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WEAKLY $\oplus$-SUPPLEMENTED MODULES AND WEAKLY $D2$ MODULES

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This paper is dedicated to Professor Le Van Thuyet on his 65th birthday

Abstract. In this paper, we introduce and study the notions of weakly $\oplus$-supplemented modules, weakly $D2$ modules and weakly $D2$-covers. A right R-module M is called weakly $\oplus$-supplemented if every non-small submodule of M has a supplement that is not essential in M, and module M_R is called weakly $D2$ if it satisfies the condition: for every $s \in S$ and $s \neq 0$, if there exists $n \in \mathbb{N}$ such that $s^n \neq 0$ and $\mathrm{Im}(s^n)$ is a direct summand of M, then $\mathrm{Ker}(s^n)$ is a direct summand of M. The class of weakly $\oplus$-supplemented-modules and weakly $D2$ modules contains $\oplus$-supplemented modules and $D2$ modules, respectively, and they are equivalent in case M is uniform, and projective, respectively.

1. Introduction

Throughout this paper, rings R are associative with unity and modules are unitary. For an R-module M, we denote by $rad(M)$, $Soc(M)$ and $E(M)$ the Jacobson radical, the socle, and the injective hull of M, respectively. If $M = R$, we write $J = J(R) = rad(R)$. We denote by $End(M)$ the endomorphism ring of M, $M_n(R)$ the $n \times n$ matrix ring over R, $M^{(I)}$ a direct sum of I-copies of M, M^I a direct product of I-copies of M, and Mod-R the category of right R-modules. Let $\mathbb{Z}$ be the ring of integers and $\mathbb{Z}_n$ be the ring of $\mathbb{Z}$ modulo n. We also use $\mathbb{N}$ to denote the set of natural numbers. We also write $N \leq M$ if N is a submodule of M, $N \leq_{ess} M$ if N is an essential submodule of M, $N \ll M$ if N is a small submodule of M, and $N \leq^{\oplus} M$ if N is a (direct) summand of M. If $A \leq M$, by a complement of A in M, we mean a submodule C of M maximal with respect to $C \cap A = 0$. Let N and L be submodules of M, the module N is called a supplement of L in M if $N + L = M$ and N is minimal with respect to this property, equivalently, $M = N + L$ and $N \cap L \ll N$. M is called supplemented if every submodule of M has a supplement in M. The

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module M is called $\oplus$-supplemented if every submodule of M has a supplement that is a direct summand of M.

A module M is called NCS if no nonzero complement submodule is small ([6] and [16]). In this paper, we will focus on modules whose non-small submodules have supplements which are not essential (such modules are called weakly $\oplus$-supplemented). This notion is a non-trivial dualization of NCS modules, since we have the following hierarchy:

$$\oplus\text{-supplemented} \Longrightarrow \text{weakly } \oplus\text{-supplemented} \Longrightarrow \text{supplemented.}$$

In Section 2 of this paper, examples for the reverse inclusions of the hierarchy are given. Also in Section 2, several interesting characterizations of weakly $\oplus$-supplemented are established. The connection between (semi)perfect rings R and ($\oplus$-)supplemented modules R_R has been investigated by many authors before. In this direction we show, in Corollaries 2.4 and 2.5, that R is semiperfect if and only if R_R is $\oplus$-supplemented if and only if R_R is weakly $\oplus$-supplemented if and only if R_R is supplemented, and R is right perfect if and only if every projective right R-module is weakly $\oplus$-supplemented, respectively.

For R-modules M and N, M is called N-injective if every R-homomorphism from any submodule of N to M can be extended to an R-homomorphism from N into M. The module M is called quasi-injective if M is M-injective. Every quasi-injective module M satisfies the ($C2$)-condition, i.e., every submodule isomorphic to a (direct) summand of M is itself a summand of M ([20]). Dually, M is called N-projective if every homomorphism from M to a homomorphic image of N can be lifted to a homomorphism from M into N. A module M is called quasi-projective if M is M-projective. The module M is called direct-projective ([12]) if, for every submodule A of M with M/A isomorphic to a direct summand of M, A is a direct summand of M. Every quasi-projective module is direct-projective, but the converse does not hold. Direct-projective modules are also called $D2$-modules. Every $D2$-module M is a $D3$-module, i.e., for any direct summands M_1 and M_2 of M with $M = M_1 + M_2$, $M_1 \cap M_2$ is a direct summand of M. A module M is called $D1$ if, for every submodule A of M, there is a decomposition $M = M_1 \oplus M_2$ with $M_1 \leq A$ and $A \cap M_2 \ll M$. $D1$-modules are called lifting by Oshiro in [13]. Finally, M is called a $D3$-module if M_1 and M_2 are direct summands of M and $M = M_1 + M_2$, then $M_1 \cap M_2$ is a direct summand of M. Modules satisfying the conditions $D1$ and $D2$ are called discrete and satisfying the conditions $D1$ and $D3$ are called quasi-discrete. For a full account, we refer the reader to [11] and [22].

Let M be a module and $S = End(M)$. As a new characterization of $D2$ module, we obtain that M is a $D2$ module if and only if, for any $s \in S$, if $\mathrm{Im}(s)$ is a direct summand of M, then $\mathrm{Ker}(s)$ is a direct summand of M. Because of this fact, the module M_R is called weakly $D2$ if it satisfies the condition: for every $s \in S$ and $s \neq 0$, if there exists $n \in \mathbb{N}$ such that $s^n \neq 0$ and $\mathrm{Im}(s^n)$ is a direct summand of M, then $\mathrm{Ker}(s^n)$ is a direct summand of M.

