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POSITIVE SOLUTIONS OF A REACTION-DIFFUSION SYSTEM WITH DIRICHLET BOUNDARY CONDITION

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Abstract. In this article, we study a reaction-diffusion system with homogeneous Dirichlet boundary conditions, which describing a three-species food chain model. Under some conditions, the predator-prey subsystem ($u_1 \equiv 0$) has a unique positive solution $(\overline{u_2}, \overline{u_3})$. By using the birth rate of the prey r_1 as a bifurcation parameter, a connected set of positive solutions of our system bifurcating from semi-trivial solution set $(r_1, (0, \overline{u_2}, \overline{u_3}))$ is obtained. Results are obtained by the use of degree theory in cones and sub and super solution techniques.

1. Introduction

Ecological systems are characterized by the interactions of different species within a fluctuating natural environment. Among various models describing different interactions, the three species models are fundamental building blocks of large scale ecosystems. To clarify the local or global and short-term or long-term behavior of ecosystems, it is essential to understand the interacting dynamics of three species models. Krikorian [16] has classified all three-species Lotka-Volterra models into four types in all 34 cases: food chains, two predators competing for one prey, one predator acting on two preys, and loops. In particular, the interest in three-species food chain models stems from the seminal work of Hastings and Powell [10] in which they show chaotic dynamics in a food chain model. Since then, there have been some interesting and significant results on the dynamics of three-species food chain systems with spatially homogeneous situations [4, 8, 9, 13] and spatially inhomogeneous situations [19, 20, 22, 23, 27] for last two decades. It is known in literature that the dynamics of the three-species model is much more complicated than that of the two-species model in a relative sense. Even for the ODE system, the dynamic behavior of positive solutions can be very complicated (see [8]). Consequently, multiple-species models will continue to be one of dominant themes

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in both ecology and mathematical ecology due to its universal existence and importance.

In this paper, we consider the following Lotka-Volterra food chain model

$$\left\{\begin{array}{l} u_{1t}-\Delta u_1=u_1\left(r_1-a_{11}u_1-a_{12}u_2\right), \\ u_{2t}-\Delta u_2=u_2\left(r_2+a_{21}u_1-a_{22}u_2-a_{23}u_3\right), \\ u_{3t}-\Delta u_3=u_3\left(r_3+a_{32}u_2-a_{33}u_3\right) \quad \text{in } \ \Omega\times(0,\infty), \end{array}\right. \tag{1}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, Δ is the Laplacian operator. r_j, a_{jj}, $j=1,2,3$, a_{12}, a_{21}, a_{23}, a_{32} are all positive constants. u_j, $j=1,2,3$, stands for the population density of prey, mid-level predator and top predator species, respectively. r_j is the birth rate of the prey, mid-level predator and top predator, respectively; a_{jj} measures the intra-specific competition of the prey, mid-level predator and top predator, respectively; a_{12} and a_{23} denote the predation rate of per capita of the mid-level predator and top predator, respectively; a_{21} and a_{32} represent the conversion rate of the prey to the mid-level predator and the mid-level predator to the top one, respectively. When the system (1) is subjected to homogeneous Neumann boundary conditions, by Lyapunov functional arguments, Xie [25] proved that the unique constant positive steady-state is globally asymptotically stable even if u_1, u_2 and u_3 possess different diffusion coefficients, which indicated that no spatiotemporal patterns happen in the system (1). Whereas, if some cross-diffusion terms are introduced in the system (1), Ma et al. [21] investigated the existence of non-constant positive steady-states as well as the Hopf bifurcation. They proved that the stationary patterns and inhomogeneous periodic oscillatory patterns emerged.

In this paper we assume that the boundary is hostile and hence no individuals would choose to leave there and consequently, we shall subsequently consider homogeneous Dirichlet boundary conditions:

$$u_1=u_2=u_3=0 \quad \text{on } \ \partial\Omega\times(0,\infty). \tag{2}$$

One aspect of great interest for a model with multi species interactions is whether the various species can coexist. An important early discovery on the problem of positive coexistence of 2×2 systems is the following: the instability of the marginal densities, i.e., the individual species with the other species absent, implies the positive coexistence of both species provided that the interacting species are a priori bounded. See, for instance [2,3,5,7,26]. However, there is little concern on the problem of 3×3 systems [11,17,18]. The goal of this paper is to establish sufficient conditions for the existence of componentwise strictly positive steady-state solutions of (1) with boundary conditions (2). Thus we will concentrate on the following elliptic system

$$\left\{\begin{array}{l} -\Delta u_1=u_1\left(r_1-a_{11}u_1-a_{12}u_2\right), \\ -\Delta u_2=u_2\left(r_2+a_{21}u_1-a_{22}u_2-a_{23}u_3\right), \\ -\Delta u_3=u_3\left(r_3+a_{32}u_2-a_{33}u_3\right) \quad \text{in } \Omega, \\ u_1=u_2=u_3=0 \quad \text{on } \partial\Omega. \end{array}\right. \tag{3}$$

Our analysis is based on the degree theory in cones and sub- and super solution techniques.

The plan of the paper is as follows: In Section 2, we give some known results which are required later. In Section 3, we discuss the unique positive solution $(\overline{u_2}, \overline{u_3})$ of the predator-prey subsystem $(u_1 \equiv 0)$. In Section 4, By regarding r_1 as a bifurcation parameter, we study the bifurcation solutions of (3) which are relative to $(0, \overline{u_2}, \overline{u_3})$. We end with concluding remarks in Section 5.

