Bull. Korean Math. Soc. **57** (2020), No. 3, pp. 649–660
https://doi.org/10.4134/BKMS.b190402
pISSN: 1015-8634 / eISSN: 2234-3016

THE 3D BOUSSINESQ EQUATIONS WITH REGULARITY IN THE HORIZONTAL COMPONENT OF THE VELOCITY

Qiao Liu

Abstract. This paper proves a new regularity criterion for solutions to the Cauchy problem of the 3D Boussinesq equations via one directional derivative of the horizontal component of the velocity field (i.e., $(\partial_i u_1; \partial_j u_2; 0)$ where $i, j \in \{1, 2, 3\}$) in the framework of the anisotropic Lebesgue spaces. More precisely, for $0 < T < \infty$, if

$$\int_0^T \Big(\Big\| \|\partial_i u_1(t)\|_{L^\alpha_{x_i}} \Big\|^\gamma_{L^\beta_{x_{\hat{i}} x_{\tilde{i}}}} + \Big\| \|\partial_j u_2(t)\|_{L^\alpha_{x_j}} \Big\|^\gamma_{L^\beta_{x_{\hat{j}} x_{\tilde{j}}}} \Big) \mathrm{d}t < \infty,$$

where $\frac{2}{\gamma} + \frac{1}{\alpha} + \frac{2}{\beta} = m \in [1, \frac{3}{2})$ and $\frac{3}{m} \le \alpha \le \beta < \frac{1}{m-1}$, then the corresponding solution (u, θ) to the 3D Boussinesq equations is regular on $[0, T]$. Here, $(i, \hat{i}, \tilde{i})$ and $(j, \hat{j}, \tilde{j})$ belong to the permutation group on the set $\mathbb{S}_3 := \{1, 2, 3\}$. This result reveals that the horizontal component of the velocity field plays a dominant role in regularity theory of the Boussinesq equations.

1. Introduction

The paper is concerned with regularity of solutions to the Cauchy problem of the following 3D Boussinesq equations

$$\begin{cases} u_t + u \cdot \nabla u = \nu \Delta u - \nabla P + \theta e_3, & x \in \mathbb{R}^3,\ t > 0, \\ \theta_t + u \cdot \nabla \theta = \eta \Delta \theta, & x \in \mathbb{R}^3,\ t > 0, \\ \nabla \cdot u = 0, & x \in \mathbb{R}^3,\ t > 0, \\ u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x), & x \in \mathbb{R}^3. \end{cases} \tag{1.1}$$

Here, u is the fluid velocity, θ is the scalar temperature and P is the pressure. u_0, θ_0 with $\nabla \cdot u_0 = 0$ in the sense of distribution are given initial data. $e_3 = (0, 0, 1)$. ν and η are two positive constants. Since the concrete values of ν and η play no role in our discussion, for simplicity, we set $\nu = \eta = 1$.

Received April 16, 2019; Revised September 24, 2019; Accepted November 6, 2019.
2010 *Mathematics Subject Classification.* 35Q355, 35B65, 76D03.
Key words and phrases. Boussinesq equations, regularity criterion, horizontal component, anisotropic Lebesgue spaces.
This work is partially supported by the National Natural Science Foundation of China (11401202).

The Boussinesq equations (1.1) is a simple model widely used in the modeling of oceanic and atmospheric motions and play an important role in the atmospheric sciences (see, e.g., [15]). It has received significant attention in mathematical fluid dynamics due to its connection to three-dimensional incompressible flows. When θ_0 is identically zero (or some constant), the equations (1.1) reduces to the well-known Navier–Stokes equations. It is currently unknown whether solutions of the initial value problem of the 3D Navier–Stokes equations or the 3D MHD equations can develop finite time singularities even if the initial data is sufficiently smooth. The global regularity issue has been thoroughly investigated for the 3D Navier–Stokes equations and many important regularity criteria have been established. The well-known Prodi-Serrin condition (see [8,18,22]) states that if a weak solution u to the 3D NS equations is in the class of $u \in L^r(0,T;L^s(\mathbb{R}^3))$ with $\frac{2}{r}+\frac{3}{s} \le 1, 3 < s \le \infty$, then u is regular on $(0,T]$. Beirão da Veiga [4] established another Prodi-Serrin type criterion on gradient of the velocity, i.e., $\nabla u \in L^r(0,T;L^s(\mathbb{R}^3))$ with $\frac{2}{r}+\frac{3}{s} \le 2, \frac{3}{2} < s \le \infty$. For more regularity results on solutions to the 3D NS equations subject to periodic boundary conditions or in the whole space via one component of the velocity field or via some components of the gradient of velocity, we refer the readers to see [2,3,5–7,9,16,17,19,26,27] and the references therein. Here we would like to mention the paper of Zhou-Pokorný [27], the authors established that the local smooth solution u to the 3D NS equations can be continued past any time $T > 0$ provided that there holds

$$\int_0^T \|u_3(t)\|_{L^p}^q \mathrm{d}t < +\infty \quad \text{with } \frac{2}{q}+\frac{3}{p} \le \frac{3}{4}+\frac{1}{2p} \text{ and } \frac{10}{3} < p < \infty,$$

or

$$\int_0^T \|\nabla u_3(t)\|_{L^p}^q \mathrm{d}t < +\infty \quad \text{with } \frac{2}{q}+\frac{3}{p} \le \begin{cases} \frac{19}{12}+\frac{1}{2p}, & p \in (\frac{30}{19}, 3], \\ \frac{3}{2}+\frac{3}{4p}, & p \in (3,\infty]. \end{cases}$$

Later, Qian [19] proved the regularity criteria in terms of only one of nine components of the gradient of velocity field in the framework of anisotropic Lebesgue spaces, precisely, by using the method introduced in papers [5,6], the author established that the local smooth solution u of the 3D NS equations satisfies

$$\int_0^T \Big\| \|\partial_i u_j(t)\|_{L^\alpha_{x_i}} \Big\|^q_{L^\beta_{x_j x_k}} \mathrm{d}t < +\infty$$

where $\frac{2}{q}+\frac{1}{\alpha}+\frac{2}{\beta} \le \frac{2\alpha\beta+5\beta+\alpha}{4\alpha\beta}$, $1 \le \alpha \le \beta$ and $\frac{7\alpha}{2\alpha+1} < \beta < \infty$, or

$$\int_0^T \Big\| \|\partial_j u_j(t)\|_{L^\alpha_{x_i}} \Big\|^q_{L^\beta_{x_j x_k}} \mathrm{d}t < +\infty$$

where $\frac{2}{q}+\frac{1}{\alpha}+\frac{2}{\beta} \le \frac{3\alpha\beta+4\beta+2\alpha}{4\alpha\beta}$, $1 \le \alpha \le \beta$ and $2 < \beta \le \infty$,
with $i,j = 1,2,3$ and $i \ne j$, then u is smooth up to time T. Here, we notice

that the repeated subscripts like j, j do not mean summation in the above inequalities.