In Section 3, we show some basic properties of weakly $D2$ modules and a non-trivial generalization of both the $D2$ modules as well as the $GD2$-modules (i.e., if for every submodule N of M for which M/N is isomorphic to M, then N is a direct summand of M). Also, interestingly, we obtain that this new notion is equivalent to $\oplus$-supplemented modules, weakly $\oplus$-supplemented modules, supplemented modules and lifting modules under the additional assumption of being projective (Theorem 3.3). Also in Section 3, several interesting characterizations of weakly $D2$-modules are established. According to Bass [1], an R-homomorphism $f : P \to M$ is called a projective cover of the R-module M, if P is projective, f is an epimorphism, and $\mathrm{Ker}(f) \ll P$.

In Section 4, following the work of Bass in [1], we introduce the notion of a weakly $D2$-cover and extend Basscharacterizations of (semi)perfect rings in terms of projective covers to weakly $D2$-covers. We show that R is semiperfect if and only if every finitely generated right R-module has a weakly $D2$-cover (Theorem 4.5), R is right perfect if and only if every flat right R-module is weakly $D2$ if and only if every right R-module has a weakly $D2$-cover (Theorem 4.7). Finally, by Theorem 4.8, R is a semiregular ring if and only if every finitely presented right R-module has a weakly $D2$-cover.

Section 5 is devoted to rings whose certain classes of modules have weakly $D2$-property. Among the other results, we show that R is a right PP-ring if and only if every principal right ideal of $M_2(R)$, generated by a diagonal matrix, is a weakly $D2$ module (Theorem 5.5), R is a semiprimary hereditary ring if and only if every torsionless right R-module is projective if and only if every torsionless left R-module is weakly $D2$ (Theorem 5.12), and R is a semilocal ring if and only if every semiprimitive R-module is weakly $D2$ (Theorem 5.13).

2. On modules whose non-small submodules have supplements which are not essential

A right R-module M will be called weakly $\oplus$-supplemented if every non-small submodule of M has a supplement which are not essential in M.

We begin with several examples of weakly $\oplus$-supplemented modules to initiate the reader and to motivate our study.

Example 2.1. (1) $\oplus$-supplemented modules are weakly $\oplus$-supplemented. Indeed, let M be a $\oplus$-supplemented module and N be a non-small submodule of M. There exists a direct summand K of M such that K is a supplement of N in M. If $K \leq_{ess} M$, then $K = M$. Thus $N \cap M = N \ll M$, a contradiction.

(2) Weakly $\oplus$-supplemented modules are supplemented.

(3) Semisimple modules are weakly $\oplus$-supplemented. Furthermore, every module over a semisimple ring is weakly $\oplus$-supplemented. Indeed, let H be a non-small submodule of a semisimple module M. Assume that H has a supplement N which is essential in M, i.e., $M = H + N$ with

$H \cap N \ll N$ and $N \leq_{ess} M$. Moreover, since N is also semisimple, $H \cap N = 0$ and so $H = 0$, a contradiction.

A nonzero module M is said to be hollow if every proper submodule is small in M, and it is said to be local if it is hollow and is finitely generated.

Lemma 2.2. *The following are equivalent for a uniform R-module M:*

(1) *M is weakly $\oplus$-supplemented;*
(2) *M is hollow;*
(3) *M is $\oplus$-supplemented.*

Proof. (1) $\Rightarrow$ (2) Let N be a non-small submodule of M. By the assumption, there exists a submodule K of M such that K is a supplement of N in M and K is not essential in M. Since M is uniform, we get $K = 0$. Hence $N + K = M = N$. Therefore every proper submodule of M is small in M, i.e., M is hollow.

(2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) are clear. □

The following example shows that there exist supplemented modules which are not weakly $\oplus$-supplemented modules, and weakly $\oplus$-supplemented modules which are not $\oplus$-supplemented, respectively.

Example 2.3. (1) Let F be a field and $R = F[[X, Y]]$ be the ring of formal power series over F in the indeterminates X and Y. Then R is a commutative local ring. Consider the ideal $I = RX + RY$ of R. Clearly, I is the unique maximal ideal of R and is also uniform. By [7, page 473], the R-module I is not $\oplus$-supplemented. Therefore I is not weakly $\oplus$-supplemented by Lemma 2.2(1). By [11, Theorem 4.41], I is supplemented.

(2) Let R be a commutative local ring which is not a valuation ring. Let a and b be elements of R, neither of them divides the other. By taking a suitable quotient ring, we may assume $aR \cap bR = 0$ and $a\mathfrak{m} = b\mathfrak{m} = 0$, where $\mathfrak{m}$ is the maximal ideal of R. Let F be a free module with generators x_1 and x_2. Let K be the submodule of F generated by $x_1a - x_2b$ and let $M := F/K$. Hence,

$$M = (x_1R \oplus x_2R)/(x_1a - x_2b)R = \overline{x_1}R + \overline{x_2}R.$$

By [21, Theorem 2], M is an indecomposable module that cannot be generated by fewer that 2 elements. Thus, M is not hollow. Hence, M is not $\oplus$-supplemented. Since R is local, the R-modules $M_1 = \overline{x_1}R$ and $M_2 = \overline{x_2}R$ are local modules. Now let us show that M_1 is not essential in M. Consider the submodule

$$N = \overline{x_2a}R = [(x_2a)R + (x_1a - x_2b)R]/(x_1a - x_2b)R.$$

Note that

$$\begin{aligned} M_1 &= [x_1R + (x_1a - x_2b)R]/(x_1a - x_2b)R \\ &= [x_1R \oplus (x_2b)R]/(x_1a - x_2b)R. \end{aligned}$$

We have $aR \cap bR = 0$ and obtain

$$N \cap M_1 = [(x_1a - x_2b)R + ((x_1R \oplus (x_2b)R) \cap (x_2a)R)]/(x_1a - x_2b)R = 0.$$