2. Preliminaries

Let λ_1 be the principal eigenvalue of the following problem

$$\begin{cases} -\Delta\phi = \lambda\phi & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega. \end{cases}$$

Let $m(x) \in C\left(\overline{\Omega}\right)$ satisfy that $m(x_0) > 0$ for some $x_0 \in \Omega$. Then by Theorem 1 of [12], the boundary value problem

$$\begin{cases} -\Delta\varphi = \lambda m\varphi & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega \end{cases} \tag{4}$$

has a principal eigenvalue $\widetilde{\lambda_1}(m) > 0$, and it is the only positive eigenvalue of (4) with a positive eigenfunction. For every $p > 0$ such that $m + p > 0$ in Ω, define an operator $L_p = (-\Delta + p)^{-1}(m + p)$. Let $R(L_p)$ denote the spectral radius of L_p.

Lemma 2.1. *If* $\widetilde{\lambda_1}(m) = 1$, *then* $R(L_p) = 1$.

Proof. If $\widetilde{\lambda_1}(m) = 1$, then there exists $\varphi > 0$ such that (4) holds for $\lambda = 1$. This implies that $\varphi = L_p\varphi$. Therefore, $R(L_p) = 1$. □

Consider the boundary value problem

$$\begin{cases} -\Delta u = au - u^2 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5}$$

Lemma 2.2. (i) *If* $a \le \lambda_1$, *then* (5) *has no nontrivial solution.*

(ii) *If* $a > \lambda_1$, *then there exists a unique positive solution* θ_a *of* (5) *and* $0 < \theta_a < a$. *Also,* $a < b$ *implies that* $\theta_a < \theta_b$.

Proof. See [3] for the proof. □

For $a > \lambda_1$ and $0 < k\ (< 1)$, we consider

$$\begin{cases} -\Delta u = u\left(a - u \pm k\theta_a\right) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6}$$

Lemma 2.3. $(1 \pm k)\,\theta_a$ *is the unique positive solution of* (6).

Proof. The existence is obvious. For the uniqueness, we only give the proof of the first case. The second case can be shown similarly. Assume that there exists another positive solution u_0 of (6) with $u_0 \not\equiv (1+k)\,\theta_a$.

Let $u = u_0 - (1+k)\,\theta_a$. Then $u \not\equiv 0$ satisfies the following boundary value problem

$$\begin{cases} -\Delta u = u\,(a - u_0 - \theta_a) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then

$$\int_\Omega (-\Delta u - u\,(a - u_0 - \theta_a))\, u dx = 0. \tag{7}$$

On the other hand, since θ_a is the unique positive solution of (5), it follows that for the eigenvalue problem

$$\begin{cases} -\Delta u - u\,(a - \theta_a) = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

zero is the least eigenvalue. Due to the variational characterization of the least eigenvalue, we have that

$$\int_\Omega (-\Delta u - u\,(a - \theta_a))\, u dx \geq 0, \quad \forall u \in C_0^2\left(\overline{\Omega}\right).$$

So (7) implies that $\int_\Omega u_0 u^2 dx \leq 0$, which is a contradiction. $\square$

3. Two-species subsystem

In this section, we assume that $a_{22} = a_{33} = 1$ in (3) and study the unique positive solution of the following predator-prey subsystem ($u_1 \equiv 0$):

$$\begin{cases} -\Delta u_2 = u_2\,(r_2 - u_2 - a_{23} u_3)\,, \\ -\Delta u_3 = u_3\,(r_3 + a_{32} u_2 - u_3) \quad \text{in } \Omega, \\ u_2 = u_3 = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{8}$$

Let

$$\mathscr{K} = \frac{a_{23}^2 + a_{32}^2\,(1+a_{32})^2}{(4+2a_{23}a_{32})\,(1-a_{23}\,(1+a_{32}))}, \quad \mathscr{L} = \frac{a_{23}^2 + a_{32}^2\,(1+a_{32})^2}{4+2a_{23}a_{32}} + a_{23}\,(1+a_{32})\,.$$

It is easy to see that $0 < \mathscr{K} < 1$ if and only if $0 < \mathscr{L} < 1$. It is easy to show that if $(a_{23},\, a_{32})$ falls into the shadow region of Fig. 1, then $0 < \mathscr{K} < 1$.

Assume that $0 < \mathscr{K} < 1$, $r_3 > \lambda_1$. Define

$$\underline{r_2} = \inf\left\{r_2 \in [\lambda_1, r_3]\,; \theta_{r_2} \geq \mathscr{L}\theta_{r_3}\right\}, \ \overline{r_2} = \sup\left\{r_2 \in [r_3, \infty]\,; \mathscr{K}\theta_{r_2} \leq \theta_{r_3}\right\},$$

where θ_{r_i} $(i = 2, 3)$ is the unique positive solution of the following boundary value problem

$$\begin{cases} -\Delta u_i = u_i\,(r_i - u_i) & \text{in } \Omega, \\ u_i = 0 & \text{on } \partial\Omega. \end{cases}$$

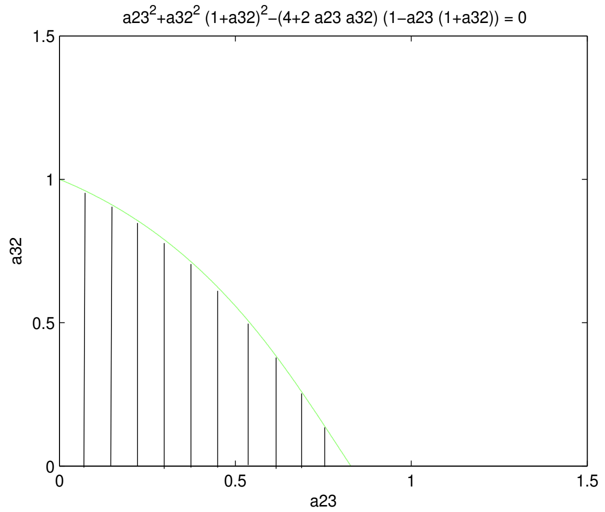

FIGURE 1. In the shadow region of plane $a_{23} - a_{32}$, $0 < \mathscr{K} < 1$ holds.