Since (1.1) contains the NS equations as a subsystem, the global regularity with large initial data is still a challenging open problem although some efforts have been made for small initial data or axi-symmetric data [1,15]. The development of regularity criterion is also of major importance for both theoretical and practical purposes. Inspired by regularity results of the NS equations and the MHD equations, regularity criteria via the velocity field of weak solutions to the Boussinesq equations in different spaces have been obtained in papers [3,10,11,13,20,21,23,25] and the reference therein. We also refer to [12,14,24] for more regularity criteria for equations (1.1) via the pressure.

Motivated by the papers cited above, we shall study regularity criterion for solutions to the Cauchy problems of the 3D Boussinesq equations (1.1) via one directional derivative of derivative of the horizontal component of the velocity field (i.e., $(\partial_i u_1; \partial_j u_2; 0)$ where $i, j \in \{1, 2, 3\}$) in framework of the anisotropic Lebesgue spaces in this paper. Before stating our main result, let us recall the definition of weak solution to (1.1) and the definition of the anisotropic Lebesgue spaces.

Definition 1.1 (Weak solution)**.** Let $(u_0, \theta_0) \in L^2(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$, and $T > 0$. A function pair (u, θ) satisfying $(u, \theta) \in C_w(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3))$ is said to be a weak solution of (1.1) if it solves $(1.1)_{1,2,3}$ in the sense of distributions, and for all $0 < t < T$, the following energy inequality holds

$$\begin{aligned}&\|u(\cdot,t)\|_{L^2}^2+\|\theta(\cdot,t)\|_{L^2}^2+2\int_0^t(\|\nabla u(\cdot,\tau)\|_{L^2}^2+\|\nabla\theta(\cdot,\tau)\|_{L^2}^2)\mathrm{d}\tau\\ \le\ &\|u_0\|_{L^2}^2+\|\theta_0\|_{L^2}^2+2\int_0^t\int_{\mathbb{R}^3}\theta u_3\mathrm{d}x\mathrm{d}\tau.\end{aligned}$$

Definition 1.2. For $1 \le \alpha, \beta < \infty$, we say that a function f belongs to $L^\beta((\mathbb{R}^2_{x_1x_2}); L^\alpha(\mathbb{R}_{x_3})$ if f is measurable on $\mathbb{R}^3$ and the following norm is finite:

$$\Big\|\|f\|_{L^\alpha_{x_3}}\Big\|_{L^\beta_{x_1x_2}} := \left(\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}}|f(x_1,x_2,x_3)|^\alpha\mathrm{d}x_3\right)^{\frac{\beta}{\alpha}}\mathrm{d}x_1\mathrm{d}x_2\right)^{\frac{1}{\beta}}.$$

Now, our main result reads as:

Theorem 1.3. *Assume the initial data $(u_0, \theta_0) \in H^1(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$. Let $T > 0$ be given, and (u, θ) be the corresponding weak solution of the 3D Boyssinesq equations (1.1) on $(0, T)$. Then the solution (u, θ) remains smooth up to time $t = T$, provide that the following condition holds*:

$$\int_0^T\Big(\Big\|\|\partial_i u_1(t)\|_{L^\alpha_{x_i}}\Big\|^\gamma_{L^\beta_{x_{\hat{i}}x_{\tilde{i}}}}+\Big\|\|\partial_j u_2(t)\|_{L^\alpha_{x_j}}\Big\|^\gamma_{L^\beta_{x_{\hat{j}}x_{\tilde{j}}}}\Big)dt<\infty$$

$$\text{with } \frac{2}{\gamma}+\frac{1}{\alpha}+\frac{2}{\beta}=m\in[1,\frac{3}{2}) \text{ and } \frac{3}{m}\leq\alpha\leq\beta<\frac{1}{m-1}. \tag{1.2}$$

Here, $(i,\hat{i},\tilde{i})$ and $(j,\hat{j},\tilde{j})$ belong to the permutation group on the set $\mathbb{S}_3 := \{1,2,3\}$.

The proof of the regularity criterion (1.2) is provided in the next section. Throughout the paper, the directional derivatives of a function f are denoted by $\partial_i f = \frac{\partial f}{\partial x_i}(i=1,2,3)$, the $L^p(\mathbb{R}^3)$-norm of a function f is denoted by $\|f\|_{L^p}$. C is the generic positive constant which may depend on the norms of the initial data and may change from line to line.

2. Proof of Theorem 1.3

In this section, we shall prove our main Theorem 1.3. We first integrating $(1.1)_2$ over $\mathbb{R}^3$, and noticing the divergence-free condition $(1.1)_3$, it follows that

$$\frac{d}{dt}\int_{\mathbb{R}^3}\theta(x,t)\mathrm{d}x=\int_{\mathbb{R}^3}\Delta\theta\mathrm{d}x=0,$$

from which

$$\int_{\mathbb{R}^3}\theta(x,t)\mathrm{d}x=\int_{\mathbb{R}^3}\theta_0(x)\mathrm{d}x.$$

This provides the L^1-estimate of θ. Applying the standard maximum principle type argument to $(1.1)_2$, one gets the L^∞-estimate, i.e.,

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}^3}\theta(x,t)\leq\operatorname*{ess\,sup}_{x\in\mathbb{R}^3}\theta_0(x) \quad \text{for all } 0<t\leq T.$$

Thus we have the following global a priori bound for θ

$$\|\theta(\cdot,t)\|_{L^p}\leq\|\theta_0\|_{L^p} \quad \forall t\in[0,T],\forall 1\leq p\leq\infty. \tag{2.1}$$

