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THE FROBENIUS PROBLEM FOR NUMERICAL SEMIGROUPS GENERATED BY THE THABIT NUMBERS OF THE FIRST, SECOND KIND BASE b AND THE CUNNINGHAM NUMBERS

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Abstract. The greatest integer that does not belong to a numerical semigroup S is called the Frobenius number of S. The Frobenius problem, which is also called the coin problem or the money changing problem, is a mathematical problem of finding the Frobenius number. In this paper, we introduce the Frobenius problem for two kinds of numerical semigroups generated by the Thabit numbers of the first kind, and the second kind base b, and by the Cunningham numbers. We provide detailed proofs for the Thabit numbers of the second kind base b and omit the proofs for the Thabit numbers of the first kind base b and Cunningham numbers.

1. Introduction

Let $\mathbb{N}$ be the set of nonnegative integers. At first, we introduce a numerical semigroup and submonoid generated by a nonempty subset.

Definition 1.1 ([21, 24])**.** A *numerical semigroup* is a subset S of $\mathbb{N}$ that is closed under addition, contains 0 and for which $\mathbb{N}\backslash S$ is finite.

Definition 1.2 ([21,24])**.** Given a nonempty subset A of a numerical semigroup $\mathbb{N}$, we will denote by $\langle A\rangle$ the submonoid of $(\mathbb{N},+)$ generated by A, that is,

$$\langle A\rangle = \{\lambda_1 a_1 + \cdots + \lambda_n a_n \mid n \in \mathbb{N}\backslash\{0\}, a_i \in A, \lambda_i \in \mathbb{N} \text{ for all } i \in \{1,\ldots,n\}\}.$$

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Also, we introduce a theorem and definition directly related to the definitions above.

Theorem 1.3 ([21, 24]). *Let* $\langle A\rangle$ *be the submonoid of* $(\mathbb{N}, +)$ *generated by a nonempty subset* A *of a numerical semigroup* $\mathbb{N}$ *as in Definition 1.2. Then* $\langle A\rangle$ *is a numerical semigroup if and only if* $\gcd(A) = 1$.

Definition 1.4 ([21,24]). If S is a numerical semigroup and $S = \langle A\rangle$, then we say that A is a *system of generators of* S. Moreover, if $S \neq \langle X\rangle$ for all $X \subsetneq A$, we say that A is a *minimal system of generators of* S.

The greatest integer that does not belong to a numerical semigroup S is called the Frobenius number of S and is denoted by $F(S)$. In other words, the Frobenius number is the largest integer that cannot be expressed as a sum $\sum_{i=1}^{n} t_i a_i$, where $t_1, t_2, \ldots, t_n$ are nonnegative integers and $a_1, a_2, \ldots, a_n$ are given positive integers such that $\gcd(a_1, a_2, \ldots, a_n) = 1$. Finding the Frobenius number is called the Frobenius problem, the coin problem or the money changing problem. The Frobenius problem is not only interesting for pure mathematicians but is also connected with graph theory in [10, 11] and the theory of computer science in [17], as introduced in [16]. There are explicit formulas for calculating the Frobenius number when only two relatively prime numbers are present [29]. Recently, semi-explicit formula [19] for the Frobenius number for three relatively prime numbers are presented. An improved semi-explicit formula was presented for this case in 2017 [31].

F. Curtis proved in [6] that the Frobenius number for three or more relatively prime numbers cannot be given by a finite set of polynomials and Ramírez-Alfonsín proved in [18] that the problem is NP-hard. Currently, only algorithmic methods for determining the general formula for the Frobenius number of a set that has three or more relatively prime numbers in [2,3] exist. Some recent studies have reported that the running time for the fastest algorithm is $O(a_1)$, with the residue table in memory in [5] and $O(na_1)$ with no additional memory requirements in [3]. In addition, research on the limiting distribution in [28] and lower bound in [1, 7] of the Frobenius number were presented. From an algebraic viewpoint, rather than finding the general formula for three or more relatively prime numbers, the formulae for special cases were found such as the Frobenius number of a set of integers in a geometric sequence in [15], a Pythagorean triples in [8] and three consecutive squares or cubes in [12]. Recently, various methods for solving the Frobenius problem for numerical semigroups have been suggested in [4, 20, 24, 25], etc. In particular, a method for computing the Apéry set and obtaining the Frobenius number using the Apéry set is an efficient tool for solving the Frobenius problem of numerical semigroups as reported in [14,24,26]. Furthermore, in recent articles presenting the Frobenius problems for Fibonacci numerical semigroups in [13], Mersenne numerical semigroups in [23], Thabit numerical semigroups in [21] and repunit numerical semigroups in [22], this method is used to obtain the Frobenius number.

The Frobenius problem in the numerical semigroups $\langle \{3 \cdot 2^{n+i} - 1 \,|\, i \in \{0, 1, \ldots\}\} \rangle$ for $n \in \{0, 1, \ldots\}$ was presented in [21]. In [21], the authors recall the Thabit number $3 \cdot 2^n - 1$ and Thabit numerical semigroups $T(n) = \langle \{3 \cdot 2^{n+i} - 1 \,|\, i \in \{0, 1, \ldots\}\} \rangle$ for a nonnegative integer n and they used the definition of the minimal system of generators for $T(n)$ as the smallest subset of $\langle \{3 \cdot 2^{n+i} - 1 \,|\, i \in \mathbb{N}\} \rangle$ that equals $T(n)$. In [21], it is proved that the minimal system of generators for $T(n)$ is $\langle \{3 \cdot 2^{n+i} - 1 \,|\, i \in \{0, 1, \ldots, n+1\}\} \rangle$. The embedding dimension is the cardinality of the minimal system of generators. By the minimality of the system $\langle \{3 \cdot 2^{n+i} - 1 \,|\, i \in \{0, 1, \ldots, n+1\}\} \rangle$ for $T(n)$, the embedding dimension for $T(n)$ is $n + 2$. For any set S and $x \in S \backslash \{0\}$, the Apéry set was defined by $Ap(S, x) = \{s \in S \,|\, s - x \notin S\}$. Let $s_i = 3 \cdot 2^{n+i} - 1$ for each nonnegative integer i. Then, the Apéry set is defined by $Ap(T(n), s_0) = \{s \in T(n) \,|\, s - s_0 \notin T(n)\}$ for s_0. In [21], $Ap(T(n), s_0)$ was described explicitly leading to a solution to the Frobenius problem. Let $R(n)$ be the set of sequences $(t_1, \ldots, t_{n+1}) \in \{0, 1, 2\}^{n+1}$ that satisfy the following conditions:

(1) $t_{n+1} \in \{0, 1\}$,
(2) If $t_j = 2$, then $t_i = 0$ for all $i < j \leq n$,
(3) If $t_n = 2$, then $t_{n+1} = 0$,
(4) If $t_n = t_{n+1} = 1$, $t_i = 0$ for all $1 \leq i < n$.

Then [21] concludes that $Ap(T(n), s_0) = \{t_1 s_1 + \cdots + t_{n+1} s_{n+1} \,|\, (t_1, \ldots, t_{n+1}) \in R(n)\}$. The Frobenius number of the numerical semigroups was presented by $F(S) = \max(Ap(S, x)) - x$ in [24] and therefore the Frobenius number of Thabit numerical semigroups is $s_n + s_{n+1} - s_0 = 9 \cdot 2^{2n} - 3 \cdot 2^n - 1$. Also, an extended result of [21] has been suggested in 2017 which dealt the numerical semigroups $\langle \{(2^k - 1) \cdot 2^{n+i} - 1 \,|\, i \in \{0, 1, \ldots\}\} \rangle$ for $n \in \{0, 1, \ldots\}$ and $2 \leq k \leq 2^n$ [9]. In other words, the coefficient 3 in Thabit numerical semigroups was extended. Also, we gave on a result of the numerical semigroups. The result is the extension of the result reported in [21], the numerical semigroup is $\langle \{(2^k + 1) \cdot 2^{n+i} - (2^k - 1) \,|\, i \in \mathbb{N}\} \rangle$ for $n \in \mathbb{N}$ and $k \in \mathbb{N} \backslash \{0\}$ [30].

In this paper, we aim to solve the Frobenius problem for the numerical semigroups generated by the Thabit numbers of the first and second kind base b defined by $\{(b + 1) \cdot b^{n+i} - 1 \,|\, i \in \{0, 1, \ldots\}\}$ for $n \in \{0, 1, \ldots\}$ and $\{(b+1) \cdot b^{n+i} + 1 \,|\, i \in \{0, 1, \ldots\}\}$ for $n \in \{0, 1, \ldots\}$ with $b \not\equiv 1 \pmod 3$, and the Cunningham numbers defined by $\{b^{n+i} + 1 \,|\, i \in \{0, 1, \ldots\}\}$ for $n \in \{0, 1, \ldots\}$ with even positive integer b. To do this, we first determine the minimal system of generators and the Apéry set of the Thabit numerical semigroups of the second kind base b. Then, we compute the Frobenius number, genus, pseudo-Frobenius number, and type in these numerical semigroups. The major part of this paper has been motivated by [21], but the generalizations in our work require some additional tools. For example, in Theorem 3.8, we introduced an inductively defined sequence related to the number of elements in $\mathrm{Ap}(T_{b,2}(n), s_0)$. Also, we have to use the modular arithmetic more deeply because the elements of Apéry set are not sorted naturally like in the case of Thabit numerical semigroups (see Lemma 13 in [21]).

This paper is organized as follows. In Section 2, we compute the minimal system of generators and the embedding dimension for the Thabit numerical semigroups of the second kind base b. In Section 3, we propose a method for obtaining the Apéry set, the Frobenius number, and the genus for the Thabit numerical semigroups of the second kind base b. In Section 4, we present a method for obtaining a pseudo-Frobenius number, which is a type of the Thabit numerical semigroup of the second kind base b. Finally, in Section 5 and 6, we summarize the results related to the Thabit numerical semigroups of the first kind base b and the Cunningham numerical semigroups without the proofs because their proofs are similar to those of the Thabit numerical semigroup of the second kind base b. Some theorems and definitions essential to understanding this paper are provided below.