Lemma 3.1. *Let* (u_2, u_3) *be a positive solution of* (8). *Then* $u_2 < \theta_{r_2}$ *and* $\theta_{r_3} < u_3$. *If* $0 < \mathscr{K} < 1$, $\underline{r_2} \le r_2 \le r_3 \le \overline{r_2}$, *then* $u_2 > \left(1 - \frac{a_{23}(1+a_{32})}{\mathscr{L}}\right)\theta_{r_2}$, $u_3 < (1 + a_{32})\theta_{r_3}$.

Proof. If (u_2, u_3) is a nonnegative solution of (8) such that u_2 is not identically zero, then $r_2 > \lambda_1$ [3]. So

$$\begin{cases} -\Delta u_2 = u_2 (r_2 - u_2) & \text{in } \Omega, \\ u_2 = 0 & \text{on } \partial\Omega \end{cases} \tag{9}$$

has a unique positive solution θ_{r_2}. Clearly, u_2 is a subsolution of (9), and moreover large positive constant can be a supersolution of (9). Hence a sub-super solution argument and the uniqueness of θ_{r_2} deduce that $u_2 \le \theta_{r_2}$. Since $r_3 > \lambda_1$,

$$-\Delta u_3 = u_3 (r_3 + a_{32}u_2 - u_3) \ge u_3 (r_3 - u_3) \quad \text{in } \Omega,$$

a simple sub-super solution argument deduces that $u_3 \ge \theta_{r_3}$.

It is easy to show from $r_2 \le r_3$ that

$$-\Delta u_3 = u_3 (r_3 + a_{32}u_2 - u_3) < u_3 (r_3 + a_{32}\theta_{r_2} - u_3) \le u_3 (r_3 + a_{32}\theta_{r_3} - u_3),$$

u_3 is a subsolution of (6) with $k = a_{32}$. Since any sufficiently large constant is a supersolution of (6), together with Lemma 2.3, we have that $u_3 < (1 + a_{32})\,\theta_{r_3}$.

Since $\underline{r_2} \le r_2 \le r_3$, we have that

$$\begin{cases} -\Delta u_2 = u_2 (r_2 - u_2 - a_{23}u_3) > u_2 \left(r_2 - u_2 - \frac{a_{23}(1+a_{32})}{\mathscr{L}}\theta_{r_2}\right) & \text{in } \Omega, \\ u_2 = 0 \ \ \text{on } \partial\Omega. \end{cases}$$

Then u_2 is a supersolution of (6) with $k = \frac{a_{23}(1+a_{32})}{\mathscr{L}}$. On the other hand, there exists a sufficiently small constant $\varepsilon > 0$ such that $\varepsilon\theta_{r_2} < u_2$ and $\varepsilon < 1 - \frac{a_{23}(1+a_{32})}{\mathscr{L}}$. Then we have

$$\begin{cases} -\Delta\left(\varepsilon\theta_{r_2}\right) = \varepsilon\theta_{r_2}\left(r_2 - \theta_{r_2}\right) \leq \varepsilon\theta_{r_2}\left(r_2 - \left(\varepsilon + \dfrac{a_{23}\left(1+a_{32}\right)}{\mathscr{L}}\right)\theta_{r_2}\right) \\ \qquad\qquad = \varepsilon\theta_{r_2}\left(r_2 - \dfrac{a_{23}\left(1+a_{32}\right)}{\mathscr{L}}\theta_{r_2} - \varepsilon\theta_{r_2}\right) \quad \text{in } \Omega, \\ \quad \varepsilon\theta_{r_2} = 0 \quad \text{on } \partial\Omega, \end{cases}$$

hence $\varepsilon\theta_{r_2}$ is a subsolution of (6) with $k = \frac{a_{23}(1+a_{32})}{\mathscr{L}}$. By Lemma 2.3, we have that $u_2 > \left(1 - \frac{a_{23}(1+a_{32})}{\mathscr{L}}\right)\theta_{r_2}$, we have to take $\varepsilon \to 0$ and still we have the strict inequality $u_2 > \left(1 - \frac{a_{23}(1+a_{32})}{\mathscr{L}}\right)\theta_{r_2}$. $\square$

Lemma 3.2. *If* $0 < \mathscr{K} < 1$, $\underline{r_2} \leq r_2 \leq r_3 \leq \overline{r_2}$, *then there exists a unique positive solution* $(\overline{u_2}, \overline{u_3})$ *of* (8).

Proof. By Lemma 3.1 and a sub-super solution argument, we can show that (8) has a positive solution. To prove the uniqueness, we assume that there exist two different positive solutions (u_{21}, u_{31}) and (u_{22}, u_{32}) of (8). Let $p = u_{21} - u_{22}$, $q = u_{31} - u_{32}$. Then $(p, q) \not\equiv (0, 0)$ satisfies

$$\begin{cases} -\Delta p = p\left(r_2 - u_{21} - a_{23}u_{31}\right) - u_{22}p - a_{23}u_{22}q, \\ -\Delta q = q\left(r_3 + a_{32}u_{21} - u_{31}\right) + a_{32}u_{32}p - u_{32}q \quad \text{in } \Omega, \\ p = q = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Therefore, it follows that

$$\int_\Omega \left[-\Delta p - p\left(r_2 - u_{21} - a_{23}u_{31}\right)\right] p dx + \int_\Omega \left(u_{22}p + a_{23}u_{22}q\right) p dx = 0, \tag{10}$$

$$\int_\Omega \left[-\Delta q - q\left(r_3 + a_{32}u_{21} - u_{31}\right)\right] q dx + \int_\Omega \left(u_{32}q - a_{32}u_{32}p\right) q dx = 0. \tag{11}$$