Next, multiplying both sides of $(1.1)_1$ by u, and using the condition $(1.1)_3$, one gets after integration by parts,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(\cdot,t)\|_{L^2}^2+\|\nabla u(\cdot,t)\|_{L^2}^2&=\int_{\mathbb{R}^3}(\theta e_3\cdot u)(x,t)\mathrm{d}x\\&\leq\|\theta(\cdot,t)\|_{L^2}\|u(\cdot,t)\|_{L^2}\\&\leq\|\theta_0\|_{L^2}\|u(\cdot,t)\|_{L^2},\end{aligned}$$

which together with Gronwall's inequality yields that

$$u\in L^\infty(0,T;L^2(\mathbb{R}^3))\cap L^2(0,T;H^1(\mathbb{R}^3)). \tag{2.2}$$

Multiplying $(1.1)_1$ by Δu, and $(1.1)_2$ by $\Delta\theta$, respectively, and then integrating over $\mathbb{R}^3$, one obtains after integrating by parts

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\nabla u(\cdot,t)\|_{L^2}^2+\|\nabla\theta(\cdot,t)\|_{L^2}^2)+\|\Delta u(\cdot,t)\|_{L^2}^2+\|\Delta\theta(\cdot,t)\|_{L^2}^2\\=&\int_{\mathbb{R}^3}(u\cdot\nabla)u\cdot\Delta u\mathrm{d}x-\int_{\mathbb{R}^3}\theta e_3\cdot\Delta u\mathrm{d}x+\int_{\mathbb{R}^3}(u\cdot\nabla)\theta\cdot\Delta\theta\mathrm{d}x\end{aligned}$$

$$:= I_1 + I_2 + I_3, \tag{2.3}$$

where we have used that $(1.1)_3$ implies the identity $\int_{\mathbb{R}^3} \nabla P \cdot \Delta u \mathrm{d}x = 0$. In what follows, we shall estimate $I_i (i = 1, 2, 3)$ term by term. By using Hölder's inequality, it follows that

$$I_2 = -\sum_{i=1}^{3} \int_{\mathbb{R}^3} \partial_i \theta \partial_i u_3 \mathrm{d}x \le \|\nabla\theta\|_{L^2} \|\nabla u\|_{L^2} \le C(\|\nabla\theta\|_{L^2}^2 + \|\nabla u\|_{L^2}^2)$$

and

$$\begin{aligned} I_3 &\le C\|u\|_{L^6} \|\nabla\theta\|_{L^3} \|\Delta\theta\|_{L^2} \le C\|\nabla\theta_0\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2} \|\Delta\theta\|_{L^2}^{\frac{3}{2}} \\ &\le \frac{1}{4}\|\Delta\theta\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4, \end{aligned}$$

where we have used the following inequality

$$\|\nabla\theta\|_{L^3} \le C\|\theta\|_{L^6}^{\frac{1}{2}} \|\Delta\theta\|_{L^2}^{\frac{1}{2}} \le C\|\theta_0\|_{L^6}^{\frac{1}{2}} \|\Delta\theta\|_{L^2}^{\frac{1}{2}} \le C\|\nabla\theta_0\|_{L^2}^{\frac{1}{2}} \|\Delta\theta\|_{L^2}^{\frac{1}{2}},$$

due to the interpolation inequality and the energy inequality in Definition 1.1. For the term I_1, noticing that it holds that

$$\begin{aligned} I_1 &= \sum_{i,j,k=1}^{3} \int_{\mathbb{R}^3} u_i \partial_i u_j \partial_k^2 u_j \mathrm{d}x \\ &= \sum_{i=1}^{2} \sum_{j,k=1}^{3} \int_{\mathbb{R}^3} u_i \partial_i u_j \partial_k^2 u_j \mathrm{d}x + \sum_{j,k=1}^{2} \int_{\mathbb{R}^3} u_3 \partial_3 u_j \partial_k^2 u_j \mathrm{d}x \\ &\quad + \sum_{k=1}^{2} \int_{\mathbb{R}^3} u_3 \partial_3 u_3 \partial_k^2 u_3 \mathrm{d}x + \sum_{j=1}^{3} \int_{\mathbb{R}^3} u_3 \partial_3 u_j \partial_3^2 u_j \mathrm{d}x \\ &= \underbrace{\sum_{i=1}^{2} \sum_{j,k=1}^{3} \int_{\mathbb{R}^3} u_i \partial_i u_j \partial_k^2 u_j \mathrm{d}x}_{I_{11}} \\ &\quad \underbrace{- \sum_{j,k=1}^{2} \Big(\int_{\mathbb{R}^3} \partial_k u_3 \partial_3 u_j \partial_k u_j \mathrm{d}x + \int_{\mathbb{R}^3} u_3 \partial_3 \partial_k u_j \partial_k u_j \mathrm{d}x \Big)}_{I_{12}} \\ &\quad \underbrace{- \sum_{k=1}^{2} \Big(\int_{\mathbb{R}^3} \partial_k u_3 \partial_3 u_3 \partial_k u_3 \mathrm{d}x + \int_{\mathbb{R}^3} u_3 \partial_3 \partial_k u_3 \partial_k u_3 \mathrm{d}x \Big) + \sum_{j=1}^{3} \int_{\mathbb{R}^3} u_3 \partial_3 u_j \partial_3^2 u_j \mathrm{d}x}_{I_{13}}, \end{aligned}$$

and

$$I_{11} \le C\Big(\int_{\mathbb{R}^3} |u_1| |\nabla u| |\nabla^2 u| \mathrm{d}x + \int_{\mathbb{R}^3} |u_2| |\nabla u| |\nabla^2 u| \mathrm{d}x \Big);$$