Moreover, $N \neq 0$, since otherwise $x_2a \in (x_1a - x_2b)R$. Then there exists $\alpha \in R$ such that $x_2a = (x_1a - x_2b)\alpha$. So, $a\alpha = 0$ and $a + b\alpha = 0$. It follows that $a = -b\alpha$. This contradicts the fact that b does not divide a. In the same manner we can see that M_2 is not essential in M. Note that $[(M_i + Rad(M)/Rad(M)]$ $(i = 1, 2)$ are simple modules and

$$M/Rad(M) = [(M_1 + Rad(M)/Rad(M)] \oplus [(M_2 + Rad(M)/Rad(M)].$$

Now let L be a proper non-small submodule of M. Since $M/Rad(M)$ is semisimple, we have

$$M/Rad(M) = [(L + Rad(M))/Rad(M)] \oplus [(M_1 + Rad(M))/Rad(M)]$$

or

$$M/Rad(M) = [(L + Rad(M))/Rad(M)] \oplus [(M_2 + Rad(M))/Rad(M)].$$

We have $Rad(M) \ll M$ and obtain $M = L + M_1$ or $M = L + M_2$. Therefore M_1 is a supplement of L or M_2 is a supplement of L in M. It follows that M is weakly ⊕-supplemented.

Recall that an epimorphism $f : P \to M$ with P projective, is called a projective cover of M if $\mathrm{Ker}(f) \ll P$. A ring R is right (semi)perfect if every (finitely generated) R-module has a projective cover.

Corollary 2.4. *The following are equivalent for a ring R:*

(1) *R is semiperfect;*
(2) *R_R is ⊕-supplemented;*
(3) *R_R is weakly ⊕-supplemented;*
(4) *R_R is supplemented.*

Corollary 2.5. *The following conditions are equivalent for a ring R:*

(1) *R is right perfect;*
(2) *Every projective right R-module is weakly ⊕-supplemented.*

Proof. (1) ⇒ (2) This is obvious.

(2) ⇒ (1) By hypothesis, we can say that $R^{(\mathbb{N})}$ is weakly ⊕-supplemented. Then $R^{(\mathbb{N})}$ is ⊕-supplemented by [5, Lemma 1.2]. Thus R is right perfect by [7, Theorem 2.10]. □

A module M over an arbitrary ring is called π-projective if for every two submodules U, V of M such that $U + V = M$, there exists an endomorphism f of M with $f(M) \leq U$ and $(1 - f)(M) \leq V$ ([22]). Remark that projective modules are π-projective and, by [22, 41.12 and 41.15], M is lifting if and only if M is supplemented and π-projective if and only if M is ⊕-supplemented and π-projective.

Corollary 2.6. *A module M is lifting if and only if M is weakly $\oplus$-supplemented and π-projective.*

In the following observation, we collect a few basic easily-proven properties of weakly $\oplus$-supplemented modules analogous to $\oplus$-supplemented modules.

Proposition 2.7. *For a right R-module M over a ring R, the following hold:*

(1) *Any finite direct sum of weakly $\oplus$-supplemented R-modules is weakly $\oplus$-supplemented;*

(2) *If M is weakly $\oplus$-supplemented, then M/N is weakly $\oplus$-supplemented for every fully invariant submodule N of M (i.e., if $f(N) \subseteq N$ for all endomorphisms f of M). Furthermore, if M is a weakly $\oplus$-supplemented module, then so are $M/Rad(M)$ and $M/Soc(M)$;*

(3) *Let $M = M_1 \oplus M_2$. Then M_2 is weakly $\oplus$-supplemented if and only if for every submodule N of M with $M_1 \leq N$ and $N \cap M_2 \not\ll M_2$, there exists a submodule K of M_2 such that $K \not\leq_{ess} M_2$, $M = K + N$ and $N \cap K \ll K$;*

(4) *Let D be a direct summand of a weakly $\oplus$-supplemented module M such that for every non-essential submodule K of M with $M = K+D$, $K \cap D$ is not essential in M_2. Then D is a weakly $\oplus$-supplemented module;*

(5) *Let M be a weakly $\oplus$-supplemented module and K be a direct summand of M such that M/K is K-projective. Then K is weakly $\oplus$-supplemented;*

(6) *Let $M = M_1 \oplus \cdots \oplus M_n$ and M_i, M_j are relative projective for all $i \neq j$. Then M is weakly $\oplus$-supplemented if and only if each M_i is weakly $\oplus$-supplemented for all i.*

Remark 2.8. We recall that the class of $\oplus$-supplemented modules is not closed under taking direct summands. Because of this fact, authors of [5] introduced and studied the notion of "completely $\oplus$-supplemented modules".

3. Weakly $D2$ modules

We begin with a new characterization of $D2$ modules.

Lemma 3.1. *Let M be a module and $S = End(M)$. Then M is a $D2$ module if and only if, for any $s \in S$, if* $\mathrm{Im}(s)$ *is a direct summand of M, then* $\mathrm{Ker}(s)$ *is a direct summand of M.*

Proof. We show that M is a $D2$ module. Let $N \leq M$ and consider the isomorphism $f : M/N \to A$, where A is a direct summand of M, the canonical projection $\pi : M \to M/N$ and the inclusion map $i : A \to M$. Let $s = if\pi$. Then $\mathrm{Im}(s) = A$ is a direct summand of M. By the assumption, $\mathrm{Ker}(s)$ is a direct summand of M. Let $m \in \mathrm{Ker}(s)$. Then $s(m) = if\pi(m) = 0$ or $\pi(m) = 0$ because f is an isomorphism. It follows that $m \in N$, i.e., $\mathrm{Ker}(s) \leq N$. We deduce that $N = \mathrm{Ker}(s)$ is a direct summand of M. The converse is clear. □

Let M be a right R-module and $S = End(M)$. The module M_R is called weakly $D2$ if it satisfies the condition: for every $s \in S$ and $s \neq 0$, if there exists $n \in \mathbb{N}$ such that $s^n \neq 0$ and $\mathrm{Im}(s^n)$ is a direct summand of M, then $\mathrm{Ker}(s^n)$ is a direct summand of M.

Example 3.2. (1) $D2$ modules are weakly $D2$.

