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ON THE CONVERGENCE OF SERIES FOR ROWWISE SUMS OF NEGATIVELY SUPERADDITIVE DEPENDENT RANDOM VARIABLES

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Abstract. In the paper, some probability convergence properties of series for rowwise sums of negatively superadditive dependent (NSD) random variables are discussed. We establish some sharp results on these convergence for NSD random variables under some general settings, which generalize and improve the corresponding ones of some known literatures.

1. Introduction

Existing methods and algorithms appeared in some literatures assume that variables are independent, but in the real world, that is not always satisfied, in most cases they are dependent. Dependent structures play an important role in all areas of computational and applied mathematics, such as the computational efficiency (e.g., the convergence, stability, accuracy, ...) for solving scientific or engineering problems. Therefore, many statisticians have proposed several kinds of dependent structures in order to some specific practical problems, such as negatively associated (NA) random variables, negatively orthant dependent (NOD) random variables, extended negatively dependent (END) random variables, negatively superadditive dependent (NSD) random variables, and many others. In the following, we will recall the concept of NSD structures, which is weaker than NA.

Definition 1.1. A function ϕ: $\mathbb{R}^n \to \mathbb{R}$ is called superadditive if $\phi(\mathrm{x} \vee \mathrm{y}) + \phi(\mathrm{x} \wedge \mathrm{y}) \geq \phi(\mathrm{x}) + \phi(\mathrm{y})$ for all $\mathrm{x}, \mathrm{y} \in \mathbb{R}^n$, where $\vee$ stands for componentwise maximum and $\wedge$ stands for componentwise minimum.

Based on the class of superadditive functions introduced by Kemperman [12], Hu [9] introduced the following concept of NSD random variables.

Definition 1.2. A random vector $X = (X_1, X_2, \ldots, X_n)$ is said to be NSD if

$$E\phi(X_1, X_2, \ldots, X_n) \leq E\phi(X_1^*, X_2^*, \ldots, X_n^*), \tag{1.1}$$

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where $X_1^*, X_2^*, \dots, X_n^*$ are independent such that X_i^* and X_i have the same distribution for each i and ϕ is a superadditive function such that the expectations in (1.1) exist.

A sequence of random variables $\{X_n; n \geq 1\}$ is said to be NSD if for all $n \geq 1$, $(X_1, X_2, \dots, X_n)$ is NSD.

An array of random variables $\{X_{ni}; 1 \leq i \leq n, n \geq 1\}$ is called rowwise NSD if for all $n \geq 1$, $\{X_{ni}; 1 \leq i \leq n\}$ is NSD.

As a matter of fact, Hu [9] gave an example for illustrating that NSD does not imply NA (introduced by Alam and Saxena [1], carefully studied by Joag-Dev and Proschan [11]), and posed an open problem whether NA implies NSD. Furthermore, Hu [9] provided some basic properties of NSD random variables. Christofides and Vaggelatou [3] solved this open problem and indicated that NA implies NSD. Therefore, NSD structure is an extension of NA structure and sometimes more useful than the latter in probability theory and mathematical statistics. Consequently, investigating the convergence properties of NSD random variables is of much significance.

Since the concept of NSD random variables introduced by Hu [9], many applications have been found in various aspects by many authors. Eghbal et al. [5] for two maximal inequalities and a strong law of large numbers of quadratic forms of nonnegative NSD random variables. Eghbal et al. [6] for some Kolmogorov inequalities for quadratic forms and weighted quadratic forms of nonnegative and uniformly bounded NSD random variables. Shen et al. [15] for the almost sure convergence and strong stability for weighted sums of NSD random variables. Wang et al. [18] for the complete convergence of arrays of rowwise NSD random variables and the complete consistency for the estimator of nonparametric regression model based on NSD errors. Naderi et al. [14] for the rate of complete convergence for weighted sums of NSD random variables. Wang et al. [19] for the complete convergence of NSD random variables and its application in the EV regression model. Shen et al. [17] for some applications of the Rosenthal-type inequality for NSD random variables, Shen et al. [16], Deng et al. [4], Meng et al. [13] for some strong convergence properties for weighted sums of NSD random variables. Wu et al. [22] for an exponential inequality and the general results on the complete convergence, Wang et al. [20] for some strong laws of large numbers of NSD random variables and the strong consistency and weak consistency of the LS estimators in the EV regression model with NSD errors, among others.

