f(ex_1e,\dots,ex_ne)a_4f(ex_1e,\dots,ex_ne)\\
&\quad + f(ex_1e,\dots,ex_ne)a_5f(ex_1e,\dots,ex_ne)a_2\\
&\quad - f(ex_1e,\dots,ex_ne)^2a_6 - a_7f(ex_1e,\dots,ex_ne)^2\Big\}e
\end{aligned} \tag{3}
$$

the subring eRe satisfies

$$
\begin{aligned}
&ea_1ef(x_1,\dots,x_n)^2ea_2e + ea_1ef(x_1,\dots,x_n)ea_3ef(x_1,\dots,x_n)\\
&+ f(x_1,\dots,x_n)ea_4ef(x_1,\dots,x_n) + f(x_1,\dots,x_n)ea_5ef(x_1,\dots,x_n)ea_2e\\
&- f(x_1,\dots,x_n)^2ea_6e - ea_7ef(x_1,\dots,x_n)^2.
\end{aligned} \tag{4}
$$

Then by the above finite dimensional case, either ea_1e or ea_3e and either ea_2e or ea_5e are central elements of eRe. Thus either $a_1h_1 = (ea_1e)h_1 = h_1ea_1e = h_1a_1$ or $a_3h_2 = (ea_3e)h_2 = h_2(ea_3e) = h_2a_3$ and either $a_5h_3 = (ea_5e)h_3 = h_3(ea_5e) = h_3a_5$ or $a_2h_4 = (ea_2e)h_4 = h_4(ea_2e) = h_4a_2$, a contradiction. □

Lemma 3.7. *Let R be a prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C and $f(x_1,\dots,x_n)$ a multilinear polynomial over C, which is not central valued on R. Suppose that for some $a, b, a', b', c, w, p \in R$ such that $af(r)^2b' + af(r)bf(r) + f(r)wf(r) + f(r)a'f(r)b' - f(r)^2c - pf(r)^2 = 0$ for all $r = (r_1,\dots,r_n) \in f(R)$. Then one of the following holds*:

(1) $a, a' \in Z(R)$; (2) $a, b' \in Z(R)$; (3) $b, a' \in Z(R)$; (4) $b, b' \in Z(R)$.

Proof. First, we shall prove that either $a \in C$ or $b \in C$. We shall prove this by contradiction. Suppose that $a \notin C$ and $b \notin C$. By hypothesis, we have

$$
\begin{aligned}
h(x_1,\dots,x_n) = {}& af(r_1,\dots,r_n)^2b' + af(r_1,\dots,r_n)bf(r_1,\dots,r_n)\\
&+ f(r_1,\dots,r_n)wf(r_1,\dots,r_n) + f(r_1,\dots,r_n)a'f(r_1,\dots,r_n)b'\\
&- f(r_1,\dots,r_n)^2c - pf(r_1,\dots,r_n)^2
\end{aligned} \tag{5}
$$

for all $r_1, \dots, r_n \in R$. Since R and U satisfy same generalized polynomial identity (GPI) (see [8]), U satisfies $h(r_1, \dots, r_n) = 0_T$. Suppose that $h(r_1, \dots, r_n)$ is a trivial GPI for U. Let $T = U *_C C\{r_1, \dots, r_n\}$, the free product of U and $C\{r_1, \dots, r_n\}$, the free C-algebra in non commuting indeterminates $r_1, \dots, r_n$. Then, $h(r_1, \dots, r_n)$ is zero element in $T = U *_C C\{r_1, \dots, r_n\}$. Since $a \notin C$ and $b \notin C$, the term $af(r_1, \dots, r_n)bf(r_1, \dots, r_n)$ appears nontrivially in $h(r_1, \dots, r_n)$. This gives a contradiction that is we have either $a \in C$ or $b \in C$.

Let $a \in C$. Then we shall show that either $a' \in C$ or $b' \in C$. Suppose that $a' \notin C$ and $b' \notin C$. Since $a \in C$, U satisfies

$$P(x_1, \dots, x_n) = f(x_1, \dots, x_n)(ab + w)f(x_1, \dots, x_n) + f(x_1, \dots, x_n)^2(ab' - c) + f(x_1, \dots, x_n)a'f(x_1, \dots, x_n)b' - pf(x_1, \dots, x_n)^2$$

for all $x_1, \dots, x_n \in R$. This is again a trivial GPI. Since $P(x_1, \dots, x_n) = 0_T$, the term $f(x_1, \dots, x_n)a'f(x_1, \dots, x_n)b'$ appears non trivially in $P(x_1, \dots, x_n)$. This implies that either $a' \in C$ or $b' \in C$, a contradiction. Thus we have either $a \in C$, $a' \in C$ or $a \in C$, $b' \in C$, which is our conclusion either (1) or (2).

Similarly, we can show that when $b \in C$ either $a' \in C$ or $b' \in C$, which is our conclusion either (3) and (4).

Next, suppose that $h(x_1, \dots, x_n)$ is a non trivial GPI for U. If C is infinite, then we have $h(x_1, \dots, x_n) = 0$ for all $x_1, \dots, x_n \in U \otimes_C \overline{C}$, where $\overline{C}$ is the algebraic closure of C. Since both U and $U \otimes_C \overline{C}$ are prime and centrally closed [14, Theorems 2.5 and 3.5], we may replace R by U or $U \otimes_C \overline{C}$ according to C finite or infinite. Then R is centrally closed over C and $h(x_1, \dots, x_n) = 0$ for all $x_1, \dots, x_n \in R$. By Martindale's theorem [25], R is then a primitive ring with non zero socle $soc(R)$ and C as its associated division ring. Then, by Jacobson's theorem [18, p. 75], R is isomorphic to a dense ring of linear transformations of a vector space V over C.

Assume first that V is finite dimensional over C, say $\dim_C V = m$. By density of R, we have $R \cong M_m(C)$. Since $f(r_1, \dots, r_n)$ is not central valued on R, R must be non commutative and so $m \geq 2$. By Proposition 3.3, we get that either $a, a' \in C$ or $a, b' \in C$ or $b, a' \in C$ or $b, b' \in C$.

If V is infinite dimensional over C, we use Proposition 3.6 to get the conclusions. □

Lemma 3.8. *Let R be a prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C and $f(x_1, \dots, x_n)$ a multilinear polynomial over C, which is not central valued on R. Suppose that for some $a, b, c, p, q \in U$ such that $af(r)^2b - f(r)cf(r) - f(r)^2q - pf(r)^2 = 0$ for all $r = (r_1, \dots, r_n) \in f(R)$. Then $c \in C$.*

Proof. By using similar argument as we have used above, we can get our conclusion. □

Lemma 3.9 ([9, Lemma 2.9])**.** *Let R be a non commutative prime ring of $char(R) \neq 2$, $a, b, c, c' \in U$, let $p(x_1, \ldots, x_n)$ be any polynomial over C which is not an identity for R. If $ap(r)+p(r)b+cp(r)c' = 0$ for all $r = (r_1, \ldots, r_n) \in R^n$, then one of the following holds*:

(1) $b, c' \in C$ *and* $a + b + cc' = 0$;
(2) $a, c \in C$ *and* $a + b + cc' = 0$;
(3) $a + b + cc' = 0$ *and* $p(x_1, \ldots, x_n)$ *is a central valued on* R.

The following Lemma is a particular case of Lemma 3 of [1].

Lemma 3.10. *Let R be a non commutative prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C and let $f(x_1, \ldots, x_n)$ be a multilinear polynomial over C which is not central valued on R. Suppose that there exist $a, b, c \in U$ such that $f(r)af(r)+f(r)^2b-cf(r)^2 = 0$ for all $r \in R^n$. Then one of the following conditions holds*:

(1) $b, c \in C$, $c - b = a = \alpha \in C$,
(2) $f(x_1, \ldots, x_n)^2$ *is central valued and there exists* $\alpha \in C$ *such that* $c-b = a = \alpha$.

Proof of Proposition 3.1. By our hypothesis, we have

$$\Big(af(r) + f(r)a'\Big)\Big(bf(r) + f(r)b'\Big) - f(r)\Big(cf(r) + f(r)c'\Big) = pf(r)^2 + f(r)^2p'$$

for all $r = (r_1, \ldots, r_n) \in R^n$. From Lemma 3.7, we get either $a, a' \in C$ or $a, b' \in C$ or $b, a' \in C$ or $b, b' \in C$. Now we will consider the following cases.