Definition 1.5. A positive integer x is a *Thabit number of the first kind base* b if $x = (b+1)\cdot b^n - 1$ for some $n, b \in \mathbb{N}$ and $b \geq 2$.

Definition 1.6. A positive integer x is a *Thabit number of the second kind base* b if $x = (b+1)\cdot b^n + 1$ for some $n, b \in \mathbb{N}$ and $b \geq 2$.

Definition 1.7. A numerical semigroup S is called a *Thabit numerical semigroup of the first kind base* b if there exist $n, b \in \mathbb{N}$ and $b \geq 2$ such that $S = \langle \{(b+1)\cdot b^{n+i} - 1 \mid i \in \mathbb{N}\} \rangle$. We will denote by $T_{b,1}(n)$ the Thabit numerical semigroup of the first kind base b $\langle \{(b+1)\cdot b^{n+i} - 1 \mid i \in \mathbb{N}\} \rangle$.

Definition 1.8. A numerical semigroup S is called a *Thabit numerical semigroup of the second kind base* b if there exist $n, b \in \mathbb{N}$, $b \geq 2$ and $b \not\equiv 1 \pmod 3$ such that $S = \langle \{(b+1)\cdot b^{n+i} + 1 \mid i \in \mathbb{N}\} \rangle$. We will denote by $T_{b,2}(n)$ the Thabit numerical semigroup of the second kind base b $\langle \{(b+1)\cdot b^{n+i} + 1 \mid i \in \mathbb{N}\} \rangle$.

Definition 1.9. We call a positive integer x a *Cunningham number* if $x = b^n + 1$ for some $n, b \in \mathbb{N}$, $2 \mid b$.

Definition 1.10. A numerical semigroup S is called a *Cunningham numerical semigroup* if there exist $n, b \in \mathbb{N}$ and $2 \mid b$ such that $S = \langle \{b^{n+i} + 1 \mid i \in \mathbb{N}\} \rangle$. We denote by $SC^+(b,n)$ the Cunningham numerical semigroup $\langle \{b^{n+i} + 1 \mid i \in \mathbb{N}\} \rangle$.

Theorem 1.11 ([24])**.** *Every numerical semigroup admits a unique minimal system of generators, which in addition is finite.*

Definition 1.12 ([21, 24])**.** The cardinality of a minimal system of generators S is called the *embedding dimension of* S and is denoted by $e(S)$.

Definition 1.13 ([21, 24])**.** We call the cardinality of $\mathbb{N}\backslash S$ the *genus of* S and denoted by $g(S)$ for a numerical semigroup S.

Definition 1.14 ([24])**.** An integer x is a pseudo-Frobenius number if $x \notin S$ and $x + s \in S$ for all $s \in S\backslash\{0\}$. The set of pseudo-Frobenius numbers of S is denoted by $PF(S)$. Also, we call its cardinality the type of S and denote it by $t(S)$.

2. The embedding dimension for $T_{b,2}(n)$

Let $T_{b,2}(n) = \langle\{(b+1)\cdot b^{n+i}+1 \mid i \in \mathbb{N}\}\rangle$ for $n, b \in \mathbb{N}$, $b \geq 2$ and $b \not\equiv 1 \pmod 3$. Then $T_{b,2}(n)$ is a submonoid of $(\mathbb{N}, +)$. Moreover we have $\{(b+1)\cdot b^n + 1, (b+1)\cdot b^{n+1}+1\} \subseteq T_{b,2}(n)$ and if we let $g = \gcd((b+1)\cdot b^n+1, (b+1)\cdot b^{n+1}+1)$, $g = \gcd((b+1)\cdot b^{n+1}+b, (b+1)\cdot b^{n+1}+1)$ and it divides $b-1$. But $(b+1)\cdot b^n+1 \equiv 3 \pmod{b-1}$ implies that g divides 3 and if $b \not\equiv 1 \pmod 3$, 3 does not divide $(b+1)\cdot b^n + 1$ and hence $T_{b,2}(n)$ is a numerical semigroup.

Lemma 2.1. *Let A be a nonempty set of positive integers, $n, b \in \mathbb{N}, b \geq 2, b \not\equiv 1 \pmod 3$ and $M = \langle A\rangle$. Then the following conditions are equivalent*:

(1) $ba - (b-1) \in M$ *for all* $a \in A$,
(2) $bm - (b-1) \in M$ *for all* $m \in M\backslash\{0\}$.

The proof of the above lemma is similar to that of Lemma 1 in [21], and it is a special case of Lemma 2 in [22].

Proposition 2.2. *For $n, b \in \mathbb{N}$, $b \geq 2$ and $b \not\equiv 1 \pmod 3$, we have $bt-(b-1) \in T_{b,2}(n)$ for all $t \in T_{b,2}(n)\backslash\{0\}$.*

The proof of the above proposition is similar to that of Proposition 2 in [21]. We need some preliminary results to find out the minimal system of generators of $T_{b,2}(n)$.

Lemma 2.3. *Let $n, b \in \mathbb{N}$, $b \geq 2$ and $b \not\equiv 1 \pmod 3$ and $S = \langle\{(b+1)\cdot b^{n+i} + 1 \mid i \in \{0, 1, \ldots, n+1\}\}\rangle$. Then $bt - (b-1) \in S$ for all $t \in S\backslash\{0\}$.*

Proof. The proof of the above lemma is similar to that of Lemma 3 in [21]. □

We show a conclusion for a minimal system of generators of $T_{b,2}(n)$ in the following theorem.

Theorem 2.4. *For $n, b \in \mathbb{N}$, $b \geq 2$ and $b \not\equiv 1 \pmod 3$, we have $\langle\{(b+1)\cdot b^{n+i}+1 \mid i \in \{0, 1, \ldots, n+1\}\}\rangle$ is the minimal system of generators.*

Proof. $T_{b,2}(n) = \langle\{(b+1)\cdot b^{n+i}+1 \mid i \in \{0, 1, \ldots, n+1\}\}\rangle$ by Lemma 2.3 in this paper and Lemma 4 in [21], and it suffices to show that the minimality holds. Let us suppose conversely, that $(b+1)\cdot b^{2n+1}+1 \in \langle\{(b+1)\cdot b^{n+i}+1 \mid i \in \{0, 1, \ldots, n\}\}\rangle$. Then there exist $a_0, \ldots, a_n \in \mathbb{N}$ such that

$$(b+1)\cdot b^{2n+1} + 1 = \sum_{j=0}^{n} a_j\left((b+1)\cdot b^{n+j}+1\right) = \sum_{j=0}^{n}(b+1)a_j b^{n+j} + \sum_{j=0}^{n} a_j$$

and consequently, $\sum_{j=0}^{n} a_j \equiv 1 \mod (b+1)\cdot b^n$. Hence $\sum_{j=0}^{n} a_j = 1+t\cdot(b+1)\cdot b^n$ for some $t \in \mathbb{N}$. In addition, it is clear that $t \neq 0$ and thus $\sum_{j=0}^{n} a_j \geq 1+(b+1)\cdot b^n$. Combining these results, we obtain the inequality

$$\sum_{j=0}^{n} a_j \left((b+1)\cdot b^{n+j}+1\right) > (b+1)\cdot b^{2n+1}+1$$

and similarly, we obtain

$$\sum_{j=0}^{n-l} a_j \left((b+1)\cdot b^{n+j}+1\right) > (b+1)\cdot b^{2n+1-l}+1$$

for $0 < l \leq n$. This completes the proof. □

By Theorem 2.4, we can identify the embedding dimension of $T_{b,2}(n)$ for all $n, b \in \mathbb{N}$, $b \geq 2$ and $b \not\equiv 1 \pmod 3$, which turns out that the embedding dimension of $T_{b,2}(n)$ is independent of b.

Corollary 2.5. *Let $n, b \in \mathbb{N}$, $b \geq 2$ and $b \not\equiv 1 \pmod 3$ and let $T_{b,2}(n)$ be a Thabit numerical semigroup of the second kind base b associated with n and b. Then we obtain that $e(T_{b,2}(n)) = n+2$.*

We propose an example related to the Thabit numerical semigroup of the second kind base b, $T_{b,2}(n)$.

Example 2.6. Let $b \in \mathbb{N}, b \geq 2$ and $b \not\equiv 1 \pmod 3$. Then $T_{b,2}(3) = \left\langle\{(b+1)\cdot b^3+1, (b+1)\cdot b^4+1, (b+1)\cdot b^5+1, (b+1)\cdot b^6+1, (b+1)\cdot b^7+1\}\right\rangle = \left\langle\{b^4+b^3+1, b^5+b^4+1, b^6+b^5+1, b^7+b^6+1, b^8+b^7+1\}\right\rangle$ is a Thabit numerical semigroup of the second kind base b with embedding dimension $3+2=5$.

3. The Apéry set for $T_{b,2}(n)$

Definition 3.1 ([24])**.** Let S be a numerical semigroup and let $x \in S\backslash\{0\}$. Then, we have the Apéry set of x in S defined as $Ap(S,x) = \{s \in S \mid s-x \notin S\}$.

From the definition above, we have the following lemma.

Lemma 3.2 ([24])**.** *Let S be a numerical semigroup and let $x \in S\backslash\{0\}$. Then $Ap(S,x)$ has cardinality equal to x. Moreover $Ap(S,x) = \{w(0), w(1), \ldots, w(x-1)\}$ where $w(i)$ is the least element of S congruent with i modulo x for all $i \in \{0, \ldots, x-1\}$.*

Example 3.3. Let $S = \langle\{7, 11, 13\}\rangle$. Then $S = \{0, 7, 11, 13, 14, 18, 20, 21, 22, 24, 25, 26, 27, 28, 29, 31, \rightarrow\}$ where the symbol $\rightarrow$ means that every integer greater than 31 belongs to the set.