$$
\begin{aligned}
I_{12} =& -\sum_{j,k=1}^{2}\int_{\mathbb{R}^3} u_j\partial_3\partial_k u_3\partial_k u_j \mathrm{d}x - \sum_{j,k=1}^{2}\int_{\mathbb{R}^3} u_j\partial_k u_3\partial_3\partial_k u_j \mathrm{d}x \\
& -\frac{1}{2}\sum_{j,k=1}^{2}\int_{\mathbb{R}^3} \partial_3 u_3\partial_k u_j\partial_k u_j \mathrm{d}x \\
=& -\sum_{j,k=1}^{2}\int_{\mathbb{R}^3} u_j\partial_3\partial_k u_3\partial_k u_j \mathrm{d}x - \sum_{j,k=1}^{2}\int_{\mathbb{R}^3} u_j\partial_k u_3\partial_3\partial_k u_j \mathrm{d}x \\
& +\frac{1}{2}\sum_{i,j,k=1}^{2}\int_{\mathbb{R}^3} \partial_i u_i\partial_k u_j\partial_k u_j \mathrm{d}x \\
=& -\sum_{j,k=1}^{2}\int_{\mathbb{R}^3} u_j\partial_3\partial_k u_3\partial_k u_j \mathrm{d}x - \sum_{j,k=1}^{2}\int_{\mathbb{R}^3} u_j\partial_k u_3\partial_3\partial_k u_j \mathrm{d}x \\
& -\sum_{i,j,k=1}^{2}\int_{\mathbb{R}^3} u_i\partial_i\partial_k u_j\partial_k u_j \mathrm{d}x \\
\leq& C\Big(\int_{\mathbb{R}^3} |u_1||\nabla u||\nabla^2 u|\mathrm{d}x + \int_{\mathbb{R}^3} |u_2||\nabla u||\nabla^2 u|\mathrm{d}x\Big); \\
2I_{13} =& \sum_{k=1}^{2}\int_{\mathbb{R}^3} \partial_k u_3\partial_3 u_3\partial_k u_3 \mathrm{d}x + \sum_{j=1}^{3}\int_{\mathbb{R}^3} \partial_3 u_3\partial_3 u_j\partial_3 u_j \mathrm{d}x \\
=& -\sum_{i,k=1}^{2}\int_{\mathbb{R}^3} \partial_k u_3\partial_i u_i\partial_k u_3 \mathrm{d}x - \sum_{i,j=1}^{3}\int_{\mathbb{R}^3} \partial_i u_i\partial_3 u_j\partial_3 u_j \mathrm{d}x \\
=& 2\sum_{i,k=1}^{2}\int_{\mathbb{R}^3} u_i\partial_i\partial_k u_3\partial_k u_3 \mathrm{d}x + 2\sum_{i,j=1}^{3}\int_{\mathbb{R}^3} u_i\partial_i\partial_3 u_j\partial_3 u_j \mathrm{d}x \\
\leq& C\Big(\int_{\mathbb{R}^3} |u_1||\nabla u||\nabla^2 u|\mathrm{d}x + \int_{\mathbb{R}^3} |u_2||\nabla u||\nabla^2 u|\mathrm{d}x\Big),
\end{aligned}
$$

where we have used the fact that $(1.1)_3$ yields $\partial_3 u_3 = -\sum_{i=1}^{2}\partial_i u_i$. Thus one obtains

$$
\begin{aligned}
I_1 \leq& C\Big(\int_{\mathbb{R}^3} |u_1||\nabla u||\nabla^2 u|\mathrm{d}x + \int_{\mathbb{R}^3} |u_2||\nabla u||\nabla^2 u|\mathrm{d}x\Big) \\
\leq& \frac{1}{8}\|\nabla^2 u\|_{L^2}^2 + C\Big(\int_{\mathbb{R}^3} |u_1|^2|\nabla u|^2\mathrm{d}x + \int_{\mathbb{R}^3} |u_2|^2|\nabla u|^2\mathrm{d}x\Big).
\end{aligned}
$$

Let us now turn to estimate the term $\int_{\mathbb{R}^3}|u_1|^2|\nabla u|^2\mathrm{d}x$, the straight calculus yields that for $1 < r \leq \infty$

$$
\int_{\mathbb{R}^3} |u_1|^2|\nabla u|^2\mathrm{d}x
$$

$$
\begin{aligned}
&\leq \int_{\mathbb{R}^2} \left(\max_{x_i \in \mathbb{R}} |u_1|^2 \cdot \int_{\mathbb{R}} |\nabla u|^2 \mathrm{d}x_i \right) \mathrm{d}x_{\hat{i}} \mathrm{d}x_{\tilde{i}} \\
&\leq C \left(\int_{\mathbb{R}^2} \max_{x_i \in \mathbb{R}} |u_1|^{2r} \mathrm{d}x_{\hat{i}} \mathrm{d}x_{\tilde{i}} \right)^{\frac{1}{r}} \left(\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}} |\nabla u|^2 \mathrm{d}x_i \right)^{\frac{r}{r-1}} \mathrm{d}x_{\hat{i}} \mathrm{d}x_{\tilde{i}} \right)^{\frac{r-1}{r}} \\
&\leq C \left(\int_{\mathbb{R}^3} |u_1|^{2r-1} |\partial_i u_1| \mathrm{d}x \right)^{\frac{1}{r}} \left(\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}} |\nabla u|^2 \mathrm{d}x_i \right)^{\frac{r}{r-1}} \mathrm{d}x_{\hat{i}} \mathrm{d}x_{\tilde{i}} \right)^{\frac{r-1}{r}}, \qquad (2.4)
\end{aligned}
$$

where we have used the following fact

$$
|f(x)|^{2r} \leq C \int_{-\infty}^{x} |f(\tau)|^{2r-1} |f'(\tau)| \mathrm{d}\tau \leq C \int_{\mathbb{R}} |f(\tau)|^{2r-1} |f'(\tau)| \mathrm{d}\tau.
$$

By using Hölder's inequality and interpolation inequality, one gets

$$
\begin{aligned}
&\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}} |u_1|^{2r-1} |\partial_i u_1| \mathrm{d}x_i \right) \mathrm{d}x_{\hat{i}} \mathrm{d}x_{\tilde{i}} \\
&\leq C \int_{\mathbb{R}^2} \|\partial_i u_1\|_{L^{\alpha}_{x_i}} \|u_1\|^{2r-1}_{L_{x_i}^{\frac{(2r-1)\alpha}{\alpha-1}}} \mathrm{d}x_{\hat{i}} \mathrm{d}x_{\tilde{i}} \\
&\leq C \int_{\mathbb{R}^2} \|\partial_i u_1\|_{L^{\alpha}_{x_i}} \|\partial_i u_1\|^{(2r-1)\theta}_{L^{\alpha}_{x_i}} \|u_1\|^{(2r-1)(1-\theta)}_{L^{\xi}_{x_i}} \mathrm{d}x_{\hat{i}} \mathrm{d}x_{\tilde{i}} \\
&\leq C \left\| \|\partial_i u_1\|_{L^{\alpha}_{x_i}} \right\|_{L^{\beta}_{x_{\hat{i}} x_{\tilde{i}}}} \left\| \|\partial_i u_1\|^{(2r-1)\theta}_{L^{\alpha}_{x_i}} \|u_1\|^{(2r-1)(1-\theta)}_{L^{\xi}_{x_i}} \right\|_{L^{\frac{\beta}{\beta-1}}_{x_{\hat{i}} x_{\tilde{i}}}} \\
&\leq C \left\| \|\partial_i u_1\|_{L^{\alpha}_{x_i}} \right\|_{L^{\beta}_{x_{\hat{i}} x_{\tilde{i}}}} \left\| \|\partial_i u_1\|_{L^{\alpha}_{x_i}} \right\|^{(2r-1)\theta}_{L^{(2r-1)\theta t}_{x_{\hat{i}} x_{\tilde{i}}}} \left\| \|u_1\|_{L^{\xi}_{x_i}} \right\|^{(2r-1)(1-\theta)}_{L^{(2r-1)(1-\theta)a}_{x_{\hat{i}} x_{\tilde{i}}}}, \qquad (2.5)
\end{aligned}
$$