(2) A module M is called $GD2$ if for every submodule N of M for which M/N is isomorphic to M, then N is a direct summand of M [15]. Weakly $D2$ modules are $GD2$. Indeed, let M be a weakly $D2$ module and $N \leq M$ such that $M/N \cong M$. Then there exists an isomorphism $f : M/N \to M$. Consider $s = f\pi$, where $\pi : M \to M/N$ is the canonical projection. Then $\mathrm{Ker}(s^n) = N$. We note that $\mathrm{Im}(s^n) = M$ for all $n \in \mathbb{N}$ because s is an epimorphism. We have that M is weakly $D2$ and obtain that $N = \mathrm{Ker}(s^n)$ is a direct summand of M. Therefore M is $GD2$.

(3) Let $M_{\mathbb{Z}} = \mathbb{Z}_p \oplus \mathbb{Z}_{p^2}$ with p a prime number. Then M is $GD2$ which is not weakly $D2$.

(4) A quasi-discrete module is discrete if and only if it is weakly $D2$.

(5) Let M be a lifting, weakly $D2$ module. If M is dual automorphism-invariant ([18]), then M is discrete by (4).

The following observation unifies weakly ⊕-supplemented modules and weakly $D2$ modules with variants of supplements.

Theorem 3.3. *The following conditions are equivalent for a projective module M:*

(1) *M is ⊕-supplemented;*
(2) *M is weakly ⊕-supplemented;*
(3) *M is supplemented;*
(4) *M is lifting;*
(5) *M is lifting and weakly $D2$.*

Proof. (1) ⇒ (2) and (2) ⇒ (3) are clear by definitions and the hierarchy.

(3) ⇒ (1) This follows from [22, 41.15].

(3) ⇒ (4) and (4) ⇒ (5) are trivial.

(5) ⇒ (3) This follows from [11, Proposition 4.8]. □

We continue with equivalent conditions of weakly $D2$ modules.

Proposition 3.4. *The following statements are equivalent for a right R-module M with $S = End(M)$:*

(1) *M is weakly $D2$;*
(2) *For every $s \in S$ and $s \neq 0$, there exists $n \in \mathbb{N}$ such that $s^n \neq 0$ and if $\mathrm{Im}(s^n) = \mathrm{Im}(e)$ with $e^2 = e \in S$, then $e \in s^n S$;*
(3) *For every $0 \neq s \in S$, there exists $n \in \mathbb{N}$ such that $s^n \neq 0$ and if $\mathrm{Im}(s^n) = \mathrm{Im}(e)$ with $e^2 = e \in S$, then $eS = s^n S$;*

(4) *For every* $0 \neq s \in S$, *there exists* $n \in \mathbb{N}$ *such that* $s^n \neq 0$ *and if* $\mathrm{Im}(s^n) = \mathrm{Im}(e)$ *with* $e^2 = e \in S$, *then* $\mathrm{Ker}(es^n)$ *is a direct summand of* M;

(5) *For every* $0 \neq s \in S$, *there exists* $n \in \mathbb{N}$ *such that* $s^n \neq 0$ *and if* $s^nS \leq eS \leq \{f \in S \mid f(M) \leq s^n(M)\}$ *with* $e^2 = e \in S$, *then* $eS = s^nS$;

(6) *For every* $0 \neq s \in S$, *there exists* $n \in \mathbb{N}$ *such that* $s^n \neq 0$ *and if* $\mathrm{Im}(s^n) = \mathrm{Im}(e)$ *with* $e^2 = e \in S$, *then* $\{f \in S \mid f(M) \leq s^n(M)\} = s^nS$.

Proof. (1) $\Leftrightarrow$ (4) Let M be a weakly $D2$ module and $0 \neq s \in S$. There exists $n \in \mathbb{N}$ such that $s^n \neq 0$ and if $\mathrm{Im}(s^n)$ is a direct summand of M, implies that $\mathrm{Ker}(s^n)$ is a direct summand of M. Assume that $\mathrm{Im}(s^n) = \mathrm{Im}(e) = \mathrm{Ker}(1-e)$ for some idempotent e of R. Then $\mathrm{Ker}(s^n)$ is a direct summand of M and $s^n = es^n$. Thus $\mathrm{Ker}(es^n) \leq^{\oplus} M$.

Conversely, for every $0 \neq s \in S$, assume that $\mathrm{Im}(s^n)$ is a direct summand of M. Then $\mathrm{Im}(s^n) = \mathrm{Im}(f) = \mathrm{Ker}(1-f)$ for some $f^2 = f \in S$, i.e., $\mathrm{Ker}(fs^n) \leq^{\oplus} M$. Since $\mathrm{Ker}(s^n) = \mathrm{Ker}(fs^n)$, we obtain that $\mathrm{Ker}(s^n)$ is a direct summand of M.

(2) $\Rightarrow$ (3) This is clear.

(3) $\Rightarrow$ (4) Let $0 \neq s \in S$. Assume that $\mathrm{Im}(s^n) = \mathrm{Im}(e)$ for some idempotent e of R. By (3), $eS = s^nS$. Therefore $e = s^nt$ and $s^n = es^n$ for some $t \in S$, and so $s^nts^n = s^n$. Then $\mathrm{Ker}(s^n)$ is a direct summand of M, i.e., $\mathrm{Ker}(s^n) = \mathrm{Ker}(es^n)$ is a direct summand of M.

(4) $\Rightarrow$ (2) Let $0 \neq s \in S$. Assume that $\mathrm{Im}(s^n) = \mathrm{Im}(e)$ for some idempotent e of R. By (4), $\mathrm{Ker}(es^n)$ is a direct summand of M. We have $\mathrm{Im}(s^n) = \mathrm{Im}(e)$ and obtain that $es^n = s^n$ and so $\mathrm{Ker}(s^n)$ is a direct summand of M. There exists a homomorphism $\phi : s^n(M) \to M$ such that $s^n\phi = 1_{s^n(M)}$. Note that $s^n(M) = e(M)$. Hence $e = s^n(\phi e) \in s^nS$.