Hence $\mathrm{Ap}(S, 7) = \{0, 11, 13, 22, 24, 26, 37\}$.

The relation among the Frobenius number, genus and Apéry set of a numerical semigroup is provided in the following lemma.

Lemma 3.4 ([24, 27])**.** *Let S be a numerical semigroup and let $x \in S\backslash\{0\}$. Then,*

(1) $F(S) = \max(Ap(S,x)) - x$.
(2) $g(S) = \frac{1}{x}(\sum_{w \in Ap(S,x)} w) - \frac{x-1}{2}$.

Henceforth, we will denote by s_i the elements $(b+1)\cdot b^{n+i}+1$ for each $i \in \{0,1,\ldots,n+1\}$. Thus, with this notation, $\{s_0, s_1, \ldots, s_{n+1}\}$ is the minimal system of generators of $T_{b,2}(n)$.

Lemma 3.5. *Let $n, b \in \mathbb{N}$, $b \geq 2$ and $b \not\equiv 1 \pmod 3$. Then:*

(1) *If $0 < i \leq j < n+1$, then $s_i + bs_j = bs_{i-1} + s_{j+1}$.*
(2) *If $0 < i \leq n+1$, then*

$$s_i + bs_{n+1} = (b^{n+2} - b^n - b - 1)s_0 + b^{n-1}s_1 + bs_{i-1}.$$

Proof. (1) The proof is similar to that of (1) of Lemma 9 in [21].

(2) It can be derived directly since $s_i + bs_{n+1} = bs_{i-1} + s_{n+2}$ and $s_{n+2} = (b^{n+2} - b^n - b - 1)s_0 + b^{n-1}s_1$. □

In Lemma 3.5, we can consider the set of coefficients $(t_1, \ldots, t_{n+1})$ such that the expressions $\sum_{j=1}^{n+1} t_j s_j$ represent all elements in $\mathrm{Ap}(T_{b,2}(n), s_0)$. We follow a step-by-step approach to establish the set of coefficients $(t_1, \ldots, t_{n+1})$. First, we obtain the set of coefficients $(t_1, \ldots, t_{n+1})$ such that $\sum_{j=1}^{n+1} t_j s_j$, which contains all elements that are in $\mathrm{Ap}(T_{b,2}(n), s_0)$, but that might not be equal. We obtain the set by the following lemma.

Lemma 3.6. *Let $A_{b,2}(n)$ be the set of $(t_1, \ldots, t_{n+1}) \in \{0,1,\ldots,b\}^{n+1}$ such that if $t_j = b$, then $t_i = 0$ for all $i < j$. Then*

$$Ap(T_{b,2}(n), s_0) \subseteq \{\sum_{j=1}^{n+1} t_j s_j \mid (t_1, \ldots, t_{n+1}) \in A_{b,2}(n)\}.$$

Proof. The overall proof is the same as that of Lemma 10 in [21]. □

We define $R_{b,2}(n)$ for $b \geq 2$ and $b \not\equiv 1 \pmod 3$, as follows:

Definition 3.7. Let $b \geq 2$ and $b \not\equiv 1 \pmod 3$. Then

$$R_{b,2}(n) = \{(t_1, t_2, \ldots, t_{n+1}) \mid t_i \in \{0,1,\ldots,b\}\}$$

is defined by

(1) If $t_i = b$, $t_j = 0$ for all $1 \leq j < i$.
(2) $t_{n+1} \leq b-1$.
(3) If $t_{n+1} = b-1$, then $t_n \leq b-1$ and if $(t_n, t_{n+1}) = (b-1, b-1)$, $t_1 \leq 2$ and all $t_i = 0$ for $i \neq 1, n, n+1$.

Then we obtain the following theorem:

Theorem 3.8. *Let $n, b \geq 2$ and $b \not\equiv 1 \pmod 3$. Then we obtain*

$$Ap(T_{b,2}(n), s_0) = \left\{ \sum_{i=1}^{n+1} t_i s_i \,|\, (t_1, \ldots, t_{n+1}) \in R_{b,2}(n) \right\}.$$

Proof. We obtain the number of nonzero elements in

$$\left\{ \sum_{i=1}^{n+1} t_i s_i \,|\, (t_1, \ldots, t_{n+1}) \in R_{b,2}(n) \right\}$$

in the following manner:

(1) The number of $t_1 \neq 0$ is b since $0 \leq t_1 \leq b$.
(2) a_i = (The number of $(t_1, \ldots, t_i) \neq (0, \ldots, 0)$) for $i < n+1$ can be defined inductively by the formula $a_i = ba_{i-1} + b$ in three parts:
 (a) If $t_i = 0$, the number of the cases is a_{i-1}.
 (b) If $t_i = j$, where $1 \leq j \leq b-1$, the number of the cases is $a_{i-1}+1$ for each j.
 (c) If $t_i = b$, the number of the case is 1.

Hence, we obtain $a_n = \frac{b^{n+1}-b}{b-1}$. The number of all elements satisfying $t_{n+1} \leq b-2$ is

$$\left(1 + \frac{b^{n+1}-b}{b-1}\right)(b-1) = b^{n+1} - 1$$

and note that the difference of s_0 and $b^{n+1}-1$ is b^n+2. Also, we obtain $a_{n-1} = \frac{b^n-b}{b-1}$ and the number of all elements satisfying $t_n \leq b-2$ for fixed $t_{n+1} = b-1$ is

$$\left(1 + \frac{b^n-b}{b-1}\right)(b-1) = b^n - 1$$

and note that the difference of b^n+2 and b^n-1 is 3. Hence the number of the elements in the set $\left\{ \sum_{i=1}^{n+1} t_i s_i \,|\, (t_1, \ldots, t_{n+1}) \in R_{b,2}(n) \right\}$ is equal to s_0.

Notice that all elements in $R_{b,2}(n)$ were chosen in $A_{b,2}(n)$ by the smallest elements in $A_{b,2}(n)$ (see Lemma 3.6). Hence the remaining part is to show that $\left\{ \sum_{i=1}^{n+1} t_i s_i \,|\, (t_1, \ldots, t_{n+1}) \in R_{b,2}(n) \right\}$ is a complete system of residues modulo s_0 by Lemma 3.2. It can be shown as follows:

(1) $0, s_1, \ldots, bs_1 \equiv 0, -(b-1), \ldots, -b(b-1) \pmod{s_0}$.
(2) For $(t_1, t_2, 0, \ldots, 0) \in R_{b,2}(n)$, $t_1 s_1 + t_2 s_2 \equiv -(b+1)(b-1), \ldots, -(b^2 + b)(b-1) \pmod{s_0}$ for $t_2 \neq 0$.
(3) For $(t_1, t_2, t_3, 0, \ldots, 0) \in R_{b,2}(n)$, $t_1 s_1 + t_2 s_2 + t_3 s_3 \equiv -(b^2+b+1)(b-1), \ldots, -(b^3+b^2+b)(b-1) \pmod{s_0}$ for $t_3 \neq 0$.

And so on, finally $2s_1 + (b-1)s_n + (b-1)s_{n+1} \equiv -2(b-1) - (b-1)(b^n-1) - (b-1)(b^{n+1}-1) \equiv (-b^{n+1} - b^n)(b-1) \equiv b-1 \pmod{s_0}$. Since $(b-1, s_0) = 1$ if $b \not\equiv 1 \pmod 3$, $\{0, -(b-1), \ldots, (-b^{n+1} - b^n)(b-1)\}$ is a complete system of residues modulo s_0 and it completes the proof. □

In a similar way, we obtain the explicit form of the Apéry set of Thabit numerical semigroups of the second kind base b for $n = 0$ and $n = 1$ and we obtain the genus of Thabit numerical semigroups of the second kind base b for $n = 0$ and $n = 1$.

Theorem 3.9. (1) *For* $n = 0$ *and* $b \not\equiv 1 \pmod 3$, *we obtain*

$$\begin{aligned} Ap(T_{b,2}(0), s_0) &= \{t_1 s_1 \,|\, t_1 \in \{0, 1, \ldots, b+1\}\} \\ &= \{b+2, b^2+b+1, \ldots, b^3+2b^2+2b+1\} \end{aligned}$$

since $s_1 \equiv -(b-1) \pmod{b+2}$ *and* $\{0, -(b-1), \ldots, -(b+1)(b-1)\}$ *is a complete system of residues modulo* $b+2$. *Hence, we obtain*

$$\begin{aligned} \sum_{(t_1)\in R_{b,2}(0)} t_1 s_1 &= \sum_{k=1}^{b+1} k s_1 \\ &= (b+2)\left(\frac{1}{2}(b+1)(b^2+b+1)\right) \end{aligned}$$

and $g(T_{b,2}(0)) = \frac{1}{2}(b+1)(b^2+b+1) - \frac{b+1}{2} = \frac{b^3+2b^2+b}{2}$.

(2) *For* $n = 1$ *and* $b \not\equiv 1 \pmod 3$, *we obtain*

$$\begin{aligned} Ap(T_{b,2}(1), s_0) &= \{t_1 s_1 + t_2 s_2 \,|\, t_1 \in \{0, 1, \ldots, b\}, t_2 \in \{0, 1, \ldots, b-1\}\} \bigcup \{b s_2\} \\ &= \{0, s_1, \ldots, b s_1, s_2, s_1 + s_2, \ldots, b s_1 + s_2, \ldots, \\ &\qquad (b-1)s_2, s_1 + (b-1)s_2, \ldots, b s_1 + (b-1)s_2, b s_2\} \end{aligned}$$

since $s_1 \equiv -(b-1) \pmod{b^2+b+1}$ *and* $\{0, -(b-1), \ldots, -(b^2+b)(b-1)\}$ *is a complete system of residues modulo* b^2+b+1. *Hence, we obtain*

$$\begin{aligned} \sum_{(t_1,t_2)\in R_{b,2}(1)} (t_1 s_1 + t_2 s_2) &= b\sum_{k=1}^{b} k s_1 + (b+1)\sum_{k=1}^{b-1} k s_2 + b s_2 \\ &= \frac{1}{2}(b^2+b+1)(b^5+b^4+b^3+b) \end{aligned}$$

and $g(T_{b,2}(1)) = \frac{1}{2}(b^5+b^4+b^3+b) - \frac{b^2+b}{2} = \frac{b^5+b^4+b^3-b^2}{2}$.