For a triangular array of rowwise random variables $\{X_{ni}; 1 \leq i \leq n, n \geq 1\}$, let $\{a_n; n \geq 1\}$ be a sequence of positive real numbers with $a_n \uparrow \infty$. Suppose that $\{\psi_n(t), n \geq 1\}$ is a sequence of nonnegative even functions such that

$$\frac{\psi_n(|t|)}{|t|^q} \uparrow \text{ and } \frac{\psi_n(|t|)}{|t|^p} \downarrow \quad \text{as } |t| \uparrow \tag{1.2}$$

for some $1 \leq q < p$.

Introduced the assumptions as follows

$$EX_{ni} = 0, \quad 1 \le i \le n, \ n \ge 1, \tag{1.3}$$

$$\sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{E\psi_i\left(X_{ni}\right)}{\psi_i\left(a_n\right)} < \infty, \tag{1.4}$$

$$\sum_{n=1}^{\infty} \left(\sum_{i=1}^{n} \frac{E|X_{ni}|^r}{a_n^r} \right)^s < \infty, \tag{1.5}$$

where $0 < r \le 2$ and $s > 0$.

Based on the above conditions (1.2)-(1.5), there are some articles for investigating the convergence properties for independent and dependent random variables. For example, Hu and Taylor [10], Wu [21], Wu and Zhu [23], among others. Especially, Gan and Chen [7] established some complete convergence results for weighted sums of NA random variables under the cases $1 < p \le 2$ and $p > 2$ with $q = 1$.

In this paper, our main purpose is to investigate the convergence properties of series for rowwise sums of NSD random variables. We establish some sharp results of the complete convergence, the complete moment convergence and the mean convergence of series for rowwise sums of NSD random variables under some mild conditions. Compared with the corresponding ones of Gan and Chen [7], it is worthy pointing out that the conditions in this paper are more general and the results obtained are stronger.

Throughout this paper, let $I\left(A\right)$ be the indicator function of the set A. C denotes a generic positive constant, whose value may be different in various places, and $a_n = O\left(b_n\right)$ stands for $a_n \le Cb_n$. $[x]$ stands for the integer part of x.

2. Main results

The concept of complete convergence was firstly introduced by Hsu and Robbins [8] as follows: A sequence of random variables $\{X_n; n \ge 1\}$ is said to converge completely to a constant λ if $\sum_{n=1}^{\infty} P\left(|X_n - \lambda| > \varepsilon\right) < \infty$ for all $\varepsilon > 0$. In view of the Borel-Cantelli lemma, this implies that $X_n \to \lambda$ almost surely (a.s.). The complete convergence plays an important role in establishing almost sure convergence of random variables.

Subsequently, Chow [2] generalized this notion by showing the following concept of complete moment convergence: Let $\{Z_n; n \ge 1\}$ be a sequence of random variables, and $a_n > 0$, $b_n > 0$, $q > 0$. If $\sum_{n=1}^{\infty} a_n E\left(b_n^{-1}\left|Z_n\right| - \varepsilon\right)_+^q < \infty$ for all $\varepsilon \ge 0$, then $\{Z_n; n \ge 1\}$ is said to be in the sense of complete moment convergence. It is well known that the complete moment convergence implies the complete convergence.

In these five theorems, suppose that $\{X_{ni}; 1 \le i \le n, n \ge 1\}$ is an array of rowwise NSD random variables. Let $\{a_n; n \ge 1\}$ be a sequence of positive real numbers with $a_n \uparrow \infty$, and let $\{\psi_n(t), n \geq 1\}$ be a sequence of nonnegative even functions such that (1.2) for some q and p to be specified in each theorem separately (of course, $q < p$). The main results are presented in this section, and the proofs will be detailed in next section.

Theorem 2.1. *If $1 \leq q < p \leq 2$, then assumptions* (1.3) *and* (1.4) *imply*

$$\sum_{n=1}^{\infty} P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_{ni}\right| > \varepsilon a_n\right) < \infty \quad \text{for } \forall\varepsilon > 0. \tag{2.1}$$

Theorem 2.2. *If $1 \leq q < p$ and $p > 2$, then assumptions* (1.3), (1.4) *and* (1.5) *imply* (2.1).

Theorem 2.3. *If $1 \leq q < p \leq 2$, then assumptions* (1.3) *and* (1.4) *imply*

$$\sum_{n=1}^{\infty} a_n^{-q} E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_{ni}\right| - \varepsilon a_n\right)_+^q < \infty \quad \text{for } \forall\varepsilon > 0. \tag{2.2}$$

Theorem 2.4. *If $1 \leq q < p$ and $p > 2$, then assumptions* (1.3), (1.4) *and* (1.5) *imply* (2.2).