where $(i, \hat{i}, \tilde{i})$ belongs to the permutation group on the set $\mathbb{S}_3 := \{1, 2, 3\}$, $\alpha, \beta, \xi, a, t \in [1, \infty]$ and $\theta \in [0, 1]$ satisfy

$$
\frac{1}{a} + \frac{1}{t} = \frac{\beta - 1}{\beta}, \qquad (2.6)
$$

and

$$
\frac{1}{(2r-1)\alpha} + \frac{\theta}{\alpha} = \frac{1-\theta}{\xi(\alpha-1)}. \qquad (2.7)
$$

On the other hand, by using Minkowski inequality, Hölder's inequality and interpolation inequality, one obtains

$$
\begin{aligned}
&\left(\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}} |\nabla u|^2 \mathrm{d}x_i \right)^{\frac{r}{r-1}} \mathrm{d}x_{\hat{i}} \mathrm{d}x_{\tilde{i}} \right)^{\frac{r-1}{r}} \\
&\leq \int_{\mathbb{R}} \left(\int_{\mathbb{R}^2} |\nabla u|^{\frac{2r}{r-1}} \mathrm{d}x_{\hat{i}} \mathrm{d}x_{\tilde{i}} \right)^{\frac{r-1}{r}} \mathrm{d}x_i \\
&\leq \int_{\mathbb{R}} \|\nabla u\|^{\frac{2(r-1)}{r}}_{L^2_{x_{\hat{i}} x_{\tilde{i}}}} \|(\partial_{\hat{i}}, \partial_{\tilde{i}}) \nabla u\|^{\frac{2}{r}}_{L^2_{x_{\hat{i}} x_{\tilde{i}}}} \mathrm{d}x_i
\end{aligned}
$$

$$\leq \|\nabla u\|_{L^2}^{\frac{2(r-1)}{r}} \|(\partial_{\hat{i}}, \partial_{\tilde{i}})\nabla u\|_{L^2}^{\frac{2}{r}} \leq \|\nabla u\|_{L^2}^{\frac{2(r-1)}{r}} \|\nabla^2 u\|_{L^2}^{\frac{2}{r}}. \tag{2.8}$$

Hence, inserting (2.8) and (2.5) into (2.4), it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^3} |u_1|^2 |\nabla u|^2 \mathrm{d}x \\
&\leq C \Big\| \|\partial_i u_1\|_{L^\alpha_{x_i}} \Big\|_{L^\beta_{x_{\hat{i}} x_{\tilde{i}}}}^{\frac{1}{r}} \Big\| \|\partial_i u_1\|_{L^\alpha_{x_i}} \Big\|_{L^{\theta(2r-1)t}_{x_{\hat{i}} x_{\tilde{i}}}}^{\frac{\theta(2r-1)}{r}} \Big\| \|u_1\|_{L^\xi_{x_i}} \Big\|_{L^{(1-\theta)(2r-1)a}_{x_{\hat{i}} x_{\tilde{i}}}}^{\frac{(1-\theta)(2r-1)}{r}} \|\nabla u\|_{L^2}^{\frac{2(r-1)}{r}} \|\nabla^2 u\|_{L^2}^{\frac{2}{r}},
\end{aligned} \tag{2.9}$$

where $1 < r \leq \infty$, $\alpha, \beta, \xi, a, t \in [1, \infty]$ and $\theta \in [0, 1]$ satisfy (2.6) and (2.7). By selecting

$$a = \frac{\xi}{(2r-1)(1-\theta)} \text{ and } t = \frac{\beta}{(2r-1)\theta}, \tag{2.10}$$

then a and t satisfy (2.6). From (2.10) and (2.9), it is easy to see that

$$\begin{aligned}
\int_{\mathbb{R}^3} |u_1|^2 |\nabla u|^2 \mathrm{d}x &\leq C \Big\| \|\partial_i u_i\|_{L^\alpha_{x_i}} \Big\|_{L^\beta_{x_{\hat{i}} x_{\tilde{i}}}}^{\frac{1+(2r-1)\theta}{r}} \|u\|_{L^\xi}^{\frac{(2r-1)(1-\theta)}{r}} \|\nabla u\|_{L^2}^{\frac{2(r-1)}{r}} \|\nabla^2 u\|_{L^2}^{\frac{2}{r}} \\
&\leq \frac{1}{8} \|\nabla^2 u\|_{L^2}^2 + C \Big\| \|\partial_i u_1\|_{L^\alpha_{x_i}} \Big\|_{L^\beta_{x_{\hat{i}} x_{\tilde{i}}}}^{\frac{1+(2r-1)\theta}{r-1}} \|u\|_{L^\xi}^{\frac{(2r-1)(1-\theta)}{r-1}} \|\nabla u\|_{L^2}^2.
\end{aligned} \tag{2.11}$$