We obtain the maximal element in the Apéry set of Thabit numerical semigroup of the second kind base b and the Frobenius number of this semigroup is obtained immediately as follows:

Corollary 3.10. (1) *If* $n = 0$, $Ap(T_{b,2}(0), s_0) = \{t_1 s_1 \,|\, t_1 \in \{0, 1, \ldots, b+1\}\} = \{b+2, b^2+b+1, \ldots, b^3+2b^2+2b+1\}$ *implies that* $\max(Ap(T_{b,2}(0), s_0)) = (b+1)s_1 = b^3+2b^2+2b+1$ *and* $F(T_{b,2}(0)) = (b+1)s_1 - s_0 = b^3+2b^2+b-1$. *If* $n = 1$, $Ap(T_{b,2}(1), s_0) = \{t_1 s_1 + t_2 s_2 \,|\, t_1 \in \{0, 1, \ldots, b\}, t_2 \in \{0, 1, \ldots, b-1\}\} \bigcup \{b s_2\}$ *implies that* $\max(Ap(T_{b,2}(1), s_0)) = b s_1 + (b-1)s_2 = b^5+b^4+2b-1$ *and* $F(T_{b,2}(1)) = b s_1 + (b-1)s_2 - s_0 = b^5+b^4-b^2+b-2$.

(2) *If* $n \geq 2$, $Ap(T_{b,2}(n), s_0) = \left\{\sum_{i=1}^{n+1} t_i s_i \,|\, (t_1, \ldots, t_{n+1}) \in R_{b,2}(n)\right\}$ *implies that*

$$\begin{aligned} \max(Ap(T_{b,2}(n), s_0)) &= 2s_1 + (b-1)s_n + (b-1)s_{n+1} \text{ and} \\ F(T_{b,2}(n)) &= 2s_1 + (b-1)s_n + (b-1)s_{n+1} - s_0 \\ &= b^{2n+3} + b^{2n+2} - b^{2n+1} - b^{2n} + 2b^{n+2} + 2b^{n+1} + 2b^2. \end{aligned}$$

Finally, we obtain the genus of Thabit numerical semigroups of the second kind base b for $n \geq 2$.

Theorem 3.11. *Let* $n, b \in \mathbb{N}$ *and* $n, b \geq 2$. *Then*

$$g(T_{b,2}(n)) = 3b + \frac{b^{2n}(b^3 + b^2 - b - 1) + b^n\left(b^2(n+1) - (n+3)\right)}{2}.$$

Proof. First, we consider

$$\begin{aligned} &\sum_{(t_1,\ldots,t_{n+1}) \in R_{b,2}(n)} (t_1 s_1 + \cdots + t_{n+1}s_{n+1}) \\ =&\sum_{k=1}^{b-1} \sum_{(t_1,\ldots,t_{n+1}) \in R_{b,2}(n), t_1 = k} ks_1 + \cdots + \sum_{k=1}^{b-1} \sum_{(t_1,\ldots,t_{n+1}) \in R_{b,2}(n), t_{n-1} = k} ks_{n-1} \\ &+ \sum_{k=1}^{b-2} \sum_{(t_1,\ldots,t_{n+1}) \in R_{b,2}(n), t_n = k} ks_n + \sum_{(t_1,\ldots,t_{n+1}) \in R_{b,2}(n), t_n = b-1} (b-1)s_n \\ &+ \sum_{k=1}^{b-2} \sum_{(t_1,\ldots,t_{n+1}) \in R_{b,2}(n), t_{n+1} = k} ks_{n+1} + \sum_{(t_1,\ldots,t_{n+1}) \in R_{b,2}(n), t_{n+1} = b-1} (b-1)s_{n+1} \\ &+ \sum_{(t_1,\ldots,t_{n+1}) \in R_{b,2}(n), t_1 = b} bs_1 + \cdots + \sum_{(t_1,\ldots,t_{n+1}) \in R_{b,2}(n), t_n = b} bs_n. \end{aligned}$$

We obtain that

$$\begin{aligned} &\sum_{(t_1,\ldots,t_{n+1}) \in R_{b,2}(n)} (t_1 s_1 + \cdots + t_{n+1}s_{n+1}) \\ =&\sum_{i=1}^{n-1} \tfrac{(b-1)b}{2} \cdot (b+1)(b^{n-1} - b^{n-i-1}) \cdot \left((b+1)\cdot b^{n+i} + 1\right) + 3\left((b+1)\cdot b^{n+1} + 1\right) \\ &+ \tfrac{(b-2)(b-1)}{2} \cdot \left(\tfrac{b^{n+1}-b}{b-1} \cdot \left((b+1)\cdot b^{2n} + 1\right) + \tfrac{b^{n+1}-1}{b-1} \cdot \left((b+1)\cdot b^{2n+1} + 1\right)\right) \\ &+ (b^n + 2)(b-1) \cdot \left((b+1)\cdot(b^{2n} + b^{2n+1}) + 2\right) \\ &+ \sum_{i=1}^{n-1} (b^2 - 1)b^{n-i}\left((b+1)\cdot b^{n+i} + 1\right) + (b-1)b\left((b+1)\cdot b^{2n} + 1\right) \\ =&\tfrac{(b-1)b(b+1)}{2}\left((b+1)\cdot \tfrac{b^{3n-1}-b^{2n}}{b-1} + (n-1)b^{n-1} - (n-1)(b+1)b^{2n-1} - \tfrac{b^{n-1}-1}{b-1}\right) \\ &+ 3\left((b+1)\cdot b^{n+1} + 1\right) \end{aligned}$$

$$+\tfrac{b-2}{2}\cdot\Big((b^{n+1}-b)\cdot\big((b+1)\cdot b^{2n}+1\big)+(b^{n+1}-1)\cdot\big((b+1)\cdot b^{2n+1}+1\big)\Big)$$
$$+(b^n+2)(b-1)\cdot\big((b+1)\cdot(b^{2n}+b^{2n+1})+2\big)$$
$$+(b^2-1)(b+1)(n-1)b^{2n}+(b+1)(b^n-b)+(b-1)b\,\big((b+1)\cdot b^{2n}+1\big)$$
$$=\tfrac{1}{2}\left((b+1)\cdot b^n+1\right)$$
$$\big(6b-b^{2n}+b^{n+1}-b^{2n+1}+b^{2n+2}+b^{2n+3}+b^{n+2}(n+1)-b^n(n+2)\big).$$

Hence,

$$\begin{aligned} g(T_{b,2}(n)) &= \tfrac{1}{2}\big(6b-b^{2n}+b^{n+1}-b^{2n+1}+b^{2n+2}+b^{2n+3} \\ &\qquad +b^{n+2}(n+1)-b^n(n+2)\big)-\tfrac{(b+1)\cdot b^n}{2} \\ &= 3b+\tfrac{b^{2n}(b^3+b^2-b-1)+b^n\left(b^2(n+1)-(n+3)\right)}{2}. \end{aligned}$$

□

We summarize all of our results by suggesting an example.

Example 3.12. Let $b=3$ and $n=2$. Then we obtain

$$\begin{aligned} \big\langle\{4\cdot 3^{2+i}+1\,|\,i\in\mathbb{N}\}\big\rangle &= \big\langle\{4\cdot 3^{2+i}+1\,|\,i\in\{0,1,2,3\}\}\big\rangle \\ &= \big\langle\{4\cdot 3^2+1, 4\cdot 3^3+1, 4\cdot 3^4+1, 4\cdot 3^5+1\}\big\rangle \\ &= \big\langle 37, 109, 325, 973\big\rangle. \end{aligned}$$

Hence, the embedding dimension is $e(T_{3,2}(2))=2+2=4$ and the Apery set is

$$\begin{aligned} &\operatorname{Ap}(T_{3,2}(2),37) \\ &=\{0,s_1,2s_1,3s_1,s_2,s_1+s_2,2s_1+s_2,3s_1+s_2,2s_2,s_1+2s_2,2s_1+2s_2, \\ &\quad 3s_1+2s_2,3s_2,s_3,s_1+s_3,2s_1+s_3,3s_1+s_3,s_2+s_3,s_1+s_2+s_3, \\ &\quad 2s_1+s_2+s_3,3s_1+s_2+s_3,2s_2+s_3,s_1+2s_2+s_3,2s_1+2s_2+s_3, \\ &\quad 3s_1+2s_2+s_3,3s_2+s_3,2s_3,s_1+2s_3,2s_1+2s_3,3s_1+2s_3,s_2+2s_3, \\ &\quad s_1+s_2+2s_3,2s_1+s_2+2s_3,3s_1+s_2+2s_3,2s_2+2s_3,s_1+2s_2+2s_3, \\ &\quad 2s_1+2s_2+2s_3\} \\ &=\{0,109,218,327,325,434,543,652,650,759,868,977,975,973,1082, \\ &\quad 1191,1300,1298,1407,1516,1625,1623,1732,1841,1950,1948,1946,2055, \\ &\quad 2164,2273,2271,2380,2489,2598,2596,2705,2814\}, \end{aligned}$$

where $s_i=4\cdot 3^{2+i}+1$. Notice that $\#\operatorname{Ap}(T_{3,2}(2))=37=s_0$ and we obtain the Frobenius number $F(T_{3,2}(2))=2814-37=2777$ and the genus $g(T_{3,2}(2))=1404$.