Since (u_{21}, u_{31}) is a positive solution of (8), zero is the least eigenvalue of the following two eigenvalue problems

$$\begin{cases} -\Delta\psi_1 - \psi_1\left(r_2 - u_{21} - a_{23}u_{31}\right) = \lambda\psi_1 \quad \text{in } \Omega, \\ \psi_1 = 0 \quad \text{on } \partial\Omega, \end{cases}$$

$$\begin{cases} -\Delta\psi_2 - \psi_2\left(r_3 + a_{32}u_{21} - u_{31}\right) = \lambda\psi_2 \quad \text{in } \Omega, \\ \psi_2 = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Arguing as in the proof of Lemma 2.3, we get that the first terms in both (10) and (11) are nonnegative. Therefore,

$$\int_\Omega \left(u_{22}p^2 + \left(a_{23}u_{22} - a_{32}u_{32}\right) pq + u_{32}q^2\right) dx \leq 0. \tag{12}$$

Let $B_1 = (a_{23}u_{22} - a_{32}u_{32})^2 - 4u_{22}u_{32}$. Since $r_2 \le r_3$,

$$\begin{aligned} B_1 &= a_{23}^2 u_{22}^2 + a_{32}^2 u_{32}^2 - (4 + 2a_{23}a_{32})\, u_{22}u_{32} \\ &< a_{23}^2\theta_{r_3}^2 + a_{32}^2 (1+a_{32})^2 \theta_{r_3}^2 - (4+2a_{23}a_{32}) \left(1 - \frac{a_{23}(1+a_{32})}{\mathscr{L}}\right) \theta_{r_2}\theta_{r_3} \\ &\le \left[a_{23}^2 + a_{32}^2 (1+a_{32})^2 - (4+2a_{23}a_{32})\,(\mathscr{L} - a_{23}(1+a_{32}))\right] \theta_{r_3}^2 = 0, \end{aligned}$$

which implies that the form in (12) is positive definite. Therefore, (12) can hold only when $p \equiv q \equiv 0$, a contradiction. □

4. Bifurcation of positive solutions related to the semi-trivial solution $(0, \overline{u_2}, \overline{u_3})$

In this section, we assume that $a_{22} = a_{33} = 1$ in (3). By regarding r_1 as the bifurcation parameter, we discuss the bifurcation of positive solutions of (3) relative to the semi-trivial solution $(0, \overline{u_2}, \overline{u_3})$.

Lemma 4.1. *Let* (u_1, u_2, u_3) *be a nonnegative solution of* (3) *such that* u_1, u_2, $u_3 \ge 0$. *Then*

$$u_1 \le \frac{r_1}{a_{11}},\ u_2 \le r_2 + \frac{a_{21}r_1}{a_{11}},\ u_3 \le r_3 + a_{32}\left(r_2 + \frac{a_{21}r_1}{a_{11}}\right).$$

Proof. If (u_1, u_2, u_3) is a nonnegative solution of (3) such that u_1 is not identically zero, then $r_1 > \lambda_1$. So

$$\begin{cases} -\Delta u_1 = u_1 (r_1 - a_{11}u_1) & \text{in } \Omega, \\ \ \ u_1 = 0 & \text{on } \partial\Omega \end{cases} \tag{13}$$

has a unique positive solution u_{1r_1} and $u_{1r_1} \le \frac{r_1}{a_{11}}$ (see [3]). A simple sub-super solution argument deduces that $u_1 \le u_{1r_1}$. Since

$$-\Delta u_2 \le u_2 \left(r_2 + \frac{a_{21}r_1}{a_{11}} - u_2\right) \ \text{ in } \Omega.$$

Then the maximum principle yields the required upper bound for u_2. Similarly,

$$-\Delta u_3 \le u_2 \left(r_3 + a_{32}\left(r_2 + \frac{a_{21}r_1}{a_{11}}\right) - u_3\right) \ \text{ in } \Omega.$$

Then the maximum principle yields the required upper bound for u_3. □

We now establish an appropriate setting that will enable us to transform problem (3) into a fixed point problem.

For $r_1 > 0$, define a set $\mathbb{T}_{r_1} \subset H = \left[C_0^+(\overline{\Omega})\right]^3$ by

$$\begin{aligned} \mathbb{T}_{r_1} = \{(u_1, u_2, u_3) \in H,\ u_1 \le \frac{2r_1}{a_{11}}, u_2 \le 2\left(r_2 + \frac{a_{21}r_1}{a_{11}}\right), \\ u_3 \le 2\left(r_3 + a_{32}\left(r_2 + \frac{a_{21}r_1}{a_{11}}\right)\right)\right\}, \end{aligned}$$

where $C_0^+(\overline{\Omega}) = \{u \in C_0(\overline{\Omega}) : u(x) \geq 0 \text{ for } x \in \overline{\Omega}\}$ and $C_0(\overline{\Omega})$ be the Banach space of continuous functions on $\overline{\Omega}$ whose values on $\partial\Omega$ are zero. By Lemma 4.1, all positive solutions of (3) lie in the interior of $\mathbb{T}_{r_1}$. Moreover, one sees that there exists a continuous and nondecreasing function of r_1, denoted by $p(r_1)$, such that for all $(u_1, u_2, u_3) \in \mathbb{T}_{r_1}$,

$$\begin{cases} r_1 - a_{11}u_1 - a_{12}u_2 + p(r_1) > 0, \\ r_2 + a_{21}u_1 - u_2 - a_{23}u_3 + p(r_1) > 0, \\ r_3 + a_{32}u_2 - u_3 + p(r_1) > 0. \end{cases} \tag{14}$$