(3) $\Leftrightarrow$ (5) and (5) $\Leftrightarrow$ (6) For any $e^2 = e \in S$, $s^nS \leq eS \leq \{f \in S \mid f(M) \leq s^n(M)\}$ if and only if $\mathrm{Im}(s^n) = \mathrm{Im}(e)$. □

Recall that the $D2$-condition is inherited by direct summands. More generally, we have:

Proposition 3.5. *The class of modules with* Di*-conditions,* $1 \leq i \leq 3$*,* $GD2$*-condition and weakly* $D2$*-condition, respectively, are closed under taking direct summands.*

Proof. We refer to [11, Lemma 4.7] for Di-conditions, and for $GD2$-modules we refer to [10, Proposition 4.3].

Assume that M is a weakly $D2$ module and $f : e(M) \to e(M)$ is a homomorphism with $e^2 = e \in S = End(M)$. Let $s = \iota f \pi$ with $\pi : M \to e(M)$ the canonical projection and $\iota : e(M) \to M$ the inclusion. Then $s(M) = \iota f \pi(M) = f(e(M))$. Since M is weakly $D2$, there exists $n \in \mathbb{N}$ such that $s^n \neq 0$ and if $\mathrm{Im}(s^n)$ is a direct summand of M, then $\mathrm{Ker}(s^n)$ is a direct summand of M. On the other hand, we have $s^n(m) = f^n(em)$ for all $m \in M$, which implies $s^n(M) = f^n(e(M))$. Now assume that if $\mathrm{Im}(f^n)$ is a direct summand of $e(M)$, then $\mathrm{Im}(s^n)$ is a direct summand of M. It follows that $\mathrm{Ker}(s^n)$ is a direct summand of M. On the other hand, $\mathrm{Ker}(s^n) = \mathrm{Ker}(f^n) \oplus (1-e)(M)$ which implies that $\mathrm{Ker}(f^n)$ is a direct summand of $e(M)$. Thus $e(M)$ is weakly $D2$. □

4. Weakly D2-covers

Proposition 4.1. *Assume that a weakly $D2$ module M has a decomposition $M = A_1 \oplus A_2$ and $f : A_1 \to A_2$ is an R-homomorphism with* $\mathrm{Im}(f) \leq^{\oplus} A_2$. *Then* $\mathrm{Ker}(f) \leq^{\oplus} A_1$.

Proof. To show that $\mathrm{Ker}(f) \leq^{\oplus} A_1$, we first assume that $f : A_1 \to A_2$ is a nonzero epimorphism. Let $s := \iota f \pi$ with the inclusion $\iota : A_2 \to M$ and the canonical projection $\pi : M \to A_1$. Then $s \neq 0$, $s(M) = (\iota f \pi)(M) = (\iota f)(A_1) = \iota(A_2) = A_2$ and $s^2(M) = \iota f \pi(A_2) = 0$. They imply that $s^2 = 0$ and $\mathrm{Im}(s)$ is a direct summand of M. We have that M is a weakly $D2$ module and obtain that $\mathrm{Ker}(s)$ is a direct summand of M, and so $\mathrm{Ker}(s) = \mathrm{Ker}(f) \oplus A_2$. Then $\mathrm{Ker}(f) \oplus A_2$ is a direct summand of M. Thus $\mathrm{Ker}(f)$ is a direct summand of A_1.

Now, let $f : A_1 \to A_2$ be an R-homomorphism and $A_2 = \mathrm{Im}(f) \oplus B$ for a submodule B of A_2. Then $A_1 \oplus \mathrm{Im}(f)$ is a direct summand of M. Since every direct summand of a weakly $D2$-module is a weakly $D2$ module (Proposition 3.5), by applying the preceding argument to the module $A_1 \oplus \mathrm{Im}(f)$, we get $\mathrm{Ker}(f) \leq^{\oplus} A_1$. □

Corollary 4.2. *Assume that P is a projective module and $P \oplus M$ is a weakly $D2$ module. If there exists an epimorphism $P \to M$, then M is also a projective module.*

Proof. Assume that $h : P \to M$ is an epimorphism. By Proposition 4.1, $\mathrm{Ker}(h)$ is a direct summand of P. Thus $M \cong P/\mathrm{Ker}(h)$ and so M is projective. □

Generally, the class of weakly $D2$ modules is not closed under taking direct sums, similar to the case of the class of $D2$ modules.

Proposition 4.3. *The following conditions are equivalent for a ring R:*

(1) *R is a semisimple ring*;
(2) *Every factor module of an injective right R-module is weakly* $D2$;
(3) *Every factor module of the right R-module $(R \oplus R)_R$ is weakly* $D2$;
(4) *Every right R-module is weakly* $D2$;
(5) *The direct sum of any two weakly $D2$-modules is weakly $D2$.*

Proof. (1) ⇒ (2), (1) ⇒ (3) and (1) ⇒ (4) ⇒ (5) are obvious.

(2) ⇒ (1) First, we note that if I is a right ideal of R, then $E(R) \oplus (E(R)/I)$ is weakly $D2$, where $E(-)$ denotes the injective hull. By Proposition 4.1, the canonical map $\eta : E(R) \to E(R)/I$ splits, and so I is a direct summand of $E(R)$. Thus I is a direct summand of R.

(3) ⇒ (1) If I is a right ideal of R, then $R \oplus R/I$ is a weakly $D2$-module. By Proposition 4.1, the canonical map $\eta : R \to R/I$ splits. Hence, I is a direct summand R.

(5) ⇒ (1) Let S be a simple right R-module and $f : R \to S$ an R-epimorphism. By Corollary 4.2, S is projective since $R \oplus S$ is a weakly $D2$-module. Thus R is semisimple. □

Definition 4.4. An R-homomorphism $\phi : P \to M$ is called weakly $D2$-cover of the right R-module M, if P is a weakly $D2$ module, ϕ is an epimorphism, and $\mathrm{Ker}(\phi) \ll P$.