4. Pseudo-Frobenius numbers and type of $T_{b,2}(n)$

Let us recall the definition of Pseudo-Frobenius numbers (Definition 1.14) and we give a definition of an order relation, maximal element of the Apéry set and a lemma which is a connection of pseudo-Frobenius numbers and the Apéry set.

Definition 4.1. (1) The order relation $\leq_S$ is defined as follows: $a \leq_S b$ if $b-a \in S$ [21]. In [24] it is proved that $\leq_S$ is an order relation.

(2) [21] Let S be a numerical semigroup. Then maximal elements in the Apery set of S is defined as follows: $maximals_{\leq_S}(\mathrm{Ap}(S,x)) = \{w \in \mathrm{Ap}(S,x) \mid w' - w \notin \mathrm{Ap}(S,x)\backslash\{0\} \text{ for all } w' \in \mathrm{Ap}(S,x)\}$.

Lemma 4.2 ([24])**.** *Let S be a numerical semigroup and let x be a nonzero element of S. Then*

$$PF(S) = \{w - x \mid w \in maximals_{\leq S}(Ap(S,x))\}.$$

Let n be an integer greater than or equal to 3. Notice that maximal elements in $R_{b,2}(n)$ are as follows:

$$\begin{aligned}\Bigg\{ &\{2s_1 + (b-1)s_n + (b-1)s_{n+1}\}\\ &\bigcup\left\{bs_i + \sum_{k=i+1}^{n}(b-1)s_k + (b-2)s_{n+1} \,\middle|\, i \in \{1,\ldots,n-2\}\right\}\\ &\bigcup\left\{bs_i + \sum_{k=i+1}^{n-1}(b-1)s_k + (b-2)s_n + (b-1)s_{n+1} \,\middle|\, i \in \{1,2,\ldots,n-2\}\right\}\\ &\bigcup\{bs_{n-1} + (b-2)s_n + (b-1)s_{n+1}\}\Bigg\}.\end{aligned}$$

Also, $s_{i+1} = bs_i - (b-1)$ for all $i \in \{0,1,\ldots,n\}$ leads to

$$\begin{aligned}&\left\{bs_i + \sum_{k=i+1}^{n-1}(b-1)s_k + (b-2)s_n + (b-1)s_{n+1} \,\middle|\, i \in \{1,2,\ldots,n-2\}\right\}\\ &\quad\bigcup\{bs_{n-1} + (b-2)s_n + (b-1)s_{n+1}\}\\ &= \Bigg\{bs_1 + \sum_{k=2}^{n-1}(b-1)s_k + (b-2)s_n + (b-1)s_{n+1},\\ &\qquad bs_1 + \sum_{k=2}^{n-1}(b-1)s_k + (b-2)s_n + (b-1)s_{n+1} - (b-1), \ldots,\\ &\qquad bs_1 + \sum_{k=2}^{n-1}(b-1)s_k + (b-2)s_n + (b-1)s_{n+1} - (n-3)(b-1),\\ &\qquad bs_1 + \sum_{k=2}^{n-1}(b-1)s_k + (b-2)s_n + (b-1)s_{n+1} - (n-2)(b-1)\Bigg\}\end{aligned}$$

and

$$\left\{bs_i + \sum_{k=i+1}^{n} (b-1)s_i + (b-2)s_{n+1} \,|\, i \in \{1,2,\ldots,n-2\}\right\}$$
$$= \left\{bs_1 + \sum_{i=2}^{n}(b-1)s_i + (b-2)s_{n+1},\right.$$
$$bs_1 + \sum_{i=2}^{n}(b-1)s_i + (b-2)s_{n+1} - (b-1), \ldots,$$
$$\left. bs_1 + \sum_{i=2}^{n}(b-1)s_i + (b-2)s_{n+1} - (n-3)(b-1)\right\}.$$

Hence we obtain the following lemma.

Lemma 4.3. *Let* $n, b \in \mathbb{N}$, $n \geq 3, b \geq 2$ *and* $b \not\equiv 1 \pmod 3$. *Then*

$$maximals_{\leq T_{b,2}(n)}(Ap(T_{b,2}(n), s_0))$$
$$= maximals_{\leq T_{b,2}(n)}\left\{2s_1 + (b-1)s_n + (b-1)s_{n+1},\right.$$
$$bs_1 + \sum_{k=2}^{n-1}(b-1)s_k + (b-2)s_n + (b-1)s_{n+1},$$
$$bs_1 + \sum_{k=2}^{n-1}(b-1)s_k + (b-2)s_n + (b-1)s_{n+1} - (b-1), \ldots,$$
$$bs_1 + \sum_{k=2}^{n-1}(b-1)s_k + (b-2)s_n + (b-1)s_{n+1} - (n-2)(b-1),$$
$$bs_1 + \sum_{i=2}^{n}(b-1)s_i + (b-2)s_{n+1},$$
$$bs_1 + \sum_{i=2}^{n}(b-1)s_i + (b-2)s_{n+1} - (b-1), \ldots,$$
$$\left. bs_1 + \sum_{i=2}^{n}(b-1)s_i + (b-2)s_{n+1} - (n-3)(b-1)\right\}.$$

As a consequence, we show the theorem related to Pseudo-Frobenius numbers of $T_{b,2}$.

Theorem 4.4. *Let* $n, b \in \mathbb{N}$, $n \geq 3, b \geq 2$ *and* $b \not\equiv 1 \pmod 3$. *Then*

$$maximals_{\leq T_{b,2}(n)}(Ap(T_{b,2}(n), s_0))$$

$$= \Bigg\{ 2s_1 + (b-1)s_n + (b-1)s_{n+1}, bs_1 + \sum_{k=2}^{n-1}(b-1)s_k + (b-2)s_n + (b-1)s_{n+1},$$

$$bs_1 + \sum_{k=2}^{n-1}(b-1)s_k + (b-2)s_n + (b-1)s_{n+1} - (b-1), \ldots,$$

$$bs_1 + \sum_{k=2}^{n-1}(b-1)s_k + (b-2)s_n + (b-1)s_{n+1} - (n-2)(b-1),$$

$$bs_1 + \sum_{i=2}^{n}(b-1)s_i + (b-2)s_{n+1} \Bigg\}.$$

Proof. At first, note that

$$\begin{aligned}(1)\qquad &\left(bs_1 + \sum_{k=2}^{n-1}(b-1)s_k + (b-2)s_n + (b-1)s_{n+1}\right) \\ &- \left(bs_1 + \sum_{i=2}^{n}(b-1)s_i + (b-2)s_{n+1}\right) \\ &= s_{n+1} - s_n \\ &= (b-1)s_n - (b-1)\end{aligned}$$

and hence

$$\begin{aligned}&\left(bs_1 + \sum_{k=2}^{n-1}(b-1)s_k + (b-2)s_n + (b-1)s_{n+1} - (i-1)(b-1)\right) \\ &- \left(bs_1 + \sum_{i=2}^{n}(b-1)s_i + (b-2)s_{n+1} - i(b-1)\right) = (b-1)s_n\end{aligned}$$

for all $1 \leq i \leq n-2$. It implies that

$$\begin{aligned}&bs_1 + \sum_{i=2}^{n}(b-1)s_i + (b-2)s_{n+1} - i(b-1) \\ &\notin maximals_{\leq T_{b,2}(n)}(\mathrm{Ap}(T_{b,2}(n), s_0))\end{aligned}$$

for all $1 \leq i \leq n-2$. Therefore, we have

$$\begin{aligned}&maximals_{\leq T_{b,2}(n)}(\mathrm{Ap}(T_{b,2}(n), s_0)) \\ &= maximals_{\leq T_{b,2}(n)} \Bigg\{ 2s_1 + (b-1)s_n + (b-1)s_{n+1}, \\ &bs_1 + \sum_{k=2}^{n-1}(b-1)s_k + (b-2)s_n + (b-1)s_{n+1},\end{aligned}$$

$$bs_1 + \sum_{k=2}^{n-1}(b-1)s_k + (b-2)s_n + (b-1)s_{n+1} - (b-1), \ldots,$$
$$bs_1 + \sum_{k=2}^{n-1}(b-1)s_k + (b-2)s_n + (b-1)s_{n+1} - (n-2)(b-1),$$
$$\left. bs_1 + \sum_{i=2}^{n}(b-1)s_i + (b-2)s_{n+1} \right\}.$$

And we will show that

$$bs_1 + \sum_{k=2}^{n-1}(b-1)s_k + (b-2)s_n + (b-1)s_{n+1} - i(b-1) \in maximals_{\leq T_{b,2}(n)}(\mathrm{Ap}(T_{b,2}(n), s_0))$$

for all $1 \leq i \leq n-2$. Note that

$$\begin{aligned} 2s_1 - s_0 < {} & 2s_1 + (b-1)s_n + (b-1)s_{n+1} \\ & - \left(bs_1 + \sum_{k=2}^{n-1}(b-1)s_k + (b-2)s_n + (b-1)s_{n+1} - i(b-1) \right) \\ = {} & 2s_1 - (n-1-i)(b-1) < 2s_1. \end{aligned}$$

Assume that

$$bs_1 + \sum_{k=2}^{n-1}(b-1)s_k + (b-2)s_n + (b-1)s_{n+1} - i(b-1) \notin maximals_{\leq T_{b,2}(n)}(\mathrm{Ap}(T_{b,2}(n), s_0))$$

for some i. Then

$$\begin{aligned} & 2s_1 + (b-1)s_n + (b-1)s_{n+1} \\ & - \left(bs_1 + \sum_{k=2}^{n-1}(b-1)s_k + (b-2)s_n + (b-1)s_{n+1} - i(b-1) \right) = xs_0 + ys_1 \end{aligned}$$

and we classify the cases to get x, y which satisfy this equation.

(1) If $y = 0, 2s_1 - s_0 < xs_0 < 2s_1$ and we obtain $\frac{x}{2} < \frac{s_1}{s_0} < \frac{x+1}{2}$ and since $b - 1 < \frac{s_1}{s_0} < b$, we get $x = 2(b-1)$ or $2(b-1)+1$.