Then we define an operator $A(r_1, \cdot)$ on $\mathbb{T}_{r_1}$ by

$$\begin{aligned} (15)\quad & A(r_1, (u_1, u_2, u_3)) \\ &= (-\Delta + p(r_1))^{-1} (u_1 (r_1 - a_{11}u_1 - a_{12}u_2 + p(r_1)), \\ &\quad u_2 (r_2 + a_{21}u_1 - u_2 - a_{23}u_3 + p(r_1)),\ u_3 (r_3 + a_{32}u_2 - u_3 + p(r_1))). \end{aligned}$$

Clearly, $A(r_1, \cdot) : \mathbb{T}_{r_1} \to H$ is completely continuous and Fréchet differentiable, and any fixed points of $A(r_1, \cdot)$ are solutions of (3). In the following, we will study equations

$$A(r_1, (u_1, u_2, u_3)) = (u_1, u_2, u_3) \tag{16}$$

instead of (3), i.e., we will study the fixed points of a one-parameter family of completely continuous map. This family $A : \mathbb{T} \to H$, where $\mathbb{T} = \cup_{r_1 \geq 0} \{r_1\} \times \mathbb{T}_{r_1}$, $\mathbb{T} \subset \mathbb{R}^+ \times H$, is completely continuous. Therefore, the solution set $\mathbb{S}$ of (16) defined by

$$\mathbb{S} = \{(r_1, (u_1, u_2, u_3)) \in \mathbb{T},\ A(r_1, (u_1, u_2, u_3)) = (u_1, u_2, u_3)\}$$

is locally compact.

Lemma 4.2. *Assume that* $0 < \mathscr{K} < 1$, $\underline{r_2} \leq r_2 \leq r_3 \leq \overline{r_2}$, $0 < a_{12}(1 - a_{23}) < \mathscr{L} < 1$. *Let* r_{1*} *be such that* $\widetilde{\lambda_1}\left[r_{1*} - a_{12}\left(1 - \frac{a_{23}(1+a_{32})}{\mathscr{L}}\right)\theta_{r_2}\right] = 1$. *Then there exists a unique* $\overline{r_1} > 0$ *such that*

$$R\left[(-\Delta + p(\overline{r_1}))^{-1} (\overline{r_1} - a_{12}\overline{u_2} + p(\overline{r_1}))\right] = 1, \tag{17}$$

$\overline{r_1} \in (r_{1*}, r_2)$. *Also, it holds that*

$$\begin{cases} R\left[(-\Delta + p(r_1))^{-1} (r_1 - a_{12}\overline{u_2} + p(r_1))\right] < 1, & \text{if } r_1 < \overline{r_1}, \\ R\left[(-\Delta + p(r_1))^{-1} (r_1 - a_{12}\overline{u_2} + p(r_1))\right] > 1, & \text{if } r_1 > \overline{r_1}. \end{cases}$$

Proof. It is easy to see from Lemma 2.1 that

$$R\left[(-\Delta + p(r_1))^{-1}\left(r_{1*} - a_{12}\left(1 - \frac{a_{23}(1 + a_{32})}{\mathscr{L}}\right)\theta_{r_2} + p(r_1)\right)\right] = 1.$$

If $r_1 \leq r_{1*}$, then

$$r_1 - a_{12}\overline{u_2} < r_1 - a_{12}\left(1 - \frac{a_{23}(1+a_{32})}{\mathscr{L}}\right)\theta_{r_2} \leq r_{1*} - a_{12}\left(1 - \frac{a_{23}(1+a_{32})}{\mathscr{L}}\right)\theta_{r_2}.$$

So,

$$\begin{aligned} &R\left[(-\Delta+p(r_1))^{-1}(r_1-a_{12}\overline{u_2}+p(r_1))\right] \\ &< R\left[(-\Delta+p(r_1))^{-1}\left(r_{1*}-a_{12}\left(1-\frac{a_{23}(1+a_{32})}{\mathscr{L}}\right)\theta_{r_2}+p(r_1)\right)\right]=1. \end{aligned}$$

Since $0<\mathscr{L}<1$, it follows that $a_{23}<1$, and $\overline{u_2}<(1-a_{23})\theta_{r_3}$. Then for $r_1\geq r_2$, $r_1-a_{12}\overline{u_2}>r_2-a_{12}(1-a_{23})\theta_{r_3}>r_2-\frac{a_{12}(1-a_{23})}{\mathscr{L}}\theta_{r_2}>r_2-\theta_{r_2}$. This implies that

$$R\left[(-\Delta+p(r_1))^{-1}(r_1-a_{12}\overline{u_2}+p(r_1))\right]>1.$$

Let $R(r_1):\mathbb{R}^+\to\mathbb{R}^+$ be defined by

$$R(r_1)=R\left[(-\Delta+p(r_1))^{-1}(r_1-a_{12}\overline{u_2}+p(r_1))\right].$$

Then Theorem 4.3.1 and §4.3.5 of [14] imply that $R(r_1)$ is a continuous function. Since $R(r_{1*})<1$ and $R(r_2)>1$, it follows that there exists a $\overline{r_1}\in(r_{1*},r_2)$ such that $R(\overline{r_1})=1$, i.e., (17) holds.