Recall the following Bass characterizations of (semi)perfect rings in terms of projective covers.

• R is a right (semi)perfect ring if and only if every (finitely generated) right R-module has a projective cover if and only if every (simple) semisimple right R-module has a projective cover.

• R is a right perfect ring if and only if every flat right R-module is projective.

Theorem 4.5. *The following statements are equivalent for a ring R:*

(1) *R is semiperfect*;
(2) *Every finitely generated right R-module has a weakly $D2$-cover*;
(3) *Every 2-generated right R-module has a weakly $D2$-cover.*

Proof. (1) ⇒ (2) ⇒ (3) Clear.

(3) ⇒ (1) Let M be a simple right R-module. There exists an epimorphism $\psi : R_R \to M$. By (3), there exists an epimorphism $\phi : X \to R_R \oplus M$ such that X is a weakly $D2$ module and $\mathrm{Ker}(\phi) \ll X$. Consider the natural projections $p_1 : R_R \oplus M \to R_R$ and $p_2 : R_R \oplus M \to M$. Then $p_1\phi : X \to R_R$ is an epimorphism. By the projectivity of R_R, we get $X = \mathrm{Ker}(p_1\phi) \oplus T$ with $T \leq X$. Let $M' := \mathrm{Ker}(p_1\phi)$. Now $X/M' \cong R_R$ and $X/M' \cong T$ and so $R_R \cong T$. Hence, we can regard $X = M' \oplus R_R$. Clearly, $f = \phi|_{M'} : M' \to M$ is an epimorphism.

Now we will show that M' is the projective cover of M. Assume that $A + \mathrm{Ker}(f) = M'$. Since $\mathrm{Ker} f \leq \mathrm{Ker}(\phi)$, we have $R_R + A + \mathrm{Ker}(\phi) = M' + R_R = X$ whence $R_R + A = R_R + M'$. Hence $A = M'$ or $\mathrm{Ker}(f) \ll M'$. On the other hand, since R_R is projective, there exists $\overline{\psi} : R_R \to M'$ such that $f\overline{\psi} = \psi$. But $\mathrm{Ker}(f) \ll M'$ and so $\overline{\psi}$ is an epimorphism. Since $X = M' \oplus R_R$ is a weakly $D2$ module, we obtain that M' is projective by Corollary 4.2. Thus R is semiperfect. □

The following example shows that, unlike projective covers, weakly $D2$-covers are not unique.

Example 4.6. Let R be a local ring that is not division. Then $\eta : R \to R/J(R)$ and $id : R/J(R) \to R/J(R)$ are non-isomorphic weakly $D2$-covers of the simple R-module $R/J(R)$. Now $id : R/J(R) \to R/J(R)$ is a weakly $D2$-cover which is not a projective cover.

Theorem 4.7. *The following statements are equivalent for a ring R:*

(1) *R is right perfect;*
(2) *Every flat right R-module is weakly $D2$;*
(3) *Every right R-module has a weakly $D2$-cover.*

Proof. (1) ⇒ (2) and (1) ⇒ (3) are clear.

(2) ⇒ (1) Let M be a flat right R-module, F a free right R-module and $f : F \to M$ an R-epimorphism. Since $T = M \oplus F$ is flat, it is a weakly $D2$ module by the hypothesis. By Proposition 4.1, f splits and M is projective. Hence R is a right perfect ring.

(3) ⇒ (1) Let M be a right R-module. There exist a free module F and an epimorphism $\psi : F \to M$. By (2), there exists an epimorphism $\phi : X \to F \oplus M$ such that X is a weakly $D2$ module and $\mathrm{Ker}(\phi) \ll X$. Now, by the same proof of (3) ⇒ (1) of Theorem 4.5, there exists a projective cover of M as desired. □

Recall that a right R-module M is said to be finitely presented if there is an exact sequence $R^m \to R^n \to M \to 0$.

A ring R is called semiregular if every finitely presented right (left) R-module has a projective cover.

Theorem 4.8. *R is a semiregular ring if and only if every finitely presented right R-module has a weakly $D2$-cover.*

Proof. This follows from Theorem 4.5 and the fact that the direct sum of any two finitely presented modules is again finitely presented. □

5. Rings whose certain classes of modules have weakly $D2$-property

In this section, we will focus on some important rings whose certain classes of modules satisfy the weakly $D2$ condition.

von Neumann regular rings:

An element a of a ring R is called von Neumann regular if there exists an element b in R such that $a = aba$. The ring is called von Neumann regular if all its elements are von Neumann regular. We note that if $a, b \in R$ and $c = a - aba$ is von Neumann regular, then a is von Neumann regular.

According to Lee, Rizvi and Roman, a module M is called d-Rickart (or dual Rickart) if the image in M of any single element of $S = End(M)$ is generated by an idempotent of S. Equivalently, $\forall \varphi \in S$, $\varphi(M) = \mathrm{Im}(\varphi) = e(M)$ for some $e^2 = e \in S$ ([9]).

Theorem 5.1. *The following conditions are equivalent for a module M with $S = End(M)$:*

(1) *S is a von Neumann regular ring;*
(2) *M is a d-Rickart, weakly $D2$ module.*

Proof. (1) ⇒ (2) This is clear.

(2) ⇒ (1) Let $x \in S,\ x \neq 0$. Since M is weakly $D2$, there exists $n \in \mathbb{N}$ such that $x^n \neq 0$ and if $\mathrm{Im}(x^n)$ is a direct summand of M, then $\mathrm{Ker}(x^n)$ is also a direct summand of M. Moreover, since M is d-Rickart, $\mathrm{Im}(x^n)$ is a direct summand of M. It follows that $\mathrm{Ker}(x^n)$ is also a direct summand of M and so x^n is a regular element of S. If $n = 1$, then x is regular. Otherwise, since x^n is regular, there exists $c \in S$ such that $x^n = x^n c x^n$. Let $y = x^{n-1} - x^{n-1}(cx)x^{n-1}$. Then $y^2 = 0$. If $y = 0$, then $x^{n-1} = x^{n-1}(cx)x^{n-1}$ or x^{n-1} is regular. If $y \neq 0$, then by above proof, we obtain that y is also regular. It follows x^{n-1} is regular. Thus, by induction on n, we have x is regular. Hence S is a von Neumann regular ring. □

We have the following corollaries.