Theorem 2.5. *Let $1 \leq q < p$.*

(1) *If $1 < p \leq 2$, then assumption*

$$\sum_{i=1}^{n} \frac{E\psi_i\left(X_{ni}\right)}{\psi_i\left(a_n\right)} \to 0 \quad \text{as } n \to \infty \tag{2.3}$$

implies

$$\frac{1}{a_n}\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_{ni}\right| \xrightarrow{L_q} 0. \tag{2.4}$$

(2) *If $p > 2$, then assumptions* (2.3) *and*

$$\sum_{i=1}^{n} \frac{E|X_{ni}|^r}{a_n^r} \to 0 \quad \text{as } n \to \infty, \text{ where } 0 < r \leq 2 \tag{2.5}$$

imply (2.4).

Remark 2.1. Since the class of NSD random variables contains independent random variables and NA random variables, these obtained theorems also hold for independent random variables and NA random variables. Taking $q = 1$ in (1.2), the conditions of Theorem 2.1 or 2.2 are the same as those of Gan and Chen [7]. In addition, the result (2.2) of Theorem 2.3 or 2.4 is more stronger than the corresponding (2.1) of Theorem 2.1 or 2.2 under the same conditions.

3. Proofs

To prove the main results, we need the following important lemmas.

Lemmas 3.1 ([9]). *If $(X_1, X_2, \ldots, X_n)$ is NSD and $f_1, f_2, \ldots, f_n$ are all non-decreasing functions, then $f_1(X_1), f_2(X_2), \ldots, f_n(X_n)$ are NSD.*

Lemmas 3.2 ([9,15]). *Let $M > 1$, $\{X_n; n \geq 1\}$ be a sequence of NSD random variables with $E|X_n|^M < \infty$ for each $n \geq 1$. Then for all $n \geq 1$,*

$$E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_i\right|^M\right) \leq 2^{3-M}\sum_{i=1}^{n} E|X_i|^M \quad \text{for } 1 < M \leq 2, \tag{3.1}$$

$$E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_i\right|^M\right) \leq 2\left(\frac{15M}{\ln M}\right)^M\left(\sum_{i=1}^{n} E|X_i|^M + \left(\sum_{i=1}^{n} EX_i^2\right)^{M/2}\right) \quad \text{for } M > 2. \tag{3.2}$$

Proof of Theorem 2.1. For fixed $n \geq 1$ and all $1 \leq i \leq n$, define $Y_{ni} = -a_n I(X_{ni} < -a_n) + X_{ni} I(|X_{ni}| \leq a_n) + a_n I(X_{ni} > a_n)$, $Z_{ni} = X_{ni} - Y_{ni}$. It is easily seen that for $\forall \varepsilon > 0$,

$$\begin{aligned} &P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_{ni}\right| > \varepsilon a_n\right) \\ &\leq P\left(\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} Y_{ni}\right| > \varepsilon a_n\right)\bigcup\left(\bigcup_{i=1}^{n}(|X_{ni}| > a_n)\right)\right) \\ &\leq P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}(Y_{ni} - EY_{ni})\right| > \varepsilon a_n - \max_{1\leq j\leq n}\left|\sum_{i=1}^{j} EY_{ni}\right|\right) \\ &\quad + P\left(\bigcup_{i=1}^{n}(|X_{ni}| > a_n)\right). \end{aligned}$$

Firstly, one will prove that

$$\frac{1}{a_n}\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} EY_{ni}\right| \to 0 \quad \text{as } n \to \infty. \tag{3.3}$$

Note that $|Z_{ni}| \leq |X_{ni}| I(|X_{ni}| > a_n)$ for fixed $n \geq 1$ and all $1 \leq i \leq n$. By (1.3), (1.4) and $q > 1$, one has that

$$\frac{1}{a_n}\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} EY_{ni}\right| = \frac{1}{a_n}\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} EZ_{ni}\right|$$

$$\leq C \sum_{i=1}^{n} \frac{E\left|X_{ni}\right| I\left(\left|X_{ni}\right| > a_n\right)}{a_n}$$
$$\leq C \sum_{i=1}^{n} \frac{E|X_{ni}|^q I\left(\left|X_{ni}\right| > a_n\right)}{a_n^q}$$
$$\leq C \sum_{i=1}^{n} \frac{E\psi_i\left(X_{ni}\right)}{\psi_i\left(a_n\right)} \to 0 \quad \text{as } n \to \infty. \tag{3.4}$$