Case (I). Let $a, a' \in C$, then $F(x) = (a + a')x$ for all $x \in R$. Then by the hypothesis, we have

$$f(r)((a + a')b - c)f(r) + f(r)^2((a + a')b' - c' - p') - pf(r)^2 = 0$$

for all $r = (r_1, \ldots, r_n) \in R^n$. By Lemma 3.10, we have one of the following:

(1) $p \in C$, $(a+a')b'-c'-p' \in C$ and $p-(a+a')b'+c'+p' = (a+a')b-c = \alpha \in C$ for some $\alpha \in C$. This implies that $T(x) = x(p + p')$ and $c = (a + a')b - \alpha$, $c' = \alpha - (p + p') + (a + a')b'$. Thus, in this case we have $F(x) = (a+a')x$, $G(x) = bx+xb'$, $H(x) = (a+a')bx-x((p+p')-(a + a')b')$ and $T(x) = x(p + p')$ for all $x \in R$, which is our conclusion (i);
(2) $f(r_1, \ldots, r_n)^2$ is central valued on R and $p - (a + a')b' + c' + p' = (a + a')b - c = \alpha \in C$, which implies that $c = (a + a')b - \alpha$ and $c' = \alpha - p + (a + a')b' - p'$. Thus we have $F(x) = (a+a')x$, $G(x) = bx+xb'$, $H(x) = (a + a')bx - x(p + p') + x(a + a')b'$, which is our conclusion (vi (a)).

Case (II). Suppose that $a \in C$ and $b' \in C$. Then we have $F(x) = x(a + a')$ and $G(x) = (b + b')x$. By the hypothesis we have

$$f(r)\Big((a + a')(b + b') - c\Big)f(r) + f(r)^2\Big(- (c' + p')\Big) - pf(r)^2 = 0$$

for all $r = (r_1, \dots, r_n) \in R^n$. By Lemma 3.10, we have one of the following:

(1) $p \in C$, $c' + p' \in C$ and $p + c' + p' = (a + a')(b + b') - c = \alpha \in C$ for some $\alpha \in C$. This implies that $T(x) = x(p + p')$ and $c = (a + a')(b + b') - \alpha$, $c' = \alpha - p - p'$. Thus, in this case we have $F(x) = x(a + a')$, $G(x) = (b + b')x$, $H(x) = (a + a')(b + b')x - x(p + p')$ and $T(x) = x(p + p')$ for all $x \in R$, which is our conclusion (ii);

(2) $f(r_1, \dots, r_n)^2$ is central valued on R and $p + c' + p' = (a + a')(b + b') - c = \alpha \in C$ for some $\alpha \in C$, which implies that $c = (a + a')(b + b') - \alpha$, $c' = \alpha - p - p'$. Thus we have $F(x) = x(a + a')$, $G(x) = (b + b')x$, $H(x) = (a + a')(b + b')x - x(p + p')$, $T(x) = px + xp'$ for all $x \in R$, which is our conclusion (vi (b)).

Case (III). Suppose that $b \in C$, $a' \in C$. That is $F(x) = (a + a')x$, $G(x) = x(b + b')$ for all $x \in R$. Hence, our hypothesis reduces to

$$(a + a')f(r)^2(b + b') - f(r)cf(r) - f(r)^2(c' + p') - pf(r)^2 = 0$$

for all $r = (r_1, \dots, r_n) \in R^n$. From Lemma 3.8, it gives that $c \in C$ that is $H(x) = x(c + c')$. Then U satisfies

$$(a + a')f(r)^2(b + b') - f(r)^2(c + c' + p') - pf(r)^2 = 0$$

for all $r = (r_1, \dots, r_n) \in R^n$. By Lemma 3.9, we have one of the following:

(1) $-c - c' - p' \in C$, $b + b' \in C$ and $-p - c - c' - p' + (a + a')(b + b') = 0$, it implies that $-c - c' - p' = \lambda \in C$ and $p = (a + a')(b + b') + \lambda$. Thus in this case, we have $F(x) = (a + a')x$, $G(x) = x(b + b') = (b + b')x$, $H(x) = x(c + c')$ and $T(x) = px + xp' = (a + a')(b + b')x + \lambda x + x(-c - c' - \lambda) = (a + a')(b + b')x - x(c + c')$ for all $x \in R$, which is our conclusion (iii);

(2) $-p \in C$, $a + a' \in C$ and $-p - c - c' - p' + (a + a')(b + b') = 0$, it implies that $F(x) = (a + a')x$, $G(x) = x(b + b')$, $H(x) = x(c + c')$ and $T(x) = px + xp' = x(p + p') = x(-c - c' + (a + a')(b + b'))$ for all $x \in R$, which is our conclusion (iv);

(3) $f(r_1, \dots, r_n)^2$ is central valued on R and $-p - c - c' - p' + (a + a')(b + b') = 0$. In this case, we have $F(x) = (a + a')x$, $G(x) = x(b + b')$, $H(x) = x(c + c')$ and $T(x) = px + xp' = px + x(-p - c - c' + (a + a')(b + b')) = [p, x] + x((a + a')(b + b') - (c + c'))$ for all $x \in R$, which is our conclusion (vi (c)).

Case (IV). Suppose that $b \in C$, $b' \in C$. Then we have $F(x) = ax + xa'$, $G(x) = (b + b')x$, $H(x) = cx + xc'$ and $T(x) = px + xp'$ for all $x \in R$. Thus our hypothesis reduces to

$$(b + b')\Big(af(r)^2 + f(r)a'f(r)\Big) - f(r)\Big(cf(r) + f(r)c'\Big) = pf(r)^2 + f(r)^2p'$$

for all $r = (r_1, \dots, r_n)$, $r_1, \dots, r_n \in R$. This can be re-written as

$$\Big((b + b')a - p\Big)f(r)^2 + f(r)\Big((b + b')a' - c\Big)f(r) - f(r)^2\Big(c' + p'\Big) = 0$$

for all $r = (r_1, \ldots, r_n)$, $r_1, \ldots, r_n \in R$. By Lemma 3.10, we have one of the following:

(1) $-c' - p' \in C$, $p - (b+b')a \in C$ and $p - (b+b')a + c' + p' = (b+b')a' - c = \alpha$ for some $\alpha \in C$. This implies that $c = (b+b')a' - \alpha$ and $p = \gamma + (b+b')a$ and $c' = -\beta - p'$, where $-c' - p' = \beta$ and $p - (b+b')a = \gamma$ for some $\beta, \gamma \in C$. We notice that $p - (b+b')a + c' + p' = \alpha$, this gives that $\alpha + \beta = p - (b+b')a = \gamma$. Thus, in this case we have $F(x) = ax + xa'$, $G(x) = (b+b')x$, $H(x) = cx + xc' = (b+b')a'x - \alpha x - x\beta - xp' = (b+b')a'x - (\alpha+\beta)x - xp' = (b+b')a'x - \gamma x - xp'$ and $T(x) = px + xp' = (b+b')ax + \gamma x + xp'$ for all $x \in R$, which is our conclusion (v);

(2) $f(r_1, \ldots, r_n)^2$ is central valued on R and $p - (b+b')a + c' + p' = (b+b')a' - c = \alpha$ for some $\alpha \in C$, which implies that $c = (b+b')a' - \alpha$, $c' = \alpha - p - p' + (b+b')a$. In this case, we have $F(x) = ax + xa'$, $G(x) = (b+b')x$, $H(x) = cx + xc' = (b+b')a'x - \alpha x + \alpha x - x(p+p') + x(b+b')a = (b+b')a'x + x(b+b')a - x(p+p')$ and $T(x) = px + xp'$ for all $x \in R$, which is our conclusion (vi (d)).