Corollary 5.2. *A ring R is von Neumann regular if and only if R is right d-Rickart right weakly $D2$.*

Corollary 5.3 ([9, Theorem 3.8])**.** *Let M be a right R-module and $S = End(M)$. Then S is a von Neumann regular ring if and only if M is a d-Rickart $D2$ module.*

S-rings:

R is called a right S-ring if every finitely generated flat right R-module is projective ([14]). R is an S-ring if it is both a left and right S-ring.

Note that every semiperfect ring is an S-ring.

Theorem 5.4. *The following statements are equivalent for a ring R:*

(1) *R is a right S-ring*;
(2) *Every finitely generated flat right R-module is quasi-projective*;
(3) *Every finitely generated flat right R-module is weakly $D2$.*

Proof. This is similar to the proof of Theorem 4.7. □

PP-rings:

We will use the following remark to study some results of right PP-rings via weakly $D2$ modules.

For a right R-module M, we consider the following condition:

($*$) If $M = A_1 \oplus A_2$ for some submodules A_1 and A_2 of M, then every R-epimorphism $f : A_1 \to A_2$ is split.

On can check that weakly $D2$ modules satisfy the ($*$)-condition by Proposition 4.1. Note that Morita equivalence preserves summands, epimorphisms, and isomorphisms. Therefore, if R and S are Morita equivalent rings with the category equivalence $F : Mod\text{-}R \longrightarrow Mod\text{-}S$, then a right R-module M_R satisfies the ($*$)-condition if and only if $F(M)_S$ satisfies the ($*$)-condition. Thus, the weakly $D2$-condition is a Morita invariant property of modules.

Recall that R is called a right PP-ring if every principal right ideal of R is projective.

Theorem 5.5. *The following statements are equivalent for a ring R:*

(1) *R is a right PP-ring*;
(2) *Every principal right ideal of $M_2(R)$, generated by a diagonal matrix, is a weakly $D2$ module.*

Proof. (1) ⇒ (2) This follows from [19, Lemma 3].

(2) ⇒ (1) Let $S = M_2(R), a \in R$ and I a principal right ideal of S generated by the diagonal matrix $\left[\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right]$. As a right S-module, I is a weakly $D2$ module. If $e = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, then S and R are Morita equivalent via $M \to Me$, where M is a right S-module. Since $Ie \cong aR \oplus R$ as right R-modules and I satisfies the condition $(*)$, we can obtain that $aR \oplus R$ satisfies condition $(*)$ as a right R-module, and so the canonical epimorphism $\eta : R \to aR$ splits. Thus aR is projective and R is a right PP-ring. □

Semihereditary rings:

A ring R is called a right (semi)hereditary if every (finitely generated) right ideal of R is projective, equivalently, if every (finitely generated) submodule of a projective right R-module is projective. R is called (semi)hereditary if it is both left and right (semi)hereditary.

Theorem 5.6. *The following statements are equivalent for a ring R:*

(1) *R is a right semihereditary ring*;
(2) *Every finitely generated submodule of a projective right R-module is weakly $D2$*;
(3) *For any finitely generated free right R-module F, every principal right ideal of $S = End(F_R)$ is a weakly $D2$ module.*

Proof. (1) ⇒ (2) This is obvious.

(2) ⇒ (1) Let K be a finitely generated submodule of a projective right R-module P and $f : F \to K$ an R-epimorphism with F_R finitely generated and free. Since $F \oplus K$ is a finitely generated submodule of the projective module $F \oplus P$, we can obtain that $F \oplus K$ is a weakly $D2$ module by the hypothesis. By Corollary 4.2, K is projective.

(1) ⇒ (3) This follows from [4, Theorem 2.4].

(3) ⇒ (1) Let F be a finitely generated free right R-module and $S = End(F_R) \cong M_n(R)$. Then $F \oplus F$ is a free right R-module such that $End(F \oplus F) = M_2(S)$. By Theorem 5.5, since each principal right ideal of $M_2(S)$ is a weakly $D2$-module, S is a right PP-ring. By [4, Theorem 2.4], R is a right semihereditary ring. □

Theorem 5.7. *The following statements are equivalent for a ring R:*

(1) *R is a right semihereditary right S-ring*;
(2) *R is a left semihereditary left S-ring*;
(3) *Every finitely generated submodule of a flat right R-module is weakly $D2$*;
(4) *Every finitely generated submodule of a flat left R-module is weakly $D2$.*

Proof. (1) ⇔ (2) This follows from [14, Proposition 4.10].

(1) ⇒ (3) Let K be a finitely generated submodule of a flat right R-module F. Since R is right semihereditary, K is flat by [8, Theorem 4.67] and so is projective. Hence K is a weakly $D2$-module.

(3) ⇒ (1) This follows from Theorems 5.4 and 5.6.

(2) ⇔ (4) This is similar to (1) ⇔ (3). □

Π-coherent rings

A module M_R is coherent if every finitely generated submodule of M is finitely presented. A ring R is called left (right) coherent if R_R (resp. $_RR$) is coherent.

Let $\Pi = \Pi R_R$ be an arbitrary product of copies of R_R. According to Camillo [2], a ring R is called right Π-coherent if every finitely generated submodule of Π is finitely presented. R is called Π-coherent if it is both left and right Π-coherent.

We say M is a Π-weakly $D2$-module if ΠM_R is a weakly $D2$-module.