Assume there exists $\widetilde{r_1}$ with $\widetilde{r_1}\neq\overline{r_1}$ such that $R(\widetilde{r_1})=1$. Then there exist $\psi_1>0$ and $\psi_2>0$ such that (4) holds for $m=\overline{r_1}-a_{12}\overline{u_2}$ and $m=\widetilde{r_1}-a_{12}\overline{u_2}$, respectively. It follows that

$$\int_\Omega -\Delta\psi_1\cdot\psi_2dx=\int_\Omega(\overline{r_1}-a_{12}\overline{u_2})\,\psi_1\psi_2dx$$

and

$$\int_\Omega -\Delta\psi_2\cdot\psi_1dx=\int_\Omega(\widetilde{r_1}-a_{12}\overline{u_2})\,\psi_2\psi_1dx.$$

Subtracting these two equalities, we get that

$$0=\int_\Omega(\overline{r_1}-\widetilde{r_1})\,\psi_1\psi_2dx\neq 0,$$

a contradiction.

Since $\overline{r_1}$ is the only value of r_1 such that $R(r_1)=1$, $R(r_1)$ is a continuous function and $R(r_{1*})<1$, $R(r_2)>1$, it follows that $R(r_1)<1$ if $r_1<\overline{r_1}$, and that $R(r_1)>1$ if $r_1>\overline{r_1}$. □

Let L_{r_1} be the linearization of $A(r_1,\cdot)$ at the point $(0,\overline{u_2},\overline{u_3})\in\mathbb{T}_{r_1}$. Then

$$\begin{aligned} L_{r_1}(l,w,h)= &\ (-\Delta+p(r_1))^{-1}\left((r_1-a_{12}\overline{u_2}+p(r_1))\,l,\right. \\ &\ a_{21}\overline{u_2}l+(r_2-2\overline{u_2}-a_{23}\overline{u_3}+p(r_1))\,w-a_{23}\overline{u_2}h, \\ &\ \left.a_{32}\overline{u_3}w+(r_3+a_{32}\overline{u_2}-2\overline{u_3}+p(r_1))\,h\right). \end{aligned}$$

Next, we would like to compute the fixed point index of $A(r_1,\cdot)$ at the point $(0,\overline{u_2},\overline{u_3})$ relative to the cone H of Banach space E. To this end, we introduce the preliminary theorem on the fixed point index.

For $y\in H$, define $W_y=\{x\in E: y+\gamma x\in H \text{ for some } \gamma>0\}$ and $S_y=\{x\in\overline{W}_y: -x\in\overline{W}_y\}$. Let y_* be a fixed point of compact operator $A:H\to H$ and

$L = A^{'}(y_*)$ be the Fréchet derivative of A at y_*. We say that A has property α on $\overline{W}_{y_*}$ if there exist $t \in (0,1)$ and $\omega \in \overline{W}_{y_*} \backslash S_{y_*}$ such that $\omega - tA^{'}\omega \in S_{y_*}$. For an open subset $U \subset H$, define $index_H(A,U) = index(A,U,H) = \deg_H(I - A, U, 0)$, where I is the identity map. Furthermore, the fixed point index of A at y_* in H is defined by $index_H(A, y_*) = index(A, y_*, H) = index(A, U(y_*), H)$, where $U(y_*)$ is a small open neighborhood of y_* in H. Then the following theorem can be obtained from the results in [6, 15].

Theorem 4.3. *Assume that $I - L$ is invertible on $\overline{W}_{y_*}$.*

(i) *If L has property α on $\overline{W}_{y_*}$, then $index_H(A, y_*) = 0$.*

(ii) *If L does not have property α on $\overline{W}_{y_*}$, then $index_H(A, y_*) = (-1)^{\sigma}$, where σ is the sum of algebraic multiplicities of the eigenvalues of L which are greater than* 1.

Lemma 4.4. *$index_H\left(A\left(r_1, \cdot\right), \left(0, \overline{u_2}, \overline{u_3}\right)\right)$ is equal to zero if $r_1 > \overline{r_1}$ and equal to ± 1 if $r_1 < \overline{r_1}$.*

Proof. Let $y = (0, \overline{u_2}, \overline{u_3})$. Then one sees that $W_y = C_0^+\left(\overline{\Omega}\right) \times C_0\left(\overline{\Omega}\right) \times C_0\left(\overline{\Omega}\right)$ and $S_y = \{0\} \times C_0\left(\overline{\Omega}\right) \times C_0\left(\overline{\Omega}\right)$. First, we have to show that L_{r_1} has no eigenvector in W_y corresponding to eigenvalue 1. Assume, on the contrary, that there exists $(l, w, h) \in W_y$ such that

$$L_{r_1}\left(l, w, h\right) = \left(l, w, h\right). \tag{18}$$

If $l \not\equiv 0$, then (18) implies that $\left(-\Delta + p\left(r_1\right)\right)^{-1}\left(r_1 - a_{12}\overline{u_2} + p\left(r_1\right)\right) l = l$, and hence,

$$R\left(\left(-\Delta + p\left(r_1\right)\right)^{-1}\left(r_1 - a_{12}\overline{u_2} + p\left(r_1\right)\right)\right) = 1.$$

This is a contradiction to the assumption that $r_1 \neq \overline{r_1}$. Therefore, it must hold that $l \equiv 0$. If $w \not\equiv 0$ and $h \not\equiv 0$, then (18) implies that

$$\begin{cases} -\Delta w - \left(r_2 - 2\overline{u_2} - a_{23}\overline{u_3}\right) w + a_{23}\overline{u_2} h = 0, \\ -\Delta h - a_{32}\overline{u_3} w - \left(r_3 + a_{32}\overline{u_2} - 2\overline{u_3}\right) h = 0 \ \text{ in } \Omega, \\ w = h = 0 \ \text{ on } \partial\Omega. \end{cases}$$