This proves Proposition 3.1. □

Lemma 3.11. *Let R be a prime ring of characteristic different from 2 and F, G, H and T generalized derivations on R. Let U be the Utumi ring of quotients of R with extended centroid C and $f(x_1, \ldots, x_n)$ be a non central multilinear polynomial over C. If any three of F, G, T, H are generalized inner derivations on R such that*

$$F(f(r))G(f(r)) - f(r)H(f(r)) = T(f(r)^2)$$

for all $r = (r_1, \ldots, r_n) \in R^n$, then one of the following holds:

(i) *there exist $a \in C$, $b, b', c \in U$ such that $F(x) = ax$, $G(x) = bx + xb'$, $H(x) = abx - x(c - ab')$ and $T(x) = xc$ for all $x \in R$;*

(ii) *there exist $a, b, c \in U$ such that $F(x) = xa$, $G(x) = bx$, $H(x) = abx - xc$ and $T(x) = xc$ for all $x \in R$;*

(iii) *there exist $a, c \in U$, $b \in C$ such that $F(x) = ax$, $G(x) = bx$, $H(x) = xc$ and $T(x) = abx - xc$ for all $x \in R$;*

(iv) *there exist $b, c \in U$, $a \in C$ such that $F(x) = ax$, $G(x) = xb$, $H(x) = xc$ and $T(x) = x(ab - c)$ for all $x \in R$;*

(v) *there exist $a, b, p \in U$, $c, \lambda \in C$ such that $F(x) = ax + xb$, $G(x) = cx$, $H(x) = bcx - \lambda x - xp$ and $T(x) = \lambda x + acx + xp$ for all $x \in R$;*

(vi) *$f(r_1, \ldots, r_n)^2$ is central valued on R and one of the following holds;*

(a) *there exist $a \in C$, $b, b', p, p' \in U$ such that $F(x) = ax$, $G(x) = bx + xb'$, $H(x) = abx + xab' - x(p + p')$ and $T(x) = px + xp'$ for all $x \in R$;*

(b) *there exist $a, b, p, p' \in U$ such that $F(x) = xa$, $G(x) = bx$, $H(x) = abx - x(p + p')$ and $T(x) = px + xp'$ for all $x \in R$;*

(c) *there exist $a, b, c, p \in U$ such that $F(x) = ax$, $G(x) = xb$, $H(x) = xc$ and $T(x) = [p, x] - xc + xab$ for all $x \in R$;*

(d) *there exist* $a, b, p, q \in U$, $c \in C$ *such that* $F(x) = ax + xb$, $G(x) = cx$, $H(x) = c(bx + xa) - x(p + q)$ *and* $T(x) = px + xq$ *for all* $x \in R$.

Proof. To prove this Lemma, we shall study the following cases.

Case 1. Let F, G, H be generalized inner derivations and T a generalized derivation on R. If T is a generalized inner derivation on R, then by Proposition 3.1, we get our conclusions. Suppose that T is not a generalized inner derivation on R. For some $a, b, u, c, p, p', q \in U$ such that $F(x) = ax + xb$, $G(x) = ux + xc$, $H(x) = px + xp'$ and $T(x) = qx + d(x)$, where d is a derivation on U. If d is an inner derivation, then T is a generalized inner derivation, a contradiction. Thus d can not be an inner derivation on R. Then U satisfies

$$\begin{aligned} &af(r_1, \ldots, r_n)uf(r_1, \ldots, r_n) + af(r_1, \ldots, r_n)^2c \\ &+ f(r_1, \ldots, r_n)buf(r_1, \ldots, r_n) + f(r_1, \ldots, r_n)bf(r_1, \ldots, r_n)c \\ &- f(r_1, \ldots, r_n)pf(r_1, \ldots, r_n) - f(r_1, \ldots, r_n)^2p' \\ =\ &qf(r_1, \ldots, r_n)^2 + d(f(r_1, \ldots, r_n))f(r_1, \ldots, r_n) \\ &+ f(r_1, \ldots, r_n)d(f(r_1, \ldots, r_n)). \end{aligned} \tag{6}$$

By using Kharchenko's theorem [19], we can replace

$$d(f(r_1, \ldots, r_n)) = f^d(r_1, \ldots, r_n) + \sum_i f(r_1, \ldots, y_i, \ldots, r_n),$$

where $d(r_i) = y_i$ in (6), we get

$$\begin{aligned} &af(r_1, \ldots, r_n)uf(r_1, \ldots, r_n) + af(r_1, \ldots, r_n)^2c \\ &+ f(r_1, \ldots, r_n)buf(r_1, \ldots, r_n) + f(r_1, \ldots, r_n)bf(r_1, \ldots, r_n)c \\ &- f(r_1, \ldots, r_n)pf(r_1, \ldots, r_n) - f(r_1, \ldots, r_n)^2p' \\ =\ &qf(r_1, \ldots, r_n)^2 + \Big(f^d(r_1, \ldots, r_n) + \sum_i f(r_1, \ldots, y_i, \ldots, r_n)\Big)f(r_1, \ldots, r_n) \\ &+ f(r_1, \ldots, r_n)\Big(f^d(r_1, \ldots, r_n) + \sum_i f(r_1, \ldots, y_i, \ldots, r_n)\Big). \end{aligned} \tag{7}$$

Then U satisfies the blended component

$$\begin{aligned} &\sum_i f(r_1, \ldots, y_i, \ldots, r_n)f(r_1, \ldots, r_n) \\ &+ f(r_1, \ldots, r_n)\sum_i f(r_1, \ldots, y_i, \ldots, r_n) = 0. \end{aligned} \tag{8}$$

Substituting $y_1 = r_1$ and $y_i = 0$ for $i \geq 2$, we get $2f(r_1, \ldots, r_n)^2 = 0$. Since characteristic of R is not 2, we get $f(r_1, \ldots, r_n)^2 = 0$, which gives a contradiction.

Case 2. Suppose that F, G, T are generalized inner derivations and H is a generalized derivation on R. By applying similar argument as we have used above(see Case 1 of Lemma 3.11), we get our conclusions.

Case 3. Suppose that F, H, T are generalized inner derivations and G is a generalized derivation on R. By applying similar argument as we have used above(see Case 1 of Lemma 3.11), we get our conclusions.

Case 4. Suppose that G, H, T are generalized inner derivations and F is a generalized derivation on R. By applying similar argument as we have used above(see Case 1 of Lemma 3.11), we get our conclusions. □

Lemma 3.12. *Let R be a prime ring of characteristic different from 2 and F, G, H and T generalized derivations on R. Let U be the Utumi ring of quotients of R with extended centroid C and $f(x_1, \ldots, x_n)$ be a non central multilinear polynomial over C. If two of F, G, T, H are generalized inner derivations on R such that*

$$F(f(r))G(f(r)) - f(r)H(f(r)) = T(f(r)^2)$$

for all $r = (r_1, \ldots, r_n) \in R^n$, then one of the following holds:

(i) *there exist $a \in C$, $b, b', c \in U$ such that $F(x) = ax$, $G(x) = bx + xb'$, $H(x) = abx - x(c - ab')$ and $T(x) = xc$ for all $x \in R$;*
(ii) *there exist $a, b, c \in U$ such that $F(x) = xa$, $G(x) = bx$, $H(x) = abx - xc$ and $T(x) = xc$ for all $x \in R$;*
(iii) *there exist $a, c \in U$, $b \in C$ such that $F(x) = ax$, $G(x) = bx$, $H(x) = xc$ and $T(x) = abx - xc$ for all $x \in R$;*
(iv) *there exist $b, c \in U$, $a \in C$ such that $F(x) = ax$, $G(x) = xb$, $H(x) = xc$ and $T(x) = x(ab - c)$ for all $x \in R$;*
(v) *there exist $a, b, p \in U$, $c, \lambda \in C$ such that $F(x) = ax + xb$, $G(x) = cx$, $H(x) = bcx - \lambda x - xp$ and $T(x) = \lambda x + acx + xp$ for all $x \in R$;*
(vi) *$f(r_1, \ldots, r_n)^2$ is central valued on R and one of the following holds;*
  (a) *there exist $a \in C$, $b, b', p, p' \in U$ such that $F(x) = ax$, $G(x) = bx + xb'$, $H(x) = abx + xab' - x(p + p')$ and $T(x) = px + xp'$ for all $x \in R$;*
  (b) *there exist $a, b, p, p' \in U$ such that $F(x) = xa$, $G(x) = bx$, $H(x) = abx - x(p + p')$ and $T(x) = px + xp'$ for all $x \in R$;*
  (c) *there exist $a, b, c, p \in U$ such that $F(x) = ax$, $G(x) = xb$, $H(x) = xc$ and $T(x) = [p, x] - xc + xab$ for all $x \in R$;*
  (d) *there exist $a, b, p, q \in U$, $c \in C$ such that $F(x) = ax + xb$, $G(x) = cx$, $H(x) = c(bx + xa) - x(p + q)$ and $T(x) = px + xq$ for all $x \in R$.*

Proof. To prove this Lemma, we shall study the following cases.