Theorem 5.8. *The following conditions are equivalent for a ring R:*

1. (1) *R is right perfect left coherent*;
2. (2) *The direct product of any family of copies of R is projective as a right R-module*;
3. (3) *All direct products of projective right R-modules are weakly $D2$*;
4. (4) *All direct products of flat right R-modules are weakly $D2$*;
5. (5) *Every projective right R-module is Π-weakly $D2$.*

Proof. (1) ⇔ (2) ⇒ (4) ⇒ (3) They follow from [3, Theorem 3.2].

(3) ⇒ (1) Let $M = \prod_{i\in I} M_i$ be a direct product of projective right R-modules and $f : F \to M$ an R-epimorphism with F free. By Corollary 4.2, M is projective since $M \times F \cong M \oplus F$ is a weakly $D2$ module. By [3, Theorem 3.3], R is right perfect and left coherent.

(3) ⇒ (5) This is clear.

(5) ⇒ (3) Let $M = \prod_{i\in I} M_i$ be a direct product of projective right R-modules, $f : F \to M$ an R-epimorphism with F free, and $g_i : M \to M_i$ the canonical projection, $i \in I$. Since each M_i is projective, the epimorphism $g_i \circ f : F \to M_i$ is split; i.e., there exist submodules $A_i, T_i \le F$ with $F = A_i \oplus T_i$ and $M_i \cong A_i, i \in I$. As

$$\prod_{i\in I} F = \prod_{i\in I}(A_i \oplus T_i) \cong \prod_{i\in I}(A_i) \oplus (\prod_{i\in I} T_i)$$

and $\prod_{i\in I} F$ is a weakly $D2$-module, it follows from Proposition 3.5 that $\prod_{i\in I} A_i$ is a weakly $D2$ module. Consequently, $M = \prod_{i\in I} M_i \cong \prod_{i\in I} A_i$ is a weakly $D2$ module. □

A right R-module M is called torsionless if it can be embedded in a direct product of copies of R_R.

Theorem 5.9. *The following statements are equivalent for a ring R:*

(1) *R is a semihereditary Π-coherent ring*;
(2) *Every finitely generated torsionless right R-module is projective*;
(3) *Every finitely generated torsionless right R-module is weakly $D2$*;
(4) *Every finitely generated torsionless left R-module is projective*;
(5) *Every finitely generated torsionless left R-module is weakly $D2$.*

Proof. (1) ⇒ (2) This follows from [17, Theorem 3.5].

(2) ⇒ (3) This is obvious.

(3) ⇒ (2) Let M be a finitely generated torsionless right R-module, F a finitely generated free right R-module and $\eta : F \to M$ an R-epimorphism. Since $F \oplus M$ is finitely generated and torsionless, the module M is projective by Corollary 4.2.

(2) ⇒ (1) Since every finitely generated torsionless right R-module is projective, the ring R is right semihereditary right Π-coherent. (1) ⇔ (4) ⇔ (5) They are symmetric. □

Semiprimary rings:

The following observation is a special case of Theorem 5.6 and [4, Theorem 2.3].

Theorem 5.10. *The following statements are equivalent for a ring R:*

(1) *R is right hereditary*;
(2) *Every submodule of a projective right R-module is weakly $D2$*;
(3) *Every principal right ideal of $S = End(F_R)$ is a weakly $D2$-module, for any free right R-module F.*

We recall that a ring R is called semiprimary if the Jacobson radical $J(R)$ of R is nilpotent and the ring R is semilocal, i.e., $R = J(R)$ has a finite length, equivalently, it is semisimple.

Theorem 5.11. *The following statements are equivalent for a ring R:*

(1) *R is a semiprimary hereditary ring*;
(2) *R is a right hereditary right perfect ring*;
(3) *R is a left hereditary left perfect ring*;
(4) *Every submodule of a flat right R-module is weakly $D2$*;
(5) *Every submodule of a flat left R-module is weakly $D2$.*

Proof. (2) ⇔ (4) and (3) ⇔ (5) They are consequences of Theorems 4.7 and 5.10.

(1) ⇔ (2) ⇔ (3) They follow from [19, Corollary 2]. □

A ring R is right semihereditary if and only if every torsionless left R-module is flat (see [3, Theorem 4.1]).

Theorem 5.12. *The following statements are equivalent for a ring R:*

(1) *R is a semiprimary hereditary ring*;

(2) *Every torsionless right R-module is projective*;
(3) *Every torsionless left R-module is projective*;
(4) *Every torsionless right R-module is weakly $D2$*;
(5) *Every torsionless left R-module is weakly $D2$.*

Proof. (1) $\Rightarrow$ (2) By Theorem 5.8, the direct product of any family of copies of R is projective as a right R-module. Since R is right hereditary, every submodule of ΠR_R is projective. This means every torsionless right R-module is projective.

(2) $\Rightarrow$ (4) This is clear.

(4) $\Rightarrow$ (2) Let M be a torsionless right R-module, F a free right R-module, and $\eta : F \to M$ an R-epimorphism. Since $F \oplus M$ is torsionless, we obtain that M is projective by Corollary 4.2.

(2) $\Rightarrow$ (1) By the hypothesis and Theorem 5.8, R is a right hereditary right perfect ring. Now, by Theorem 5.11, R is a semiprimary hereditary ring.

(1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (5) They follow by a symmetrical argument. □

Theorem 5.13. *The following conditions are equivalent for a ring R*:
(1) *R is semilocal*;
(2) *Every semiprimitive R-module is weakly $D2$*;
(3) *Every finitely generated semiprimitive R-module is weakly $D2$*;
(4) *Every 2-generated, semiprimitive R-module is weakly $D2$.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) They are obvious.

(4) $\Rightarrow$ (1) Let $R' = R/J(R)$ and P a simple R'-module. Since R' and P are J-semisimple as R-modules and $R' \oplus P$ is 2-generated, by hypothesis, $R' \oplus P$ is weakly $D2$ as an R'-module. By Corollary 4.2, the simple R'-module P is projective. Thus, R' is a semisimple ring, i.e., R is semilocal. □

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