Hence for n large enough,

$$P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_{ni}\right| > \varepsilon a_n\right) \leq P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}\left(Y_{ni} - EY_{ni}\right)\right| > \frac{\varepsilon a_n}{2}\right) + \sum_{i=1}^{n} P\left(\left|X_{ni}\right| > a_n\right).$$

To prove (2.1), it suffices to show that

$$I_1 \doteq \sum_{n=1}^{\infty} P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}\left(Y_{ni} - EY_{ni}\right)\right| > \frac{\varepsilon a_n}{2}\right) < \infty, \tag{3.5}$$

$$I_2 \doteq \sum_{n=1}^{\infty}\sum_{i=1}^{n} P\left(\left|X_{ni}\right| > a_n\right) < \infty. \tag{3.6}$$

For I_1, it follows that $\{Y_{ni} - EY_{ni}; 1 \leq i \leq n, n \geq 1\}$ is still an array of rowwise NSD random variables with zero mean by Lemma 3.1. Note that $|Y_{ni}| \leq a_n$ a.s.. For $1 \leq q < p \leq 2$, by the Markov inequality (for $1 < p \leq M \leq 2$), (3.1) and (1.4), one has that

$$I_1 \leq C \sum_{n=1}^{\infty} \frac{1}{a_n^M} E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}\left(Y_{ni} - EY_{ni}\right)\right|^M\right)$$
$$\leq C \sum_{n=1}^{\infty} \frac{1}{a_n^M} \sum_{i=1}^{n} E|Y_{ni} - EY_{ni}|^M$$
$$\leq C \sum_{n=1}^{\infty}\sum_{i=1}^{n} \frac{E|Y_{ni}|^p}{a_n^p}$$
$$\leq C \sum_{n=1}^{\infty}\sum_{i=1}^{n} \frac{E\psi_i\left(\left|Y_{ni}\right|\right)}{\psi_i\left(a_n\right)}$$
$$\leq C \sum_{n=1}^{\infty}\sum_{i=1}^{n} \frac{E\psi_i\left(X_{ni}\right)}{\psi_i\left(a_n\right)} < \infty. \tag{3.7}$$

For I_2, by some standard computations and $q \geq 1$, one has

$$\begin{aligned} I_2 &= \sum_{n=1}^{\infty}\sum_{i=1}^{n} EI\left(|X_{ni}| > a_n\right) \\ &\leq C\sum_{n=1}^{\infty}\sum_{i=1}^{n} \frac{E|X_{ni}|^q I\left(|X_{ni}| > a_n\right)}{a_n^q} \\ &\leq C\sum_{n=1}^{\infty}\sum_{i=1}^{n} \frac{E\psi_i\left(X_{ni}\right)}{\psi_i\left(a_n\right)} < \infty. \end{aligned} \tag{3.8}$$

The proof of Theorem 2.1 is completed. □

Proof of Theorem 2.2. In view of the proof of Theorem 2.1, following the same notation, (3.3) and $I_2 < \infty$ hold. It suffices to show that $I_1 < \infty$ for $1 \leq q < p$ and $p > 2$. Note that $|Y_{ni}| \leq |X_{ni}|$ and $|Y_{ni}| \leq a_n$ a.s.. Take $0 < r \leq 2$ and $s > 0$. By the Markov inequality (for $M = p > 2$ and $p > 2s$), (3.2), the c_r inequality, (1.4) and (1.5),

$$\begin{aligned} I_1 &\leq C\sum_{n=1}^{\infty}\frac{1}{a_n^p}E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}\left(Y_{ni}-EY_{ni}\right)\right|^p\right) \\ &\leq C\sum_{n=1}^{\infty}\frac{1}{a_n^p}\left(\sum_{i=1}^{n}E|Y_{ni}-EY_{ni}|^p + \left(\sum_{i=1}^{n}E(Y_{ni}-EY_{ni})^2\right)^{p/2}\right) \\ &\leq C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E|Y_{ni}|^p}{a_n^p} + C\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}\frac{E|Y_{ni}|^2}{a_n^2}\right)^{p/2} \\ &\leq C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E|X_{ni}|^p}{a_n^p} + C\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}\frac{E|X_{ni}|^r}{a_n^r}\right)^{p/2} \\ &\leq C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E\psi_i\left(X_{ni}\right)}{\psi_i\left(a_n\right)} + C\left(\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}\frac{E|X_{ni}|^r}{a_n^r}\right)^s\right)^{p/2s} < \infty. \end{aligned} \tag{3.9}$$