Case 1. Suppose that F, G are generalized inner derivations and H, T are generalized derivations on R. If one of H and T is a generalized inner derivation on R, then by Lemma 3.11, we get our conclusions. Let $F(x) = ax + xb$, $G(x) = ux + xc$, $H(x) = px + d_1(x)$ and $T(x) = qx + d_2(x)$, where d_1, d_2 are derivations on U for some $a, b, u, c, p, q \in U$. Assume that H and T both are not generalized inner derivation on R, then d_1 and d_2 can not be an inner derivations. Thus U satisfies

$$\begin{aligned} & af(r_1,\ldots,r_n)uf(r_1,\ldots,r_n)+af(r_1,\ldots,r_n)^2c \\ & + f(r_1,\ldots,r_n)buf(r_1,\ldots,r_n)+f(r_1,\ldots,r_n)bf(r_1,\ldots,r_n)c \\ & - f(r_1,\ldots,r_n)pf(r_1,\ldots,r_n)-f(r_1,\ldots,r_n)d_1(f(r_1,\ldots,r_n)) \\ = {} & qf(r_1,\ldots,r_n)^2+d_2(f(r_1,\ldots,r_n))f(r_1,\ldots,r_n) \\ & + f(r_1,\ldots,r_n)d_2(f(r_1,\ldots,r_n)). \end{aligned} \tag{9}$$

Now we shall study following two subcases:

Subcase-I. Let d_1 and d_2 be C-dependent modulo inner derivation of U. Then, for some $\alpha_1, \alpha_2 \in C$ and $P \in U$ such that $\alpha_1 d_1(x) + \alpha_2 d_2(x) = [P, x]$ for all $x \in U$. If $\alpha_1 = 0$, then α_2 can not be zero. This implies that d_2 is an inner derivation on R, a contradiction. Similarly, if $\alpha_2 = 0$, we get a contradiction. Now we assume α_1 and α_2 both are non zero. This gives $d_1(x) = \beta d_2(x) + [P', x]$ for all $x \in U$, where $\beta = -\alpha_1^{-1}\alpha_2$ and $P' = \alpha_1^{-1}P$. Hence U satisfies

$$\begin{aligned} & af(r_1,\ldots,r_n)uf(r_1,\ldots,r_n)+af(r_1,\ldots,r_n)^2c \\ & + f(r_1,\ldots,r_n)buf(r_1,\ldots,r_n)+f(r_1,\ldots,r_n)bf(r_1,\ldots,r_n)c \\ & - f(r_1,\ldots,r_n)pf(r_1,\ldots,r_n)-\beta f(r_1,\ldots,r_n)d_2(f(r_1,\ldots,r_n)) \\ & - f(r_1,\ldots,r_n)\big[P', f(r_1,\ldots,r_n)\big] \\ = {} & qf(r_1,\ldots,r_n)^2+d_2(f(r_1,\ldots,r_n))f(r_1,\ldots,r_n) \\ & + f(r_1,\ldots,r_n)d_2(f(r_1,\ldots,r_n)). \end{aligned} \tag{10}$$

By applying Kharchenko's theorem [19] to (10), U satisfies the blended component

$$\begin{aligned} & -\beta f(r_1,\ldots,r_n)\Big(\sum_i f(r_1,\ldots,y_i,\ldots,r_n)\Big) \\ = {} & \Big(\sum_i f(r_1,\ldots,y_i,\ldots,r_n)\Big)f(r_1,\ldots,r_n) \\ & + f(r_1,\ldots,r_n)\Big(\sum_i f(r_1,\ldots,y_i,\ldots,r_n)\Big) \end{aligned} \tag{11}$$

for all $r_1, \ldots, r_n \in R$. Replacing y_i with $[w, r_i]$ for some $w \notin C$ in (11), we have that U satisfies

$$\begin{aligned} -\beta f(r_1,\ldots,r_n)\Big[w, f(r_1,\ldots,r_n)\Big] = {} & \Big[w, f(r_1,\ldots,r_n)\Big]f(r_1,\ldots,r_n) \\ & + f(r_1,\ldots,r_n)\Big[w, f(r_1,\ldots,r_n)\Big] \end{aligned} \tag{12}$$

for all $r_1, \ldots, r_n \in R$. This implies that

$$-f(r_1,\ldots,r_n)\beta wf(r_1,\ldots,r_n)+f(r_1,\ldots,r_n)^2(\beta w+w)-wf(r_1,\ldots,r_n)^2=0$$

for all $r_1, \ldots, r_n \in R$. This gives that $\beta w \in C$. Since $\beta \neq 0$, hence it gives $w \in C$, a contradiction.

Subcase-II. Let d_1 and d_2 be C-independent. Then in (9) substituting the values of $d_1(f(r_1,\dots,r_n))$ and $d_2(f(r_1,\dots,r_n))$ and then applying Kharchenko's theorem [19], U satisfies the blended component

$$f(r_1,\dots,r_n)\Big\{\sum_i f(r_1,\dots,y_i,\dots,r_n)\Big\}=0. \tag{13}$$

In particular for $y_1=r_1$ and $y_2=\cdots=y_n=0$, we have $f(r_1,\dots,r_n)^2=0$ a contradiction.

Case 2. Suppose that F,H are generalized inner derivations and G, T are generalized derivations on R. By applying similar argument as we have used in above (see Case 1; Lemma 3.12), we get our conclusions.

Case 3. Suppose that F, T are generalized inner derivations and G, H are generalized derivations on R. By using similar argument as we have used in above (see Case 1; Lemma 3.12), we get our conclusions.

Case 4. Suppose that G, H are generalized inner derivations and F, T are generalized derivations on R. By using similar argument as we have used in above (see Case 1; Lemma 3.12), we get our conclusions.

Case 5. Suppose that G, T are generalized inner derivations and F, H are generalized derivations on R. By using similar argument as we have used in above (see Case 1; Lemma 3.12), we get our conclusions.

Case 6. Suppose that H, T are generalized inner derivations and F, G are generalized derivations on R. By applying similar argument as we have used above(see Case 1 of Lemma 3.12), we get our conclusions. □

Lemma 3.13. *Let R be a prime ring of characteristic different from 2 and F, G, H, T generalized derivations on R. Let U be the Utumi ring of quotients of R with extended centroid C and $f(x_1,\dots,x_n)$ be a non central multilinear polynomial over C. If one of F, G, T, H is a generalized inner derivation on R such that*

$$F(f(r))G(f(r))-f(r)H(f(r))=T(f(r)^2)$$

for all $r=(r_1,\dots,r_n)\in R^n$, then one of the following holds:

- (i) *there exist $a\in C$, $b,b',c\in U$ such that $F(x)=ax$, $G(x)=bx+xb'$, $H(x)=abx-x(c-ab')$ and $T(x)=xc$ for all $x\in R$;*
- (ii) *there exist $a,b,c\in U$ such that $F(x)=xa$, $G(x)=bx$, $H(x)=abx-xc$ and $T(x)=xc$ for all $x\in R$;*
- (iii) *there exist $a,c\in U$, $b\in C$ such that $F(x)=ax$, $G(x)=bx$, $H(x)=xc$ and $T(x)=abx-xc$ for all $x\in R$;*
- (iv) *there exist $b,c\in U$, $a\in C$ such that $F(x)=ax$, $G(x)=xb$, $H(x)=xc$ and $T(x)=x(ab-c)$ for all $x\in R$;*
- (v) *there exist $a,b,p\in U$, $c,\lambda\in C$ such that $F(x)=ax+xb$, $G(x)=cx$, $H(x)=bcx-\lambda x-xp$ and $T(x)=\lambda x+acx+xp$ for all $x\in R$;*
- (vi) *$f(r_1,\dots,r_n)^2$ is central valued on R and one of the following holds*;

(a) *there exist* $a \in C$, $b, b', p, p' \in U$ *such that* $F(x) = ax$, $G(x) = bx + xb'$, $H(x) = abx + xab' - x(p + p')$ *and* $T(x) = px + xp'$ *for all* $x \in R$;
(b) *there exist* $a, b, p, p' \in U$ *such that* $F(x) = xa$, $G(x) = bx$, $H(x) = abx - x(p + p')$ *and* $T(x) = px + xp'$ *for all* $x \in R$;
(c) *there exist* $a, b, c, p \in U$ *such that* $F(x) = ax$, $G(x) = xb$, $H(x) = xc$ *and* $T(x) = [p, x] - xc + xab$ *for all* $x \in R$;
(d) *there exist* $a, b, p, q \in U$, $c \in C$ *such that* $F(x) = ax + xb$, $G(x) = cx$, $H(x) = c(bx+xa)-x(p+q)$ *and* $T(x) = px+xq$ *for all* $x \in R$.