   (a) If $x = 2(b-1)$, $2s_1 - (n-1-i)(b-1) = 2(b-1)s_0$ and by observing with taking modulo $b-1$ to both sides, we get $6 \equiv 0 \pmod{b-1}$ and since $b \not\equiv 1 \pmod 3$ implies that $b = 2$ or 3 but $2\left(s_1 - (b-1)s_0\right) = 2\left(s_0 - (b-1)\right) > (n-1-i)(b-1)$ for $b \leq 3$ implies that $x = 2(b-1)$ is not a solution.

   (b) If $x = 2(b-1)+1$, $2s_1 - (n-1-i)(b-1) = \left(2(b-1)+1\right)s_0$ and by observing with taking modulo $b-1$ to both sides, we get $6 \equiv 3 \pmod{b-1}$ and since $b \not\equiv 1 \pmod 3$ implies that $b = 2$ but

$s_0-2(b-1) > (n-1-i)(b-1)$ for $b=2$ implies that $x = 2(b-1)+1$ is not a solution.

(2) If $y = 1, s_1 - s_0 < xs_0 < s_1$ and we obtain $x < \frac{s_1}{s_0} < x+1$ and since $b-1 < \frac{s_1}{s_0} < b$, we get $x = b-1$. Then the equation $2s_1 - (n-1-i)(b-1) = (b-1)s_0 + s_1$ is simplified as $s_0 - (b-1) = (n-1-i)(b-1)$ but the left-hand side is always larger than the right-hand side implies that there is no solution when $y = 1$.

Also, by Equation (1), we conclude that

$$\left(bs_1 + \sum_{i=2}^{n}(b-1)s_i + (b-2)s_{n+1}\right) - \left(bs_1 + \sum_{k=2}^{n-1}(b-1)s_k + (b-2)s_n + (b-1)s_{n+1} - i(b-1)\right) \notin T_{b,2}.$$

Hence,

$$bs_1 + \sum_{k=2}^{n-1}(b-1)s_k + (b-2)s_n + (b-1)s_{n+1} - i(b-1) \in maximals_{\leq T_{b,2}(n)}(\mathrm{Ap}(T_{b,2}(n), s_0))$$

for all $1 \leq i \leq n-2$.

Finally, we will show that

$$bs_1 + \sum_{i=2}^{n}(b-1)s_i + (b-2)s_{n+1} \in maximals_{\leq T_{b,2}(n)}(\mathrm{Ap}(T_{b,2}(n), s_0)).$$

By Equation (1),

$$\left(bs_1 + \sum_{k=2}^{n-1}(b-1)s_k + (b-2)s_n + (b-1)s_{n+1} - i(b-1)\right) - \left(bs_1 + \sum_{i=2}^{n}(b-1)s_i + (b-2)s_{n+1}\right) = (b-1)s_n - (i+1)(b-1)$$

and we obtain

$$(b-1)s_n - s_0 < (b-1)s_n - (i+1)(b-1) < (b-1)s_n.$$

But there is no element of the form $(b-1)s_n - (i+1)(b-1)$ which is in $T_{b,2}$, and between $(b-1)s_n - s_0$ and $(b-1)s_n$ since $(b-1)s_n = (b-2)s_n + (b-1)s_{n-1} + \cdots + (b-1)s_1 + (s_1 - (n-1)(b-1))$ and hence

$$\begin{aligned}&(b-1)s_n - (i+1)(b-1)\\ &= (b-2)s_n + (b-1)s_{n-1} + \cdots + (b-1)s_1 + (s_1 - (n+i)(b-1)).\end{aligned}$$

Note that $s_1 - (n+i)(b-1) \notin T_{b,2}$ since if it is in $T_{b,2}$, its form should be $(b-1)s_0$ but $s_1 - (b-1)s_0 = (n+i)(b-1)$ cannot be satisfied.
Hence, it completes the proof. □

Finally, we get this main theorem for pseudo-Frobenius numbers and type of $T_{b,2}(n)$.

Theorem 4.5. *Let $n, b \in \mathbb{N}$, $b \geq 2$ and $b \not\equiv 1 \pmod 3$. Then*

$$PF(T_{b,2}) = \{F(T_{b,2})\} \bigcup \left\{ bs_1 + \sum_{k=2}^{n-1}(b-1)s_k + (b-2)s_n + (b-1)s_{n+1} - i(b-1) - s_0 \,|\, i \in \{0,1,2,\dots,n-2\} \right\} \bigcup \left\{ bs_1 + \sum_{i=2}^{n}(b-1)s_i + (b-2)s_{n+1} - s_0 \right\}$$

and hence $t(T_{b,2}) = n+1$.

Proof. It is directly derived from Lemma 4.2 and Theorem 4.4. □

And we show some examples related to Pseudo-Frobenius numbers.

Example 4.6. Let $b = 3$ and $n = 2$. Then

$$\begin{aligned} T_{3,2}(2) &= \{4 \cdot 3^{2+i} + 1 \,|\, i \in \mathbb{N}\} = \langle 37, 109, 325, 973 \rangle \text{ and} \\ PF(T_{3,2}(2)) &= \{2814, 2598, 1950\} - 37 = \{2777, 2561, 1913\} \\ &= \{2s_1 + 2s_2 + 2s_3 - s_0, 3s_1 + s_2 + 2s_3 - s_0, 3s_1 + 2s_2 + s_3 - s_0\}. \end{aligned}$$

Note that

$$\begin{aligned} PF(T_{3,2}(n)) = \{&2s_1 + 2s_n + 2s_{n+1} - s_0, 3s_1 + \sum_{i=2}^{n-1} 2s_i + s_n + 2s_{n+1} - s_0, \\ &3s_2 + \sum_{i=3}^{n-1} 2s_i + s_n + 2s_{n+1} - s_0, \dots, \\ &3s_{n-2} + 2s_{n-1} + s_n + 2s_{n+1} - s_0, 3s_{n-1} + s_n + 2s_{n+1}, \\ &3s_1 + \sum_{i=2}^{n} 2s_i + s_{n+1} - s_0\}. \end{aligned}$$

Example 4.7. Let $b = 5$ and $n = 3$. Then

$$\begin{aligned} T_{5,2}(3) &= \{6 \cdot 5^{3+i} + 1 \,|\, i \in \mathbb{N}\} \\ &= \langle 6 \cdot 5^3 + 1 (= 751), 6 \cdot 5^4 + 1, 6 \cdot 5^5 + 1, \\ &\quad 6 \cdot 5^6 + 1, 6 \cdot 5^7 + 1, 6 \cdot 5^8 + 1 \rangle \text{ and} \\ PF(T_{5,2}(3)) &= \{2257510, 2250016, 2250012, 1875016\} - 751 \\ &= \{2256759, 2249265, 2249261, 1874265\} \\ &= \{2s_1 + 4s_3 + 4s_4 - s_0, 5s_1 + 4s_2 + 3s_3 + 4s_4 - s_0, \\ &\quad 5s_2 + 3s_3 + 4s_4 - s_0, 5s_1 + 4s_2 + 4s_3 + 3s_4 - s_0\}. \end{aligned}$$

Note that

$$\begin{aligned}PF(T_{5,2}(n)) = \{&2s_1 + 4s_n + 4s_{n+1} - s_0, 5s_1 + \sum_{i=2}^{n-1} 4s_i + 3s_n + 4s_{n+1} - s_0,\\ &5s_2 + \sum_{i=3}^{n-1} 4s_i + 3s_n + 4s_{n+1} - s_0, \ldots,\\ &5s_{n-2} + 4s_{n-1} + 3s_n + 4s_{n+1} - s_0, 5s_{n-1} + 3s_n + 4s_{n+1} - s_0,\\ &5s_1 + \sum_{i=2}^{n} 4s_i + 3s_{n+1} - s_0\}.\end{aligned}$$

As stated in the Introduction, we give the embedding dimension, the Apery set, the Frobenius number, genus, the Pseudo-Frobenius number and type related to the numerical semigroup generated by Thabit number of the first kind base b ($(b+1)\cdot b^{n+i} - 1$) in Section 5 and the Cunningham numbers ($b^{n+i} + 1$) in Section 6 without the proofs.

5. Results related to the $T_{b,1}(n)$

If $n, b \in \mathbb{N}$ and $b \geq 2$, then $T_{b,1}(n)$ is a submonoid of $(\mathbb{N}, +)$. Moreover we have $\{(b+1)\cdot b^n - 1, (b+1)\cdot b^{n+1} - 1\} \subseteq T_{b,1}(n)$ and $\gcd((b+1)\cdot b^n - 1, (b+1)\cdot b^{n+1} - 1) = \gcd((b+1)\cdot b^{n+1} - b, (b+1)\cdot b^{n+1} - 1) \mid b-1$. But $(b+1)\cdot b^n - 1 \equiv 1 \pmod{b-1}$ implies that if we let $(b+1)\cdot b^n - 1 = g\alpha = h(b-1) + 1$ where $g = \gcd((b+1)\cdot b^n - 1, (b+1)\cdot b^{n+1} - 1) \mid b-1$ and $h \in \mathbb{N}$ then $g \mid 1$ and hence $\gcd(T_{b,1}(n)) = 1$ and $T_{b,1}(n)$ is a numerical semigroup.

5.1. Embedding dimension for $T_{b,1}(n)$

Theorem 5.1. *If $n, b \in \mathbb{N}$ and $b \geq 2$, then $\langle\{(b+1)\cdot b^{n+i} - 1 \mid i \in \{0, 1, \ldots, n+1\}\}\rangle$ is a minimal system of generators.*

By Theorem 5.1, we can identify the embedding dimension of $T_{b,1}(n)$ for all $n, b \in \mathbb{N}$ and $b \geq 2$.