Multiplying the first equation by w, the second by h, and then integrating over Ω, we get that

$$\int_{\Omega} \left[-\Delta w \cdot w - \left(r_2 - \overline{u_2} - a_{23}\overline{u_3}\right) w^2\right] dx + \int_{\Omega} \left[\overline{u_2} w^2 + a_{23}\overline{u_2} h w\right] dx = 0,$$

$$\int_{\Omega} \left[-\Delta h \cdot h - \left(r_3 + a_{32}\overline{u_2} - \overline{u_3}\right) h^2\right] dx + \int_{\Omega} \left[\overline{u_3} h^2 - a_{32}\overline{u_3} h w\right] dx = 0.$$

Arguing as in the proof of Lemma 3.2, we get that

$$\int_{\Omega} \left[\overline{u_2} w^2 + \overline{u_3} h^2 + \left(a_{23}\overline{u_2} - a_{32}\overline{u_3}\right) w h\right] dx \leq 0.$$

The form above is equal to the form in (12), which is positive definite. Hence $w \equiv h \equiv 0$.

Let $\mathscr{M} = C_0\left(\overline{\Omega}\right) \times \{0\} \times \{0\}$. Then $\mathscr{M}$ is a closed complement of S_y in E. Let $\mathbb{T}$ be the projection onto $\mathscr{M}$, and let $\mathscr{M}_{r_1}$ denote the restriction of L_{r_1} to $\mathscr{M}$. Then

$$\Pi \circ \mathscr{M}_{r_1} = \left(-\Delta + p\left(r_1\right)\right)^{-1}\left(r_1 - a_{12}\overline{u_2} + p\left(r_1\right)\right).$$

Now, Theorem 4.3 and Lemma 2 of [6] imply that $index_H\left(A\left(r_1,\cdot\right),\left(0,\overline{u_2},\overline{u_3}\right)\right) = index\left(L_{r_1}, 0, S_y\right) = \pm 1$ if $r_1 < \overline{r_1}$, and $index_H\left(A\left(r_1,\cdot\right),\left(0,\overline{u_2},\overline{u_3}\right)\right) = 0$ if $r_1 > \overline{r_1}$. □

Let $\Sigma_0 = \{(r_1,(0,\overline{u_2},\overline{u_3})), r_1 \geq 0\}$. Then $\Sigma_0 \subset \mathbb{S}$, it is a continuum (a closed connected set) in $\mathbb{S}$, and therefore it is in $\mathbb{R}^+ \times H$.

Theorem 4.5. *$\overline{r_1}$ is a bifurcation point for (16) with respect to Σ_0, and it is the only one.*

Proof. Assume that $\overline{r_1}$ is not a bifurcation point for Σ_0. Then there exists an interval $[\rho_1,\rho_2]$ such that $\overline{r_1} \in [\rho_1,\rho_2]$ and an open set $U \subset [\rho_1,\rho_2] \times H \cap \mathbb{T}$, such that $\overline{U} \cap \mathbb{S} = [\rho_1,\rho_2] \times \{(0,\overline{u_2},\overline{u_3})\}$ and $\partial U \cap \mathbb{S} = \varnothing$. Let $U_{r_1} = \{(u_1,u_2,u_3),(r_1,(u_1,u_2,u_3)) \in U\}$. Then, by the homotopy invariance property of the fixed point [1], $index_H\left(A\left(\rho_1,\cdot\right),U_{\rho_1}\right) = index_H\left(A\left(\rho_2,\cdot\right),U_{\rho_2}\right)$. On the other hand, by the excision property and definition of local index,

$$index_H\left(A\left(\rho_1,\cdot\right),U_{\rho_1}\right) = index_H\left(A\left(\rho_1,\cdot\right),\left(0,\overline{u_2},\overline{u_3}\right)\right) = \pm 1,$$

$$index_H\left(A\left(\rho_2,\cdot\right),U_{\rho_2}\right) = index_H\left(A\left(\rho_2,\cdot\right),\left(0,\overline{u_2},\overline{u_3}\right)\right) = 0,$$

this is because $\rho_1 < \overline{r_1}$ and $\rho_2 > \overline{r_1}$, a contradiction.

Assume that there exists another bifurcation point $\widehat{r_1}$ with $\widehat{r_1} \neq \overline{r_1}$. Then there exists a sequence $\{(r_{1n},(u_{1n},u_{2n},u_{3n}))\}$ in $\mathbb{S}\backslash\Sigma_0$ such that $r_{1n} \to \widehat{r_1}$, $u_{1n} \to 0$, $u_{2n} \to \overline{u_2}$, $u_{3n} \to \overline{u_3}$. So, there exists N such that $u_{2n} > 0$, $u_{3n} > 0$ for all $n \geq N$.

Assume that $u_{1n} = 0$ for some $n \geq N$. Then (u_{2n},u_{3n}) is a positive solution of (8), and therefore, $u_{2n} = \overline{u_2}$, $u_{3n} = \overline{u_3}$. Hence, $(r_{1n},(u_{1n},u_{2n},u_{3n})) \in \Sigma_0$, contrary to the assumption. So, $u_{1n} > 0$ for all $n \geq N$. Then it holds that

$$\frac{u_{1n}}{\|u_{1n}\|} = (-\Delta)^{-1}\left(\frac{u_{1n}}{\|u_{1n}\|}\left(r_{1n} - a_{11}u_{1n} - a_{12}u_{2n}\right)\right),\ n \geq N. \tag{19}$$

Since $(-\Delta)^{-1}$ is a compact operator and the sequence in the parentheses is bounded, the sequence on the right hand side converges for some subsequence, which we relabel as the original one. Therefore, the left hand side of (19) also converges to some u_1 of norm 1. Passing to the limit in (19), we get that

$$u_1 = (-\Delta)^{-1}\left(u_1\left(\widehat{r_1} - a_{12}\overline{u_2}\right)\right).$$

Therefore,

$$u_1 = \left(-\Delta + p\left(r_1\right)\right)^{-1}\left(\widehat{r_1} - a_{12}\overline{u_2} + p\left(r_1\right)\right)u_1.$$

By Lemma 4.2, it follows that $\widehat{r_1} = \overline{r_1}$, contrary to the assumption. □

Let $\Sigma = (\mathbb{S}\backslash\Sigma_0) \cup (\overline{r_1},(0,\overline{u_2},\overline{u_3}))$. Σ is a closed subset of $\mathbb{S}$ by Theorem 4.5.