In what follows, for $m \in [1, \frac{3}{2}]$ and $\frac{3}{m} < \alpha \leq \beta \leq \frac{1}{m-1}$, by setting

$$r = \frac{(\frac{5}{2} - m)\alpha\beta}{\alpha + \alpha\beta - \beta} = \frac{(5-2m)\alpha\beta}{2(\alpha + \alpha\beta - \beta)} \text{ and } \xi = \frac{2\alpha(2-m)}{\alpha - 1}, \tag{2.12}$$

and then setting

$$\theta = \frac{(2r-1)\alpha - \xi(\alpha - 1)}{(2r-1)(\xi(\alpha-1) + \alpha)} = \frac{\beta - \alpha}{(4-2m)\alpha\beta - \alpha + \beta} \tag{2.13}$$

we have $\theta \in [0, 1)$, and r, ξ and θ satisfy (2.7). Hence

$$\begin{aligned}
&\int_{\mathbb{R}^3} |u_1|^2 |\nabla u|^2 \mathrm{d}x \\
&\quad \leq \frac{1}{8} \|\nabla^2 u\|_{L^2}^2 + C \Big\| \|\partial_i u_1\|_{L^\alpha_{x_i}} \Big\|_{L^\beta_{x_{\hat{i}} x_{\tilde{i}}}}^{\frac{1+(2r-1)\theta}{r-1}} \|u\|_{L^\xi}^{\frac{(2r-1)(1-\theta)}{r-1}} \|\nabla u\|_{L^2}^2 \\
&\quad \leq \frac{1}{8} \|\nabla^2 u\|_{L^2}^2 + C \Big\| \|\partial_i u_1\|_{L^\alpha_{x_i}} \Big\|_{L^\beta_{x_{\hat{i}} x_{\tilde{i}}}}^{\frac{2\alpha\beta}{(3-2m)\alpha\beta - 2\alpha + 2\beta}} \|u\|_{L^\xi}^{\frac{4\alpha\beta(2-m)}{(3-2m)\alpha\beta - 2\alpha + 2\beta}} \|\nabla u\|_{L^2}^2.
\end{aligned} \tag{2.14}$$

Applying Hölder's inequality with

$$\frac{m\alpha\beta - \beta - 2\alpha}{(3-2m)\alpha\beta - 2\alpha + 2\beta} + \frac{3(1-m)\alpha\beta + 3\beta}{(3-2m)\alpha\beta - 2\alpha + 2\beta} = 1,$$

where $\frac{m\alpha\beta - \beta - 2\alpha}{(3-2m)\alpha\beta - 2\alpha + 2\beta} \in (0, 1]$ by (1.2). (2.14) becomes

$$\int_{\mathbb{R}^3} |u_1|^2 |\nabla u|^2 \mathrm{d}x \tag{2.15}$$

$$\leq \begin{cases} \frac{1}{8}\|\nabla^2 u\|_{L^2}^2 + C\left(\left\|\|\partial_i u_1\|_{L^\alpha_{x_i}}\right\|_{L^\beta_{x_{\hat{i}}x_{\tilde{i}}}}^{\frac{2\alpha\beta}{m\alpha\beta-\beta-2\alpha}} + \|u\|_{L^\xi}^{\frac{4\alpha(2-m)}{3(1-m)\alpha+3}}\right)\|\nabla u\|_{L^2}^2 \\ \qquad \text{if } \frac{m\alpha\beta-\beta-2\alpha}{(3-2m)\alpha\beta-2\alpha+2\beta} \in (0,1), \text{ i.e., } \frac{3}{m} < \alpha \leq \beta < \frac{1}{m-1}, \\ \frac{1}{8}\|\nabla^2 u\|_{L^2}^2 + C\left\|\|\partial_i u_1\|_{L^\alpha_{x_i}}\right\|_{L^\beta_{x_{\hat{i}}x_{\tilde{i}}}}^{\frac{2\alpha\beta}{m\alpha\beta-\beta-2\alpha}}\|u\|_{L^2}^{\frac{4\alpha\beta(2-m)}{m\alpha\beta-\beta-2\alpha}}\|\nabla u\|_{L^2}^2 \\ \qquad \text{if } \frac{m\alpha\beta-\beta-2\alpha}{(3-2m)\alpha\beta-2\alpha+2\beta} = 1, \text{ i.e., } \alpha = \beta = \frac{1}{m-1}. \end{cases}$$

Similar estimate still holds for $\int_{\mathbb{R}^3}|u_2|^2|\nabla u|^2\mathrm{d}x$. Hence

$$I_1 \leq \frac{1}{4}\|\Delta u\|_{L^2}^2 + \begin{cases} C\left(\left\|\|\partial_i u_1\|_{L^\alpha_{x_i}}\right\|_{L^\beta_{x_{\hat{i}}x_{\tilde{i}}}}^{\frac{2\alpha\beta}{m\alpha\beta-\beta-2\alpha}} + \left\|\|\partial_j u_2\|_{L^\alpha_{x_j}}\right\|_{L^\beta_{x_{\hat{j}}x_{\tilde{j}}}}^{\frac{2\alpha\beta}{m\alpha\beta-\beta-2\alpha}} + \|u\|_{L^\xi}^{\frac{4\alpha(2-m)}{3(1-m)\alpha+3}}\right)\|\nabla u\|_{L^2}^2 \\ \qquad \text{if } \frac{m\alpha\beta-\beta-2\alpha}{(3-2m)\alpha\beta-2\alpha+2\beta} \in (0,1), \text{ i.e., } \frac{3}{m} < \alpha \leq \beta < \frac{1}{m-1}, \\ C\left(\left\|\|\partial_i u_1\|_{L^\alpha_{x_i}}\right\|_{L^\beta_{x_{\hat{i}}x_{\tilde{i}}}}^{\frac{2\alpha\beta}{m\alpha\beta-\beta-2\alpha}} + \left\|\|\partial_j u_2\|_{L^\alpha_{x_j}}\right\|_{L^\beta_{x_{\hat{j}}x_{\tilde{j}}}}^{\frac{2\alpha\beta}{m\alpha\beta-\beta-2\alpha}}\right)\|u\|_{L^2}^{\frac{4\alpha\beta(2-m)}{m\alpha\beta-\beta-2\alpha}}\|\nabla u\|_{L^2}^2 \\ \qquad \text{if } \frac{m\alpha\beta-\beta-2\alpha}{(3-2m)\alpha\beta-2\alpha+2\beta} = 1, \text{ i.e., } \alpha = \beta = \frac{1}{m-1}, \end{cases}$$

where we have used the identity $\|\nabla^2 u\|_{L^2}^2 = \|\Delta u\|_{L^2}^2$. Inserting all estimates of $I_i (i = 1, 2, 3)$ into (2.3), it follows that