Proof. To prove this Lemma, we shall study the following cases.

Case 1. Suppose that F is a generalized inner derivation on R and G, H, T are generalized derivations on R. Let $a, b, c, p, q \in U$ such that $F(x) = ax + xb$, $G(x) = cx + g(x)$, $H(x) = px + h(x)$ and $T(x) = qx + d(x)$, where g, h, d are derivations on U. If one of g, h, d is an inner, then by Lemma 3.12, we get our conclusions. Now suppose that all h, g and d are not inner derivations. Then U satisfies

$$\begin{aligned} &\Big(af(r_1, \ldots, r_n) + f(r_1, \ldots, r_n)b\Big)\Big(cf(r_1, \ldots, r_n) + g(f(r_1, \ldots, r_n))\Big) \\ &\quad - f(r_1, \ldots, r_n)pf(r_1, \ldots, r_n) - f(r_1, \ldots, r_n)h(f(r_1, \ldots, r_n)) \\ &= qf(r_1, \ldots, r_n)^2 + d(f(r_1, \ldots, r_n))f(r_1, \ldots, r_n) \\ &\quad + f(r_1, \ldots, r_n)d(f(r_1, \ldots, r_n)) \end{aligned} \tag{14}$$

for all $r_1, \ldots, r_n \in R$. Now we shall study the following cases.

Subcase-I. Let g, h and d be linearly C-dependent modulo inner derivations. Then for some $\alpha, \beta, \gamma \in C$ such that $\alpha g(x) + \beta h(x) + \gamma d(x) = [u, x]$ for all $x \in R$ and $u \in U$. If $\alpha = 0 = \beta$, then γ can not be zero. Hence, it implies that d is an inner derivation, a contradiction. If $\alpha = 0 = \gamma$, then β can not be zero, it gives h is an inner derivation, a contradiction. If $\beta = 0 = \gamma$, then α can not be zero, gives g is an inner derivation, a contradiction. Hence two of α, β, γ can not be zero.

If $\alpha = 0$, $\beta \neq 0$, $\gamma \neq 0$, then $h(x) = \gamma' d(x) + [u', x]$, where $\gamma' = -\beta^{-1}\gamma$, $u' = \beta^{-1}u$. Equation (14) reduces to

$$\begin{aligned} &\Big(af(r_1, \ldots, r_n) + f(r_1, \ldots, r_n)b\Big)\Big(cf(r_1, \ldots, r_n) + g(f(r_1, \ldots, r_n))\Big) \\ &\quad - f(r_1, \ldots, r_n)pf(r_1, \ldots, r_n) - \gamma' f(r_1, \ldots, r_n)d(f(r_1, \ldots, r_n)) \\ &\quad - f(r_1, \ldots, r_n)\Big[u', f(r_1, \ldots, r_n)\Big] \\ &= qf(r_1, \ldots, r_n)^2 + d(f(r_1, \ldots, r_n))f(r_1, \ldots, r_n) \\ &\quad + f(r_1, \ldots, r_n)d(f(r_1, \ldots, r_n)). \end{aligned} \tag{15}$$

If g and d are linearly C-dependent modulo inner derivations, then $\alpha_1 g(x) + \alpha_2 d(x) = [p', x]$ for some $\alpha_1, \alpha_2 \in C$, $p' \in U$. Since g and d are not an inner derivations, hence α_1, α_2 can not be zero. Then $g(x) = \alpha_2{}'d(x) + [p'', x]$, where $\alpha_2{}' = -\alpha_1^{-1}\alpha_2$, $p'' = \alpha_1^{-1}p'$ and then U satisfies

$$
\begin{aligned}
&\Big(af(r_1,\ldots,r_n)+f(r_1,\ldots,r_n)b\Big)\Big(cf(r_1,\ldots,r_n)+\alpha_2{}'d(f(r_1,\ldots,r_n))\\
&\quad+\Big[p'', f(r_1,\ldots,r_n)\Big]\Big) - f(r_1,\ldots,r_n)pf(r_1,\ldots,r_n)\\
&\quad-\gamma' f(r_1,\ldots,r_n)d(f(r_1,\ldots,r_n)) - f(r_1,\ldots,r_n)\Big[u', f(r_1,\ldots,r_n)\Big]\\
&= qf(r_1,\ldots,r_n)^2 + d(f(r_1,\ldots,r_n))f(r_1,\ldots,r_n)\\
&\quad+ f(r_1,\ldots,r_n)d(f(r_1,\ldots,r_n)).
\end{aligned}
\tag{16}
$$

By applying Kharchenko's theorem [19] to (16) and then U satisfies the blended component

$$
\begin{aligned}
&\Big(af(r_1,\ldots,r_n)+f(r_1,\ldots,r_n)b\Big)\Big(\alpha_2{}'\sum_i f(r_1,\ldots,y_i,\ldots,r_n)\Big)\\
&\quad-\gamma' f(r_1,\ldots,r_n)\sum_i f(r_1,\ldots,y_i,\ldots,r_n)\\
&= \sum_i f(r_1,\ldots,y_i,\ldots,r_n)f(r_1,\ldots,r_n)\\
&\quad+ f(r_1,\ldots,r_n)\sum_i f(r_1,\ldots,y_i,\ldots,r_n).
\end{aligned}
$$

In particular for $y_1 = r_1$ and $y_i = 0$ for all $i \geq 2$, then U satisfies

$$
\begin{aligned}
&\alpha_2{}'\Big(af(r_1,\ldots,r_n)+f(r_1,\ldots,r_n)b\Big)f(r_1,\ldots,r_n) - \gamma' f(r_1,\ldots,r_n)^2\\
&= 2f(r_1,\ldots,r_n)^2.
\end{aligned}
\tag{17}
$$

This relation is a particular case of Proposition 3.1, hence we get our conclusions.

If g and d are linearly C-independent, then by using Kharchenko's theorem [19] to (15) and then U satisfies the blended component

$$\Big(af(r_1,\ldots,r_n)+f(r_1,\ldots,r_n)b\Big)\sum_i f(r_1,\ldots,y_i,\ldots,r_n) = 0,$$

where $g(r_i) = y_i$ which implies that

$$\Big(af(r_1,\ldots,r_n)+f(r_1,\ldots,r_n)b\Big)f(r_1,\ldots,r_n) = 0.$$

This is a particular case of Proposition 3.1, hence we get our conclusions.

If $\beta = 0$ and $\alpha \neq 0$, $\gamma \neq 0$, then $g(x) = \gamma_1 d(x) + [u_1, x]$, where $\gamma_1 = -\alpha^{-1}\gamma$, $u_1 = \alpha^{-1}u$. Equation (14) gives that

$$
\begin{aligned}
&\Big(af(r_1,\ldots,r_n)+f(r_1,\ldots,r_n)b\Big)\Big(cf(r_1,\ldots,r_n)+\gamma_1 d(f(r_1,\ldots,r_n))\\
&\quad+\Big[u_1, f(r_1,\ldots,r_n)\Big]\Big) - f(r_1,\ldots,r_n)pf(r_1,\ldots,r_n)
\end{aligned}
\tag{18}
$$

$$\begin{aligned}&f(r_1,\ldots,r_n)h(f(r_1,\ldots,r_n))\\&= qf(r_1,\ldots,r_n)^2 + d(f(r_1,\ldots,r_n))f(r_1,\ldots,r_n)\\&\quad + f(r_1,\ldots,r_n)d(f(r_1,\ldots,r_n)).\end{aligned}$$