Theorem 4.6. *Assume that* $a_{22} = a_{33} = 1$ *and the conditions of Lemma 4.2 hold. Then an unbounded global bifurcation* $\overline{\Sigma} \subset \Sigma$ *of positive solutions of* (3) *occurs at* $(\overline{r_1}, (0, \overline{u_2}, \overline{u_3}))$.

Proof. Let $\overline{\Sigma}$ be the component of Σ containing $(\overline{r_1}, (0, \overline{u_2}, \overline{u_3}))$. Assume that $\overline{\Sigma}$ is bounded. Then there exists $\mu > \overline{r_1}$ such that $\overline{\Sigma} \subset [0, \mu] \times H \cap \mathbb{T}$ and $\overline{\Sigma} \cap \{\mu\} \times \mathbb{T}_\mu = \varnothing$. Let $\mathbf{X} = \mathbb{S} \cap [0, \mu] \times H$, $\mathbf{X}$ is obviously a compact topological space. Let $\mathbf{Y} = \overline{\Sigma} \cup \Sigma_0 \cap \mathbf{X}$, $\mathbf{Z} = \mathbb{S} \cap (\{\mu\} \times H \cup \{0\} \times H) \setminus \mathbf{Y}$. Then $\mathbf{Y}$ and $\mathbf{Z}$ are nonempty, disjoint, closed subsets of $\mathbf{X}$. By Whyburn's Lemma (see [24]) there exist two compact sets $\mathbf{V}$ and $\mathbf{W}$, such that $\mathbf{Y} \subset \mathbf{V}$, $\mathbf{Z} \subset \mathbf{W}$, $\mathbf{V} \cap \mathbf{W} = \varnothing$, $\mathbf{V} \cup \mathbf{W} = \mathbf{X}$. This implies that there exists an open set $\mathbf{U}$ in $[0, \mu] \times H$, such that $\mathbf{Y} \subset \mathbf{V} \subset \mathbf{U}$, $\mathbf{U} \cap \mathbf{W} = \varnothing$ and $\partial \mathbf{U} \cap \mathbb{S} = \varnothing$. Therefore, $index_H\left(A\left(r_1, \cdot\right), \mathbf{U}_{r_1}\right)$ is well defined for all $r_1 \in [0, \mu]$ and it is constant (with respect to r) by the homotopy invariance principle (see Theorem 11.3 [1]). On the other hand, $index_H\left(A\left(\mu, \cdot\right), \mathbf{U}_\mu\right) = index_H\left(A\left(\mu, \cdot\right), (0, \overline{u_2}, \overline{u_3})\right) = 0$, since $\mu > \overline{r_1}$. Therefore, $index_H\left(A\left(0, \cdot\right), \mathbf{U}_0\right) = 0$. Since $index_H\left(A\left(0, \cdot\right), (0, \overline{u_2}, \overline{u_3})\right) = \pm 1$, it follows that $\overline{\Sigma} \cap \{0\} \times H \neq \varnothing$. Let $(0, (u_1, u_2, u_3)) \in \overline{\Sigma} \cap \{0\} \times H$. If $u_1 \not\equiv 0$, then (14) implies that

$$\begin{cases} -\Delta u_1 = -u_1\left(a_{11} u_1 + a_{12} u_2\right) & \text{in } \Omega, \\ u_1 = 0 & \text{on } \partial\Omega \end{cases}$$

which contradicts the maximum principle. Hence $u_1 \equiv 0$. Therefore, (u_2, u_3) is a solution of (8), different from $(\overline{u_2}, \overline{u_3})$. So at least one of the components must be zero. It cannot be u_3, since $\overline{\Sigma}$ is a continuum and positive u_3 components are bounded away from zero (see Lemma 3.1). So $u_2 \equiv 0$ and $u_3 = \theta_{r_3}$. Since $\overline{\Sigma} \cap \{0\} \times H = \{(0, (0, 0, \theta_{r_3}))\}$, $\overline{\Sigma}$ must contain the whole unbounded continuum $\{(r, (0, 0, \theta_{r_3})),\ r \geq 0\}$, contrary to the assumption. □

5. Concluding remarks

In this work, we have investigated a reaction-diffusion system describing three-species food chain model with zero Dirichlet boundary conditions. We assumed conditions under which the predator-prey subsystem ($u_1 \equiv 0$) has a unique positive solution $(\overline{u_2}, \overline{u_3})$. We obtained a connected set of positive solution of (3) bifurcating from the branch $\{(r_1, (0, \overline{u_2}, \overline{u_3})) : r_1 > 0\}$. That is, if the predator-prey subsystem has a unique co-existence state, then prey, mid-level predator and top predator can co-exist provided the prey birth rate is sufficiently high. If we regard r_2 or r_3 as bifurcation parameter, it is difficult to study the bifurcation solutions of (3) which are relative to $(0, \overline{u_2}, \overline{u_3})$. Since we cannot establish the corresponding spectral radius results such as Lemma 4.2 when regarding r_2 or r_3 as bifurcation parameter. Therefore it is impossible to study the corresponding fixed point index and bifurcation point.

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