Corollary 5.2. *Let $n, b \in \mathbb{N}$ and $b \geq 2$ and let $T_{b,1}(n)$ be a Thabit numerical semigroup of the first kind base b associated with n and b. Then $e(T_{b,1}(n)) = n + 2$.*

5.2. The Apéry set for $T_{b,1}(n)$

Lemma 5.3. *Let $A_{b,1}(n)$ be the set of $(t_1, \ldots, t_{n+1}) \in \{0, 1, \ldots, b\}^{n+1}$ such that $t_{n+1} \in \{0, 1, \ldots, b-1\}$ and if $t_j = b$, then $t_i = 0$ for all $i < j$. Then $Ap(T_{b,1}(n), s_0) \subseteq \{\sum_{j=1}^{n+1} t_j s_j \mid (t_1, \ldots, t_{n+1}) \in A_{b,1}(n)\}$.*

Let $R_{b,1}(n)$ be the set of the sequences $(t_1, \ldots, t_{n+1}) \in A_{b,1}(n)$ that if $t_{n+1} = b-1$, it satisfies the following conditions:

(1) $t_n \leq b-1$.
(2) If $t_n = b-1, t_1 = \cdots = t_{n-1} = 0$.

Then, we obtain the following lemma:

Lemma 5.4.

$$Ap(T_{b,1}(n), s_0) = \left\{ \sum_{j=1}^{n+1} t_j s_j \,|\, (t_1, \ldots, t_{n+1}) \in R_{b,1}(n) \right\}.$$

Theorem 5.5.

$$F(T_{b,1}(n)) = (b^3 + b^2 - b - 1) \cdot b^{2n} - (b+1) \cdot b^n - 2b + 3.$$

Example 5.6. Let $b = 2$. Then we obtain

$$F(T_{2,1}(n)) = 9 \cdot 2^{2n} - 3 \cdot 2^n - 1.$$

It is the Frobenius number of Thabit numerical semigroups suggested in [21].

To obtain the genus of the numerical semigroups generated by Thabit number of the first kind base b, we have to check the number of elements in $R_{b,1}(n)$ when one element t_i is fixed.

Lemma 5.7. *Let $i \in \{1, 2, \ldots, n+1\}$ where $n \geq 2$ be an integer. Then,*

$$\#\{(t_1, \ldots, t_{n+1}) \in R_{b,1}(n) \,|\, t_i = b\} = \begin{cases} (b^2-1) \cdot b^{n-i-1} & \text{if } i \in \{1, \ldots, n-1\}, \\ b-1 & \text{if } i = n, \\ 0 & \text{if } i = n+1. \end{cases}$$

Lemma 5.8. *Let $i \in \{1, 2, \ldots, n-1\}$ where $n \geq 2$ be an integer. Then,*

$$\#\{(t_1, \ldots, t_{n+1}) \in R_{b,1}(n) \,|\, t_i = k\} = (b+1)(b^{n-1} - b^{n-i-1})$$

for each $k \in \{1, \ldots, b-1\}$.

Lemma 5.9. *Let $n \geq 2$ be an integer. Then,*

$$\#\{(t_1, \ldots, t_{n+1}) \in R_{b,1}(n) \,|\, t_n = k\} = \frac{b^{n+1} - b}{b-1}$$

for each $k \in \{1, \ldots, b-2\}$.

Lemma 5.10. *Let $n \geq 2$ be an integer. Then,*

$$\#\{(t_1, \ldots, t_{n+1}) \in R_{b,1}(n) \,|\, t_n = b-1\} = b^n.$$

Lemma 5.11. *Let $n \geq 2$ be an integer. Then,*

$$\#\{(t_1, \ldots, t_{n+1}) \in R_{b,1}(n) \,|\, t_{n+1} = k\} = \frac{b^{n+1} - 1}{b-1}$$

for each $k \in \{1, \ldots, b-2\}$.

Lemma 5.12. *Let $n \geq 2$ be an integer. Then,*

$$\#\{(t_1, \ldots, t_{n+1}) \in R_{b,1}(n) \,|\, t_{n+1} = b-1\} = b^n.$$

By combining the above lemmas, we obtain the genus of $T_{b,1}(n)$.

Theorem 5.13. *Let $n, b \in \mathbb{N}$, $n, b \geq 2$ and $T_{b,1}(n)$ be the Thabit numerical semigroup of the first kind base b associated to n. Then,*

$$g(T_{b,1}(n)) = \frac{(b^3+b^2-b-1)b^{2n} + \{(n-1)(b^2-1)-2\}b^n - 2b + 4}{2}.$$

We summarize all of our results by suggesting an example.

Example 5.14. In the case of $n = 1$, $T_{b,1}(1) = \langle s_0, s_1, s_2 \rangle = \langle b^2+b-1, b^3+b^2-1, b^4+b^3-1 \rangle$ and we obtain

$$\begin{aligned}\mathrm{Ap}(T_{b,1}(1), s_0) = \{&0, s_1, \ldots, bs_1, s_2, s_1+s_2, \ldots, bs_1+s_2,\\ &2s_2, s_1+2s_2, \ldots, bs_1+2s_2, \ldots,\\ &(b-1)s_2, s_1+(b-1)s_2, \ldots, (b-1)s_1+(b-1)s_2\}.\end{aligned}$$

Note that $\#\mathrm{Ap}(T_{b,1}(1), s_0) = (b+1)+(b-1)(b+1)-1 = b^2+b-1 = s_0$, $\max(\mathrm{Ap}(T_{b,1}(1)), s_0) = (b-1)s_1+(b-1)s_2$ and $F(T_{b,1}(1)) = (b-1)s_1+(b-1)s_2-s_0 = (b-1)(b^4+2b^3+b^2-2)-(b^2+b-1) = (b^3+b^2-b-1)b^2-(b+1)b^1-2b+3$ and $g(T_{b,1}(1)) = \frac{b^5+b^4-b^3-b^2-2b+4}{2}$. Let $b = 3$. Then we obtain the more detailed example as follows:

1. $T_{3,1}(1) = \langle 11, 35, 107 \rangle$. Note that $e(T_{3,1}(1)) = 3 = 1+2$.
2. $\mathrm{Ap}(T_{3,1}(1), s_0) = \{0, s_1, 2s_1, 3s_1, s_2, s_1+s_2, 2s_1+s_2, 3s_1+s_2, 2s_2, s_1+2s_2, 2s_1+2s_2\}$. Note that $\#\mathrm{Ap}(T_{3,1}(1), s_0) = 11 = s_0$.
3. $\max(\mathrm{Ap}(T_{3,1}(1)), s_0) = 2 \cdot 35 + 2 \cdot 107 = 284$ and hence $F(T_{3,1}(1)) = \max(\mathrm{Ap}(T_{3,1}(1)), s_0) - s_0 = 284 - 11 = 273$.
4. $g(T_{3,1}(1)) = \frac{3^5+3^4-3^3-3^2-2\cdot 3+4}{2} = 143$.

5.3. Pseudo-Frobenius numbers and type of $T_{b,1}(n)$

Lemma 5.15. *Let $n, b \in \mathbb{N}$, $n, b \geq 2$ and $T_{b,1}(n)$ be the Thabit numerical semigroup of the first kind base b associated to n. Then,*

$$\begin{aligned}&maximals_{\leq_{T_{b,1}(n)}}(Ap(T_{b,1}(n)))\\ &= maximals_{\leq_{T_{b,1}(n)}}\Bigg(\{(b-1)s_n + (b-1)s_{n+1} - s_0\}\\ &\bigcup \left\{bs_i + \sum_{k=i+1}^{n}(b-2)s_k + (b-1)s_{n+1} \,|\, i \in \{1, \ldots, n-1\}\right\}\\ &\bigcup \left\{bs_i + \sum_{k=i+1}^{n}(b-1)s_k + (b-2)s_{n+1} \,|\, i \in \{1, \ldots, n-1\}\right\}\Bigg).\end{aligned}$$

Finally, we get this main theorem for pseudo-Frobenius numbers and type of $T_{b,1}(n)$.

Theorem 5.16. *Let $n, b \in \mathbb{N}$, $n, b \geq 2$ and $T_{b,1}(n)$ be the Thabit numerical semigroup of the first kind base b associated to n. Then,*

$$PF(T_{b,1}(n)) = \{F_{b,1}(n) - i(b-1) \,|\, i \in \{0, 1, \ldots, n-1\}\}$$

$$\bigcup\left\{bs_1+\sum_{i=2}^{n}(b-1)s_i+(b-2)s_{n+1}-s_0\right\}$$

and hence $t(T_{b,1})=n+1$.

Example 5.17. (1) If $b=3$ and $n=4$; $PF(T_{3,1}(4))=\{209625, 209623, 209621, 209619, 157131\}=\{2s_4+2s_5-s_0, 3s_3+s_4+2s_5-s_0, 3s_2+2s_3+s_4+2s+5, 3s_1+2s_2+2s_3+s_4+2s_5, 3s_1+2s_2+2s_3+2s_4+s_5\}$.

(2) If $b=4$ and $n=3$; $PF(T_{4,1}(3))=\{306875, 306872, 306869, 245429\}=\{3s_3+3s_4-s_0, 4s_2+2s_3+3s_4-s_0, 4s_1+3s_2+2s_3+3s_4-s_0, 4s_1+3s_2+3s_3+2s_4-s_0\}$.

6. Results related to the $SC^+(b,n)$

If $n,b\in\mathbb{N}$ and $2\,|\,b$, then $SC^+(b,n)$ is a submonoid of $(\mathbb{N},+)$. Moreover, we have $\{b^n+1, b^{n+1}+1\}\subseteq SC^+(b,n)$ and $g=\gcd(b^n+1, b^{n+1}+1)=\gcd(b^n+1, b-1)\,|\,b-1$. However, $b^n+1\equiv 2 \pmod{b-1}$ implies that $g\,|\,2$ and if $2\,|\,b$, $2\nmid b^n+1$ and $SC^+(b,n)$ is a numerical semigroup.