If h and d are linearly C-dependent modulo inner derivations, then $\alpha_1 h(x) + \alpha_2 d(x) = [q', x]$ for some $\alpha_1, \alpha_2 \in C$, $q' \in U$. We notice that α_1, α_2 both will be non zero, otherwise we get a contradiction. This gives $h(x) = \alpha_2{}' d(x) + [q'', x]$, where $\alpha_2{}' = -\alpha_1^{-1}\alpha_2$, $q'' = \alpha_1^{-1}q'$. Hence U satisfies

$$\begin{aligned}&\Big(af(r_1,\ldots,r_n) + f(r_1,\ldots,r_n)b\Big)\Big(cf(r_1,\ldots,r_n) + \gamma_1 d(f(r_1,\ldots,r_n))\\&+ \Big[u_1, f(r_1,\ldots,r_n)\Big]\Big) - f(r_1,\ldots,r_n)pf(r_1,\ldots,r_n)\\&- \alpha_2{}' f(r_1,\ldots,r_n)d(f(r_1,\ldots,r_n)) - f(r_1,\ldots,r_n)\Big[q'', f(r_1,\ldots,r_n)\Big]\\&= qf(r_1,\ldots,r_n)^2 + d(f(r_1,\ldots,r_n))f(r_1,\ldots,r_n)\\&\quad + f(r_1,\ldots,r_n)d(f(r_1,\ldots,r_n)).\end{aligned} \tag{19}$$

Applying Kharchenko's theorem [19], U satisfies the blended component

$$\begin{aligned}&\gamma_1\Big(af(r_1,\ldots,r_n) + f(r_1,\ldots,r_n)b\Big)\sum_i f(r_1,\ldots,y_i,\ldots,r_n)\\&- \alpha_2{}' f(r_1,\ldots,r_n)\sum_i f(r_1,\ldots,y_i,\ldots,r_n)\\&= \sum_i f(r_1,\ldots,y_i,\ldots,r_n)f(r_1,\ldots,r_n)\\&\quad + f(r_1,\ldots,r_n)\sum_i f(r_1,\ldots,y_i,\ldots,r_n),\end{aligned} \tag{20}$$

where $y_i = d(r_i)$. This implies that

$$\begin{aligned}&\gamma_1\Big(af(r_1,\ldots,r_n) + f(r_1,\ldots,r_n)b\Big)f(r_1,\ldots,r_n) - \alpha_2{}' f(r_1,\ldots,r_n)^2\\&= 2f(r_1,\ldots,r_n)^2.\end{aligned} \tag{21}$$

Since this equations similar to the equation (17), hence we get our conclusions.

If h and d are linearly C-independent, then by applying Kharchenko's theorem [19] to (18) and then U satisfies the blended component

$$f(r_1,\ldots,r_n)\sum_i f(r_1,\ldots,z_i,\ldots,r_n) = 0,$$

where $z_i = h(r_i)$ which implies that $f(r_1,\ldots,r_n)^2 = 0$, a contradiction.

If $\gamma = 0$ and $\alpha \neq 0$, $\beta \neq 0$, then $g(x) = \beta' h(x) + [u', x]$, where $\beta' = -\alpha^{-1}\beta$, $u' = \alpha^{-1}u$. Equation (14) gives that

$$\Big(af(r_1,\ldots,r_n) + f(r_1,\ldots,r_n)b\Big)\Big(cf(r_1,\ldots,r_n) + \beta' h(f(r_1,\ldots,r_n)) \tag{22}$$

$$+\Big[u', f(r_1,\ldots,r_n)\Big]\Big) - f(r_1,\ldots,r_n)pf(r_1,\ldots,r_n)$$
$$- f(r_1,\ldots,r_n)h(f(r_1,\ldots,r_n))$$
$$= qf(r_1,\ldots,r_n)^2 + d(f(r_1,\ldots,r_n))f(r_1,\ldots,r_n)$$
$$+ f(r_1,\ldots,r_n)d(f(r_1,\ldots,r_n)).$$

If d and h are linearly C-independent, then by using Kharchenko's theorem [19], U satisfies the blended component

$$\begin{aligned}(23)\qquad &\sum_i f(r_1,\ldots,y_i,\ldots,r_n)f(r_1,\ldots,r_n)\\ &+ f(r_1,\ldots,r_n)\sum_i f(r_1,\ldots,y_i,\ldots,r_n),\end{aligned}$$

where $y_i = d(r_i)$. In particular for $y_1 = r_1$ and $y_i = 0$ for all $i = 2, 3, \ldots, n$, we get $2f(r_1,\ldots,r_n)^2 = 0$. Since $\mathrm{char}(R) \neq 2$, it gives that $f(r_1,\ldots,r_n)^2 = 0$, a contradiction.

Now we shall assume the case that none of α, β and γ is zero. Then

$$g(x) = \beta' h(x) + \gamma' d(x) + [u', x]$$

for all $x \in R$, where $\beta' = -\alpha^{-1}\beta$, $\gamma' = -\alpha^{-1}\gamma$ and $u' = \alpha^{-1}u$. Then relation (14) reduces to

$$\begin{aligned}(24)\qquad &\Big(af(r_1,\ldots,r_n) + f(r_1,\ldots,r_n)b\Big)\Big(cf(r_1,\ldots,r_n) + \beta' h(f(r_1,\ldots,r_n))\\ &+ \gamma' d(f(r_1,\ldots,r_n)) + \Big[u', f(r_1,\ldots,r_n)\Big]\Big)\\ &- f(r_1,\ldots,r_n)pf(r_1,\ldots,r_n) - f(r_1,\ldots,r_n)h(f(r_1,\ldots,r_n))\\ =\ & qf(r_1,\ldots,r_n)^2 + d(f(r_1,\ldots,r_n))f(r_1,\ldots,r_n)\\ &+ f(r_1,\ldots,r_n)d(f(r_1,\ldots,r_n))\end{aligned}$$

for all $r_1,\ldots,r_n \in R$. If d and h are linearly C-dependent modulo inner derivations on U, then for some $\alpha_1, \alpha_2 \in C$ and $p' \in U$ such that $\alpha_1 d(x) + \alpha_2 h(x) = [p', x]$ for all $x \in U$. Since none of d and h are inner, hence α_1 and α_2 both are non zero. Then $h(x) = \alpha_1' d(x) + [p'', x]$ for all $x \in U$, where $\alpha_1' = -\alpha_2^{-1}\alpha_1$ and $p'' = \alpha_2^{-1}p'$. Thus the equation (24) reduces to

$$\begin{aligned}(25)\qquad &\Big(af(r_1,\ldots,r_n)+f(r_1,\ldots,r_n)b\Big)\Big(cf(r_1,\ldots,r_n)+\beta'\alpha_1{}' d(f(r_1,\ldots,r_n))\\ &+ \beta'\Big[p'', f(r_1,\ldots,r_n)\Big] + \gamma' d(f(r_1,\ldots,r_n)) + \Big[u', f(r_1,\ldots,r_n)\Big]\Big)\\ &- f(r_1,\ldots,r_n)pf(r_1,\ldots,r_n) - \alpha_1{}' f(r_1,\ldots,r_n)d(f(r_1,\ldots,r_n))\\ &- f(r_1,\ldots,r_n)\Big[p'', f(r_1,\ldots,r_n)\Big]\\ =\ & qf(r_1,\ldots,r_n)^2 + d(f(r_1,\ldots,r_n))f(r_1,\ldots,r_n)\\ &+ f(r_1,\ldots,r_n)d(f(r_1,\ldots,r_n))\end{aligned}$$

for all $r_1, \dots, r_n \in R$. By using Kharchenko's theorem [19], we can replace $d(f(r_1, \dots, r_n))$ with $f^d(r_1, \dots, r_n) + \sum_i f(r_1, \dots, y_i, \dots, r_n)$, where $d(r_i) = y_i$, then U satisfies

$$\begin{aligned}
&\Big(af(r_1,\dots,r_n) + f(r_1,\dots,r_n)b\Big)\Big(cf(r_1,\dots,r_n) + \beta'\alpha_1{}'\Big(f^d(r_1,\dots,r_n)\\
&+\sum_i f(r_1,\dots,y_i,\dots,r_n)\Big) + \beta'\Big[p'', f(r_1,\dots,r_n)\Big] + \gamma'\Big(f^d(r_1,\dots,r_n)\\
&+\sum_i f(r_1,\dots,y_i,\dots,r_n)\Big) + \Big[u', f(r_1,\dots,r_n)\Big]\Big)\\
&- f(r_1,\dots,r_n)pf(r_1,\dots,r_n) - \alpha_1{}'f(r_1,\dots,r_n)\Big(f^d(r_1,\dots,r_n)\\
&+\sum_i f(r_1,\dots,y_i,\dots,r_n)\Big) - f(r_1,\dots,r_n)\Big[p'', f(r_1,\dots,r_n)\Big]\\
&= qf(r_1,\dots,r_n)^2 + \Big(f^d(r_1,\dots,r_n) + \sum_i f(r_1,\dots,y_i,\dots,r_n)\Big)f(r_1,\dots,r_n)\\
&+ f(r_1,\dots,r_n)\Big(f^d(r_1,\dots,r_n) + \sum_i f(r_1,\dots,y_i,\dots,r_n)\Big). \qquad (26)
\end{aligned}$$