$$\mathcal{F}(t) + (\|\Delta u\|_{L^2}^2 + \|\Delta\theta\|_{L^2}^2) \leq C\begin{cases} \left(\mathcal{H}(t) + \|u\|_{L^\xi}^{\frac{4\alpha(2-m)}{3(1-m)\alpha+3}} + \|\nabla u\|_{L^2}^2 + 1\right)(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2) \\ \qquad \text{if } \frac{m\alpha\beta-\beta-2\alpha}{(3-2m)\alpha\beta-2\alpha+2\beta} \in (0,1), \text{ i.e., } \frac{3}{m} < \alpha \leq \beta < \frac{1}{m-1}, \\ \left(\mathcal{H}(t)\|u\|_{L^2}^{\frac{4\alpha\beta(2-m)}{m\alpha\beta-\beta-2\alpha}} + \|\nabla u\|_{L^2}^2 + 1\right)(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2) \\ \qquad \text{if } \frac{m\alpha\beta-\beta-2\alpha}{(3-2m)\alpha\beta-2\alpha+2\beta} = 1, \text{ i.e., } \alpha = \beta = \frac{1}{m-1}, \end{cases} \tag{2.16}$$

where $\mathcal{F}(t) = \|\nabla u(\cdot,t)\|_{L^2}^2 + \|\nabla\theta(\cdot,t)\|_{L^2}^2$ and $\mathcal{H}(t) = \left\|\|\partial_i u_1\|_{L^\alpha_{x_i}}\right\|_{L^\beta_{x_{\hat{i}}x_{\tilde{i}}}}^{\frac{2\alpha\beta}{m\alpha\beta-\beta-2\alpha}} + \left\|\|\partial_j u_2\|_{L^\alpha_{x_j}}\right\|_{L^\beta_{x_{\hat{j}}x_{\tilde{j}}}}^{\frac{2\alpha\beta}{m\alpha\beta-\beta-2\alpha}}$ with $\frac{m\alpha\beta-\beta-2\alpha}{(3-2m)\alpha\beta-2\alpha+2\beta} \in (0,1]$, i.e., $\frac{3}{m} < \alpha \leq \beta \leq \frac{1}{m-1}$. Notice that from (2.2) together with the standard interpolation inequality yields that

$$u \in L^a(0,T;L^b(\mathbb{R}^3)) \quad \text{with } \frac{2}{a} + \frac{3}{b} = \frac{3}{2} \text{ and } 2 \leq b \leq 6.$$

On the other hand, it is easy to see that $2 < \xi = \frac{2\alpha(2-m)}{\alpha-1} < 6$ if $\frac{3}{m} < \alpha \le \beta < \frac{1}{m-1}$, and it holds that

$$\frac{3(1-m)\alpha+3}{2\alpha(2-m)} + \frac{3}{\xi} = \frac{3(1-m)\alpha+3}{2\alpha(2-m)} + \frac{3(\alpha-1)}{2\alpha(2-m)} = \frac{3}{2}.$$

Hence, one obtains that

$$u \in L^{\frac{4\alpha(2-m)}{3[(1-m)\alpha+1]}}(0,T;L^{\xi}(\mathbb{R}^3)), \tag{2.17}$$

if $\frac{3}{m} < \alpha \le \beta < \frac{1}{m-1}$. When $\alpha = \beta = \frac{1}{m-1}$, then $\xi = 2$, and one obtains from energy inequality in Definition 1.1 and the estimate (2.1) that

$$\|u(\cdot,t)\|_{L^2} \le C(\|u_0\|_{L^2} + \|\theta_0\|_{L^2}) \text{ for all } 0 \le t \le T. \tag{2.18}$$

Then, by applying Gronwall's inequality to (2.16) in the interval $[0,T)$, one can deduce that

$$\begin{aligned} &\sup_{0\le t\le T} \mathcal{F}(t) \\ &\le \mathcal{F}(0) \begin{cases} \exp\Big\{C\int_0^T\Big(\mathcal{G}(t)+\|u(t)\|_{L^\xi}^{\frac{4\alpha(2-m)}{3(1-m)\alpha+3}}+\|\nabla u\|_{L^2}^2+1\Big)dt\Big\} \\ \qquad \text{if } \frac{m\alpha\beta-\beta-2\alpha}{(3-2m)\alpha\beta-2\alpha+2\beta} \in (0,1), \text{ i.e., } \frac{3}{m}<\alpha\le\beta<\frac{1}{m-1}, \\ \exp\Big\{C\int_0^T\Big(\mathcal{G}(t)\|u(t)\|_{L^2}^{\frac{4\alpha\beta(2-m)}{m\alpha\beta-\beta-2\alpha}}+\|\nabla u\|_{L^2}^2+1\Big)dt\Big\} \\ \qquad \text{if } \frac{m\alpha\beta-\beta-2\alpha}{(3-2m)\alpha\beta-2\alpha+2\beta} = 1, \text{ i.e., } \alpha=\beta=\frac{1}{m-1}, \end{cases} \end{aligned}$$

where $\mathcal{F}(0) = \|u_0\|_{L^2}^2 + \|\nabla d_0\|_{L^2}^2 + 1$, which together with assumption (1.2), estimates (2.17) and (2.18) yields that

$$\sup_{0\le t\le T} (\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2) < +\infty. \tag{2.19}$$

Then by using the standard arguments of the continuation of local solutions, it is easy to conclude that the above estimate (2.19) implies that the solution $(u(x,t),\theta(x,t))$ can be smoothly up to time T. Thus we complete the proof of Theorem 1.3. □

Acknowledgments. The author would like to acknowledge his sincere thanks to the editor and the referees for a careful reading of the manuscript and many valuable comments and suggestions.

References

[1] H. Abidi and T. Hmidi, *On the global well-posedness for Boussinesq system*, J. Differential Equations **233** (2007), no. 1, 199–220. https://doi.org/10.1016/j.jde.2006.10.008

[2] H.-O. Bae and H. J. Choe, *L^∞-bound of weak solutions to Navier-Stokes equations*, in Proceedings of the Korea-Japan Partial Differential Equations Conference (Taejon, 1996), 13 pp, Lecture Notes Ser., **39**, Seoul Nat. Univ., Seoul, 1997.

[3] ———, *A regularity criterion for the Navier-Stokes equations*, Comm. Partial Differential Equations **32** (2007), no. 7-9, 1173–1187. https://doi.org/10.1080/03605300701257500

[4] H. Beirão da Veiga, *A new regularity class for the Navier-Stokes equations in* $\mathbf{R}^n$, Chinese Ann. Math. Ser. B **16** (1995), no. 4, 407–412.