6.1. Embedding dimension for $SC^+(b,n)$

Theorem 6.1. *If* $n,b\in\mathbb{N}$, $2\,|\,b$, *and* $n\neq 0$, *then* $\left\langle\{b^{n+i}+1 \mid i\in\{0,1,\ldots,n\}\}\right\rangle$ *is a minimal system of generators.*

By Theorem 6.1, we can identify the embedding dimension of $SC^+(b,n)$ for all $n,b\in\mathbb{N}$ and $2\,|\,b$.

Corollary 6.2. *Let* $n,b\in\mathbb{N}$, $2\,|\,b$, *and let* $SC^+(b,n)$ *be a Cunningham numerical semigroup associated with* n *and* b. *Then* $e(SC^+(b,n))=n+1$.

6.2. The Apéry set for $SC^+(b,n)$

Lemma 6.3. *Let* $A_b(n)$ *be the set of* $(t_1,\ldots,t_n)\in\{0,1,\ldots,b\}^n$ *such that if* $t_j=b$, *then* $t_i=0$ *for all* $i<j$. *Then*

$$Ap(SC^+(b,n),s_0)\subseteq\left\{\sum_{j=1}^{n}t_js_j \mid (t_1,\ldots,t_n)\in A_b(n)\right\}.$$

We define $R_b(n)$ for $b,n\geq 2$ and $2\,|\,b$ as follows.

Definition 6.4. Let $b,n\geq 2$ and $2\,|\,b$. Then $R_b(n)=\{(t_1,t_2,\ldots,t_n)\,|\,t_i\in\{0,1,\ldots,b\}\}$ is defined by:

(1) if $t_i=b$, $t_j=0$ for all $1\leq j<i$;
(2) $t_n\leq b-1$;
(3) if $t_n=b-1$, then $t_1\leq 1$ and all $t_i=0$ for $i\neq 1,n$.

Then we obtain the following lemma that defines the explicit form of the Apéry set of $SC^+(b,n)$.

Lemma 6.5. *Let $b, n \geq 2$ and $2 \mid b$. Then we obtain*

$$Ap(SC^{+}(b,n), s_0) = \left\{ \sum_{i=1}^{n} t_i s_i \mid (t_1, \ldots, t_n) \in R_b(n) \right\}.$$

Theorem 6.6. *We obtain the maximal element in the Apéry set of Cunningham numerical semigroups and the Frobenius number of this semigroup is obtained immediately as follows.*

(1) *If $n = 0$, $\max(Ap(SC^{+}(b,0), s_0)) = b+1$ and $F(SC^{+}(b,0)) = (b+1) - s_0 = b - 1$.*
(2) *If $n = 1$, $\max(Ap(SC^{+}(b,1), s_0)) = b^3 + b$ and $F(SC^{+}(b,1)) = bs_1 - s_0 = b^3 - 1$.*
(3) *If $n \geq 2$, $Ap(SC^{+}(b,n), s_0) = \{\sum_{i=1}^{n} t_i s_i \mid (t_1, \ldots, t_n) \in R_b(n)\}$ implies that*

$$\max(Ap(SC^{+}(b,n), s_0) = s_1 + (b-1)s_n \text{ and}$$
$$F(SC^{+}(b,n)) = s_1 + (b-1)s_n - s_0 = (b-1)(b^{2n} + b^n + 1).$$

To obtain the genus of the numerical semigroups generated by Cunningham numbers, we must check the number of elements in $R_b(n)$ when one element t_i is fixed.

Lemma 6.7. *Let $i \in \{1, 2, \ldots, n\}$ where $n \geq 2$ is an integer. Then,*

$$\#\{(t_1, \ldots, t_n) \in R_b(n) \mid t_i = b\} = \begin{cases} (b-1) \cdot b^{n-i-1} & \text{if } i \in \{1, \ldots, n-1\}, \\ 0 & \text{if } i = n. \end{cases}$$

Lemma 6.8. *Let $n \geq 2$ be an integer. Then,*

$$\#\{(t_1, \ldots, t_n) \in R_b(n) \mid t_1 = k\} = \begin{cases} b^{n-1} - b^{n-i-1} + 1 & \text{if } i = 1 \text{ and } k = 1, \\ b^{n-1} - b^{n-i-1} & \text{otherwise}, \end{cases}$$

for each $k \in \{1, \ldots, b-1\}$.

Lemma 6.9. *Let $n \geq 2$ be an integer. Then,*

$$\#\{(t_1, \ldots, t_n) \in R_b(n) \mid t_n = k\} = \frac{b^n - 1}{b - 1}$$

for each $k \in \{1, \ldots, b-2\}$.

Lemma 6.10. *Let $n \geq 2$ be an integer. Then,*

$$\#\{(t_1, \ldots, t_n) \in R_b(n) \mid t_n = b-1\} = 2.$$

Finally, we obtain the genus of a Cunningham numerical semigroup for $n \geq 2$.

Theorem 6.11. *Let $n, b \in \mathbb{N}$ and $n, b \geq 2$ where $2 \mid b$. Then*

$$g(SC^{+}(b,n)) = b + \frac{b^{2n}(b-1) + b^n(bn - n - 1)}{2}.$$

We summarize our results by suggesting an example.

Example 6.12. Let $b = 4$ and $n = 2$. Then we obtain

$$\begin{aligned}\left\langle\{4^{2+i} + 1 \mid i \in \mathbb{N}\}\right\rangle &= \left\langle\{4^{2+i} + 1 \mid i \in \{0, 1, 2\}\}\right\rangle \\ &= \left\langle\{4^2 + 1, 4^3 + 1, 4^4 + 1\}\right\rangle \\ &= \left\langle 17, 65, 257\right\rangle.\end{aligned}$$

Hence, the Apery set is

$$\begin{aligned}Ap(SC^+(4,2), 17) &= \{s_0, s_1, 2s_1, 3s_1, 4s_1, s_2, s_1 + s_2, 2s_1 + s_2, \\ &\qquad 3s_1 + s_2, 4s_1 + s_2, 2s_2, s_1 + 2s_2, 2s_1 + 2s_2, \\ &\qquad 3s_1 + 2s_2, 4s_1 + 2s_2, 3s_2, s_1 + 3s_2\} \\ &= \{0, 65, 130, 195, 260, 257, 322, 387, 452, \\ &\qquad 517, 514, 579, 644, 709, 774, 771, 836\},\end{aligned}$$

where $s_i = 4^{2+i} + 1$ and we obtain the Frobenius number $F(SC^+(4,2)) = 836 - 17 = 819$.

6.3. Pseudo-Frobenius numbers and type of $SC^+(b, n)$

Lemma 6.13. *Let $n, b \in \mathbb{N}$, $n \geq 3, b \geq 2$ and $2 \mid b$. Then*

$$\begin{aligned}&maximals_{\leq SC^+(b,n)}(Ap(SC^+(b,n), s_0)) \\ &= maximals_{\leq SC^+(b,n)}\Bigg\{s_1 + (b-1)s_n, bs_1 + \sum_{k=2}^{n-1}(b-1)s_k + (b-2)s_n, \\ &\quad bs_1 + \sum_{k=2}^{n-1}(b-1)s_k + (b-2)s_n - (b-1), \ldots, \\ &\quad bs_1 + \sum_{k=2}^{n-1}(b-1)s_k + (b-2)s_n - (n-2)(b-1)\Bigg\}.\end{aligned}$$

As a consequence, we show a theorem related to pseudo-Frobenius numbers of $SC^+(b, n)$.

Theorem 6.14. *Let $n, b \in \mathbb{N}$, $n \geq 3$, $b \geq 2$, and $2 \mid b$. Then*

$$\begin{aligned}&maximals_{\leq SC^+(b,n)}(Ap(SC^+(b,n), s_0)) \\ &= \Bigg\{s_1 + (b-1)s_n, bs_1 + \sum_{k=2}^{n-1}(b-1)s_k + (b-2)s_n, bs_1 + \sum_{k=2}^{n-1}(b-1)s_k \\ &\quad + (b-2)s_n - (b-1), \ldots, bs_1 + \sum_{k=2}^{n-1}(b-1)s_k + (b-2)s_n - (n-2)(b-1)\Bigg\}.\end{aligned}$$

Finally, we obtain the following main theorem for pseudo-Frobenius numbers and type of $SC^+(b, n)$.

Theorem 6.15. *Let $n, b \in \mathbb{N}$, $n \geq 3$, $b \geq 2$, and $2 \mid b$. Then*

$$\begin{aligned} &PF(SC^+(b,n)) \\ &= \{F(SC^+(b,n))\} \\ &\quad \bigcup \left\{ bs_1 + \sum_{k=2}^{n-1}(b-1)s_k + (b-2)s_n - i(b-1) - s_0 \mid i \in \{0,1,2,\ldots,n-2\} \right\} \end{aligned}$$

and, hence, $t(SC^+(b,n)) = n$.

Finally, we show two simple examples related to pseudo-Frobenius numbers.

Example 6.16. (1) Let $b = 4$ and $n = 2$. Then $SC^+(4,2) = \{4^{2+i} + 1 \mid i \in \mathbb{N}\} = \langle 17, 65, 257 \rangle$ and $PF(SC^+(4,2)) = \{836, 774\} - 17 = \{819, 757\} = \{s_1 + 3s_2 - s_0, 4s_1 + 2s_2 - s_0\}$.

(2) Let $b = 4$ and $n = 3$. Then $SC^+(4,3) = \{4^{3+i} + 1 \mid i \in \mathbb{N}\} = \langle 65, 257, 1025, 4097 \rangle$ and $PF(SC^+(4,3)) = \{12548, 12297, 12294\} - 65 = \{12483, 12232, 12229\} = \{s_1 + 3s_3 - s_0, 4s_1 + 3s_2 + 2s_3 - s_0, 4s_2 + 2s_3 - s_0\}$.

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