Hence, U satisfies the blended component

$$\begin{aligned}
&\Big(\beta'\alpha_1{}' + \gamma'\Big)\Big(af(r_1,\dots,r_n) + f(r_1,\dots,r_n)b\Big)\sum_i f(r_1,\dots,y_i,\dots,r_n)\\
&- \alpha_1{}'f(r_1,\dots,r_n)\sum_i f(r_1,\dots,y_i,\dots,r_n) \qquad (27)\\
&= \sum_i f(r_1,\dots,y_i,\dots,r_n)f(r_1,\dots,r_n) + f(r_1,\dots,r_n)\sum_i f(r_1,\dots,y_i,\dots,r_n).
\end{aligned}$$

In particular for $y_1 = r_1$ and $y_i = 0$ for all $i \geq 2$, then U satisfies

$$\begin{aligned}
&\Big(\beta'\alpha_1{}' + \gamma'\Big)\Big(af(r_1,\dots,r_n) + f(r_1,\dots,r_n)b\Big)f(r_1,\dots,r_n)\\
&- \alpha_1{}'f(r_1,\dots,r_n)^2 = 2f(r_1,\dots,r_n)^2. \qquad (28)
\end{aligned}$$

Since this is a particular case of Proposition 3.1, hence we get our conclusions.

If d and h are linearly C-independent, then by using Kharchenko's theorem [19], we have

$$\begin{aligned}
&\Big(af(r_1,\dots,r_n) + f(r_1,\dots,r_n)b\Big)\Big(cf(r_1,\dots,r_n) + \beta'f^h(r_1,\dots,r_n)\\
&+ \beta'\sum_i f(r_1,\dots,z_i,\dots,r_n) + \gamma'f^d(r_1,\dots,r_n) + \gamma'\sum_i f(r_1,\dots,y_i,\dots,r_n)\\
&+ \Big[u', f(r_1,\dots,r_n)\Big]\Big) - f(r_1,\dots,r_n)pf(r_1,\dots,r_n)\\
&- f(r_1,\dots,r_n)\Big(f^h(r_1,\dots,r_n) + \sum_i f(r_1,\dots,z_i,\dots,r_n)\Big)
\end{aligned}$$

$$= qf(r_1,\dots,r_n)^2 + \Big(f^d(r_1,\dots,r_n) + \sum_i f(r_1,\dots,y_i,\dots,r_n)\Big)f(r_1,\dots,r_n) + f(r_1,\dots,r_n)\Big(f^d(r_1,\dots,r_n) + \sum_i f(r_1,\dots,y_i,\dots,r_n)\Big)$$

for all $r_1,\dots,r_n \in R$, where $d(r_i) = y_i$ and $h(r_i) = z_i$. In particular U satisfies

$$\beta'\Big(af(r_1,\dots,r_n) + f(r_1,\dots,r_n)b\Big)\sum_i f(r_1,\dots,z_i,\dots,r_n) - f(r_1,\dots,r_n)\sum_i f(r_1,\dots,z_i,\dots,r_n) = 0.$$

In particular for $z_1 = r_1$ and $z_i = 0$ for all $i \geq 2$, then we have

$$\beta'\Big(af(r_1,\dots,r_n) + f(r_1,\dots,r_n)b\Big)f(r_1,\dots,r_n) - f(r_1,\dots,r_n)^2 = 0.$$

This relation is a particular case of Proposition 3.1, which gives our conclusions.

Subcase-II. Let g, h and d be linearly C-independent modulo inner derivation. Then by using Kharchenko's theorem [19], the equation (14) implies that

$$\begin{aligned}&\Big(af(r_1,\dots,r_n) + f(r_1,\dots,r_n)b\Big)\Big(cf(r_1,\dots,r_n) + f^g(r_1,\dots,r_n) + \sum_i f(r_1,\dots,y_i,\dots,r_n)\Big) - f(r_1,\dots,r_n)pf(r_1,\dots,r_n)\\&- f(r_1,\dots,r_n)\Big(f^h(r_1,\dots,r_n) + \sum_i f(r_1,\dots,z_i,\dots,r_n)\Big)\\&= qf(r_1,\dots,r_n)^2 + \Big(f^d(r_1,\dots,r_n) + \sum_i f(r_1,\dots,w_i,\dots,r_n)\Big)f(r_1,\dots,r_n) + f(r_1,\dots,r_n)\Big(f^d(r_1,\dots,r_n) + \sum_i f(r_1,\dots,w_i,\dots,r_n)\Big),\end{aligned} \tag{29}$$

where $g(r_i) = y_i$, $h(r_i) = z_i$ and $d(r_i) = w_i$. Then U satisfies the blended component

$$\sum_i f(r_1,\dots,w_i,\dots,r_n)\Big)f(r_1,\dots,r_n) + f(r_1,\dots,r_n)\sum_i f(r_1,\dots,w_i,\dots,r_n),$$

which is the same as the equation (23), hence we get our result.

Case 2. Suppose that G is a generalized inner derivation and F, H, T are generalized derivations on R. By using similar argument as we have used in above (see Case 1; Lemma 3.13), we get our conclusions.

Case 3. Suppose that H is a generalized inner derivation and F, G, T are generalized derivations on R. By using similar argument as we have used in above (see Case 1; Lemma 3.13), we get our conclusions.

Case 4. Suppose that T is a generalized inner derivation and F, G, H are generalized derivations on R. By using similar argument as we have used in above (see Case 1; Lemma 3.13), we get our conclusions. □

Now we are in a position to prove our main results.

Proof of Theorem 2.1. If one of F, G, H and T is a generalized inner derivation, then by Lemma 3.13, we get our conclusions. Suppose that none of F, G, H and T is a generalized inner derivation. For some $a, b, c, p \in U$ such that $F(x) = ax + d(x)$, $G(x) = bx + g(x)$, $H(x) = cx + h(x)$ and $T(x) = px + \delta(x)$, where d, g, h, δ are derivations on U. Then U satisfies

$$\begin{aligned} &\Big(af(r_1,\ldots,r_n) + d(f(r_1,\ldots,r_n))\Big)\Big(bf(r_1,\ldots,r_n) + g(f(r_1,\ldots,r_n))\Big) \\ &- f(r_1,\ldots,r_n)cf(r_1,\ldots,r_n) - f(r_1,\ldots,r_n)h(f(r_1,\ldots,r_n)) \\ = &\ pf(r_1,\ldots,r_n)^2 + \delta(f(r_1,\ldots,r_n))f(r_1,\ldots,r_n) \\ &+ f(r_1,\ldots,r_n)\delta(f(r_1,\ldots,r_n)). \end{aligned} \tag{30}$$

We shall study the following cases.

Case 1. Let d, g, h and δ be linearly C-dependent modulo inner derivations on U. Then we have $\alpha_1 d(x) + \alpha_2 g(x) + \alpha_3 h(x) + \alpha_4 \delta(x) = [u, x]$, where $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in C$ and $u \in U$. If at a time any three coefficients are zero, then we shall get a contradiction. Now we assume two coefficients are zero.

Subcase-I. If $\alpha_1 = 0 = \alpha_2$ and $\alpha_3 \neq 0$, $\alpha_4 \neq 0$, then we have $h(x) = \alpha'_4\delta(x) + [u', x]$, where $\alpha'_4 = -\alpha_3^{-1}\alpha_4$, $u' = \alpha_3^{-1}u$. Then (30) gives that

$$\begin{aligned} &\Big(af(r_1,\ldots,r_n)+d(f(r_1,\ldots,r_n))\Big)\Big(bf(r_1,\ldots,r_n)+g(f(r_1,\ldots,r_n))\Big) \\ &- f(r_1,\ldots,r_n)cf(r_1,\ldots,r_n) - \alpha'_4 f(r_1,\ldots,r_n)\delta(f(r_1,\ldots,r_n)) \\ &- f(r_1,\ldots,r_n)\Big[u', f(r_1,\ldots,r_n)\Big] \\ = &\ pf(r_1,\ldots,r_n)^2 + \delta(f(r_1,\ldots,r_n))f(r_1,\ldots,r_n) \\ &+ f(r_1,\ldots,r_n)\delta(f(r_1,\ldots,r_n)). \end{aligned} \tag{31}$$

If d, g and δ are linearly C-independent, then by applying similar argument as we have used in above (see Subcase-II, Case 1 of Lemma 3.13) we get our conclusions.