[5] C. Cao and E. S. Titi, *Global regularity criterion for the 3D Navier-Stokes equations involving one entry of the velocity gradient tensor*, Arch. Ration. Mech. Anal. **202** (2011), no. 3, 919–932. https://doi.org/10.1007/s00205-011-0439-6

[6] C. Cao and J. Wu, *Two regularity criteria for the 3D MHD equations*, J. Differential Equations **248** (2010), no. 9, 2263–2274. https://doi.org/10.1016/j.jde.2009.09.020

[7] D. Chae and H.-J. Choe, *Regularity of solutions to the Navier-Stokes equation*, Electron. J. Differential Equations **1999** (1999), No. 05, 7 pp.

[8] L. Iskauriaza, G. A. Serëgin, and V. Shverak, $L^{3,\infty}$ *solutions to the Navier–Stokes equations and backward uniqueness*, Russian Math. Surveys **58** (2003), no. 2, 211–250; translated from Uspekhi Mat. Nauk **58** (2003), no. 2(350), 3–44. https://doi.org/10.1070/RM2003v058n02ABEH000609

[9] J. Fan, S. Jiang, G. Nakamura, and Y. Zhou, *Logarithmically improved regularity criteria for the Navier-Stokes and MHD equations*, J. Math. Fluid Mech. **13** (2011), no. 4, 557–571. https://doi.org/10.1007/s00021-010-0039-5

[10] S. Gala, *On the regularity criterion of strong solutions to the 3D Boussinesq equations*, Appl. Anal. **90** (2011), no. 12, 1829–1835. https://doi.org/10.1080/00036811.2010.530261

[11] S. Gala and M. A. Ragusa, *Logarithmically improved regularity criterion for the Boussinesq equations in Besov spaces with negative indices*, Appl. Anal. **95** (2016), no. 6, 1271–1279. https://doi.org/10.1080/00036811.2015.1061122

[12] ———, *A logarithmic regularity criterion for the two-dimensional MHD equations*, J. Math. Anal. Appl. **444** (2016), no. 2, 1752–1758. https://doi.org/10.1016/j.jmaa.2016.07.001

[13] J. Geng and J. Fan, *A note on regularity criterion for the 3D Boussinesq system with zero thermal conductivity*, Appl. Math. Lett. **25** (2012), no. 1, 63–66. https://doi.org/10.1016/j.aml.2011.07.008

[14] Y. Jia, X. Zhang, and B.-Q. Dong, *Remarks on the blow-up criterion for smooth solutions of the Boussinesq equations with zero diffusion*, Commun. Pure Appl. Anal. **12** (2013), no. 2, 923–937. https://doi.org/10.3934/cpaa.2013.12.923

[15] A. Majda, *Introduction to PDEs and waves for the atmosphere and ocean*, Courant Lecture Notes in Mathematics, **9**, New York University, Courant Institute of Mathematical Sciences, New York, 2003. https://doi.org/10.1090/cln/009

[16] J. Neustupa, A. Novotný, and P. Penel, *An interior regularity of a weak solution to the Navier-Stokes equations in dependence on one component of velocity*, in Topics in mathematical fluid mechanics, 163–183, Quad. Mat., **10**, Dept. Math., Seconda Univ. Napoli, Caserta, 2002.

[17] P. Penel and M. Pokorný, *Some new regularity criteria for the Navier-Stokes equations containing gradient of the velocity*, Appl. Math. **49** (2004), no. 5, 483–493. https://doi.org/10.1023/B:APOM.0000048124.64244.7e

[18] G. Prodi, *Un teorema di unicità per le equazioni di Navier-Stokes*, Ann. Mat. Pura Appl. (4) **48** (1959), 173–182. https://doi.org/10.1007/BF02410664

[19] C. Qian, *A generalized regularity criterion for 3D Navier-Stokes equations in terms of one velocity component*, J. Differential Equations **260** (2016), no. 4, 3477–3494. https://doi.org/10.1016/j.jde.2015.10.037

[20] H. Qiu, Y. Du, and Z. Yao, *Serrin-type blow-up criteria for 3D Boussinesq equations*, Appl. Anal. **89** (2010), no. 10, 1603–1613. https://doi.org/10.1080/00036811.2010.492505

[21] ———, *Blow-up criteria for 3D Boussinesq equations in the multiplier space*, Commun. Nonlinear Sci. Numer. Simul. **16** (2011), no. 4, 1820–1824. https://doi.org/10.1016/j.cnsns.2010.08.036

[22] J. Serrin, *On the interior regularity of weak solutions of the Navier-Stokes equations*, Arch. Rational Mech. Anal. **9** (1962), 187–195. https://doi.org/10.1007/BF00253344

[23] F. Xu, Q. Zhang, and X. Zheng, *Regularity criteria of the 3D Boussinesq equations in the Morrey-Campanato space*, Acta Appl. Math. **121** (2012), 231–240. https://doi.org/10.1007/s10440-012-9705-3

[24] Z. Zhang, *A logarithmically improved regularity criterion for the 3D Boussinesq equations via the pressure*, Acta Appl. Math. **131** (2014), 213–219. https://doi.org/10.1007/s10440-013-9855-y

[25] Z. Zhang, *Global regularity criteria for the n-dimensional Boussinesq equations with fractional dissipation*, Electron. J. Differential Equations **2016** (2016), Paper No. 99, 5 pp.

[26] Y. Zhou, *On regularity criteria in terms of pressure for the Navier-Stokes equations in* $\mathbb{R}^3$, Proc. Amer. Math. Soc. **134** (2006), no. 1, 149–156. https://doi.org/10.1090/S0002-9939-05-08312-7

[27] Y. Zhou and M. Pokorný, *On the regularity of the solutions of the Navier-Stokes equations via one velocity component*, Nonlinearity **23** (2010), no. 5, 1097–1107. https://doi.org/10.1088/0951-7715/23/5/004

Qiao Liu
Key Laboratory of High Performance Computing and
Stochastic Information Processing (HPCSIP)
(Ministry of Education of China), College of Mathematics and Statistics
Hunan Normal University
Changsha, Hunan 410081, P. R. China
Email address: liuqao2005@163.com