If d, g and δ are linearly C-dependent, then by using similar argument as we have used in Subcase-I of Case 1 of Lemma 3.13, we get our conclusions.

Subcase-II. If $\alpha_1 = 0 = \alpha_3$ and $\alpha_2 \neq 0, \alpha_4 \neq 0$, then by using similar argument as we have used in above (see Subcase-I, Case 1 of proof of Theorem 2.1), we get our conclusions.

Subcase-III. If $\alpha_1 = 0 = \alpha_4$ and $\alpha_2 \neq 0, \alpha_3 \neq 0$, then by using similar argument as we have used in above (see Subcase-I, Case 1 of proof of Theorem 2.1), we get our conclusions.

Subcase-IV. If $\alpha_2 = 0 = \alpha_3$ and $\alpha_1 \neq 0, \alpha_4 \neq 0$, then by using similar argument as we have used in above (see Subcase-I, Case 1 of proof of Theorem 2.1), we get our conclusions.

Subcase-V. If $\alpha_2 = 0 = \alpha_4$ and $\alpha_1 \neq 0, \alpha_3 \neq 0$, then by using similar argument as we have used in above (see Subcase-I, Case 1 of proof of Theorem 2.1), we get our conclusions.

Subcase-VI. If $\alpha_3 = 0 = \alpha_4$ and $\alpha_1 \neq 0, \alpha_2 \neq 0$, then by using similar argument as we have used in above (see Subcase-I, Case 1 of proof of Theorem 2.1), we get our conclusions.

Now we suppose that only one coefficient is zero. Then we have the following.

Subcase-VII. If $\alpha_1 = 0$ and $\alpha_2 \neq 0, \alpha_3 \neq 0, \alpha_4 \neq 0$, then $g(x) = \alpha_3' h(x) + \alpha_4' \delta(x) + [u', x]$, where $\alpha_3' = -\alpha_2^{-1}\alpha_3$, $\alpha_4' = -\alpha_2^{-1}\alpha_4$, $u' = \alpha_2^{-1}u$. Then (30) gives that

$$\begin{aligned} &\Big(af(r_1,\ldots,r_n) + d(f(r_1,\ldots,r_n))\Big)\Big(bf(r_1,\ldots,r_n) + \alpha_3' h(f(r_1,\ldots,r_n)) \\ &+\alpha_4'\delta(f(r_1,\ldots,r_n)) + \Big[u', f(r_1,\ldots,r_n)\Big]\Big) - f(r_1,\ldots,r_n)cf(r_1,\ldots,r_n) \\ &- f(r_1,\ldots,r_n)h(f(r_1,\ldots,r_n)) \\ =\ & pf(r_1,\ldots,r_n)^2 + \delta(f(r_1,\ldots,r_n))f(r_1,\ldots,r_n) \\ &+ f(r_1,\ldots,r_n)\delta(f(r_1,\ldots,r_n)). \end{aligned} \tag{32}$$

If d, h and δ are linearly C-independent, then by applying similar argument as we have used in above (see Subcase-II, Case 1 of Lemma 3.13) we get our conclusions.

If d, h and δ are linearly C-dependent modulo inner derivations, then using parallel argument as we have used in the Subcase-I of Case 1 of Lemma 3.13, we get our conclusions.

Subcase-VIII. If $\alpha_2 = 0$ and $\alpha_1 \neq 0, \alpha_3 \neq 0, \alpha_4 \neq 0$, then by using similar argument as we have used in above (see Subcase-VII, Case 1 of proof of Theorem 2.1), we get our conclusions.

Subcase-IX. If $\alpha_3 = 0$ and $\alpha_1 \neq 0, \alpha_2 \neq 0, \alpha_4 \neq 0$, then by using similar argument as we have used in above (see Subcase-VII, Case 1 of proof of Theorem 2.1), we get our conclusions.

Subcase-X. If $\alpha_4 = 0$ and $\alpha_1 \neq 0, \alpha_2 \neq 0, \alpha_3 \neq 0$, then by using similar argument as we have used in above (see Subcase-VII, Case 1 of proof of Theorem 2.1), we get our conclusions.

Now we consider that none of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ is zero. It implies that $d(x) = \alpha_2' g(x) + \alpha_3' h(x) + \alpha_4' \delta(x) + [u', x]$, where $\alpha_2' = -\alpha_1^{-1}\alpha_2$, $\alpha_3' = -\alpha_1^{-1}\alpha_3$,

${\alpha_4}' = -\alpha_1^{-1}\alpha_4$ and $u' = \alpha_1^{-1}u$ and then (30) reduces to

$$\begin{aligned}&\Big(af(r_1,\ldots,r_n) + {\alpha_2}'g(f(r_1,\ldots,r_n)) + {\alpha_3}'h(f(r_1,\ldots,r_n))\\&+ {\alpha_4}'\delta(f(r_1,\ldots,r_n)) + \Big[u', f(r_1,\ldots,r_n)\Big]\Big)\\&\Big(bf(r_1,\ldots,r_n) + g(f(r_1,\ldots,r_n))\Big)\\&- f(r_1,\ldots,r_n)cf(r_1,\ldots,r_n) - f(r_1,\ldots,r_n)h(f(r_1,\ldots,r_n))\\=\ & pf(r_1,\ldots,r_n)^2 + \delta(f(r_1,\ldots,r_n))f(r_1,\ldots,r_n)\\&+ f(r_1,\ldots,r_n)\delta(f(r_1,\ldots,r_n)).\end{aligned} \tag{33}$$

If g, h and δ are linearly C-independent, then by applying similar argument as we have used in above (see Subcase-II, Case 1 of Lemma 3.13) we get our conclusions.

If g, h, δ are linearly C-dependent modulo inner derivations, then by applying similar argument as we have used in above (see Subcase-I, Case 1 of Lemma 3.13) we get our conclusions.

Case 2. Let d, g, h, δ be linearly C-independent. Then by using kharchenko's theorem [19] in (30), U satisfies the blended component

$$f(r_1,\ldots,r_n)\sum_i f(r_1,\ldots,z_i,\ldots,r_n) = 0,$$

where $z_i = h(r_i)$, which implies that $f(r_1,\ldots,r_n)^2 = 0$, a contradiction. Hence proof of the theorem is complete. □

The following corollaries are immediate consequences of our Theorem 2.1.

Corollary 3.14. *Let R be a prime ring with characteristic different from 2 and U be its Utumi ring of quotients, extended centroid $C = Z(U)$ and $f(x_1,\ldots,x_n)$ be a non central multilinear polynomial over C. Suppose that d_1, d_2 and d_3 are derivations on R such that $d_1(f(r))d_2(f(r)) = d_3(f(r)^2)$ for all $r = (r_1,\ldots,r_n)$, where $r_1,\ldots,r_n \in R$, then one of the following holds*:

(i) $d_1 = 0 = d_3$;
(ii) $d_2 = 0 = d_3$;
(iii) *there exists $a \in U$ such that $f(x_1,\ldots,x_n)^2$ is central valued on R and either $d_1 = 0 = d_2$, $d_3(x) = [a,x]$ or $d_2 = 0$, $d_3(x) = [a,x]$ for all $x \in R$.*

In particular for $F = H = T = d$, where d is a derivation and $G = I$, the identity mapping on R in our Theorem 2.1, we obtain the following.

Corollary 3.15. *Let R be a prime ring with characteristic different from 2 and U be its Utumi ring of quotients, extended centroid $C = Z(U)$ and $f(x_1,\ldots,x_n)$ be a non central multilinear polynomial over C. Suppose that d_1 and d_2 are two derivations on R such that $[d_1(f(r)), f(r)] = d_2(f(r)^2)$ for all $r = (r_1,\ldots,r_n)$,* *where* $r_1, \ldots, r_n \in R$*, then either* $d_1 = 0 = d_2$ *or there exists* $a \in U$ *such that* $f(x_1, \ldots, x_n)^2$ *is central valued on* R *and* $d_1 = 0$*,* $d_2 = [a, x]$ *for all* $x \in R$*.*

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