The proof of Theorem 2.2 is completed. □

Proof of Theorem 2.3. For $\forall\varepsilon > 0$ and all $t \geq 0$, noting that

$$\begin{aligned} &\sum_{n=1}^{\infty}a_n^{-q}E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}X_{ni}\right| - \varepsilon a_n\right)_+^q \\ &= \sum_{n=1}^{\infty}a_n^{-q}\int_0^{\infty}P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}X_{ni}\right| - \varepsilon a_n > t^{1/q}\right)dt \\ &= \sum_{n=1}^{\infty}a_n^{-q}\int_0^{a_n^q}P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}X_{ni}\right| > \varepsilon a_n + t^{1/q}\right)dt \end{aligned}$$

$$
\begin{aligned}
&+\sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{n} X_{ni}\right| > \varepsilon a_n + t^{1/q}\right) dt \\
&\le \sum_{n=1}^{\infty} P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} X_{ni}\right| > \varepsilon a_n\right) \\
&\quad +\sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{n} X_{ni}\right| > t^{1/q}\right) dt \\
&\doteq I_3 + I_4. \qquad (3.10)
\end{aligned}
$$

According to the proof of Theorem 2.1, the result $I_3 < \infty$ holds. To prove (2.2), it needs only to show that $I_4 < \infty$. For fixed $n \ge 1$, $1 \le i \le n$ and all $t \ge 0$, define $Y_{ni}^t = -t^{1/q} I\left(X_{ni} < -t^{1/q}\right) + X_{ni} I\left(|X_{ni}| \le t^{1/q}\right) + t^{1/q} I\left(X_{ni} > t^{1/q}\right)$, $Z_{ni}^t = X_{ni} - Y_{ni}^t$. It easily follows that

$$
\begin{aligned}
&P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} X_{ni}\right| > t^{1/q}\right) \\
&\le P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} Y_{ni}^t\right| > t^{1/q}\right) + P\left(\bigcup_{i=1}^{n}\left(|X_{ni}| > t^{1/q}\right)\right) \\
&\le P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} Y_{ni}^t\right| > t^{1/q}\right) + \sum_{i=1}^{n} P\left(|X_{ni}| > t^{1/q}\right), \qquad (3.11)
\end{aligned}
$$

which implies

$$
\begin{aligned}
I_4 \le &\sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} Y_{ni}^t\right| > t^{1/q}\right) dt \\
&+\sum_{n=1}^{\infty} a_n^{-q} \sum_{i=1}^{n} \int_{a_n^q}^{\infty} P\left(|X_{ni}| > t^{1/q}\right) dt \doteq I_5 + I_6. \qquad (3.12)
\end{aligned}
$$

For I_6, by some standard computations and (1.4),

$$
\begin{aligned}
I_6 &\le C\sum_{n=1}^{\infty} a_n^{-q} \sum_{i=1}^{n} \int_{0}^{\infty} P\left(|X_{ni}|\, I\left(|X_{ni}| > a_n\right) > t^{1/q}\right) dt \\
&\le C\sum_{n=1}^{\infty} a_n^{-q} \sum_{i=1}^{n} E|X_{ni}|^q I\left(|X_{ni}| > a_n\right) \\
&\le C\sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{E\psi_i\left(X_{ni}\right)}{\psi_i\left(a_n\right)} < \infty. \qquad (3.13)
\end{aligned}
$$

By an argument similar to that in the proof of (3.3), one has

$$
\max_{t\ge a_n^q} \frac{1}{t^{1/q}} \max_{1\le j\le n}\left|\sum_{i=1}^{j} EY_{ni}^t\right| \to 0 \quad \text{as } n \to \infty. \qquad (3.14)
$$

Hence for n is sufficiently large, $\max_{1\le j\le n}\left|\sum_{i=1}^{j} EY_{ni}^t\right| \le \frac{t^{1/q}}{2}$ holds uniformly for all $t \ge a_n^q$, which implies

$$P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} Y_{ni}^t\right| > t^{1/q}\right) \le P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}\left(Y_{ni}^t - EY_{ni}^t\right)\right| > \frac{t^{1/q}}{2}\right). \tag{3.15}$$

For I_5, let $d_n = [a_n] + 1$. By (3.15), the Markov inequality, (3.1) and the c_r inequality, one has that for $1 \le q < p \le 2$,

$$\begin{aligned}
I_5 \le\ & \sum_{n=1}^{\infty} a_n^{-q}\int_{a_n^q}^{\infty} P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}\left(Y_{ni}^t - EY_{ni}^t\right)\right| > \frac{t^{1/q}}{2}\right) dt \\
\le\ & C\sum_{n=1}^{\infty} a_n^{-q}\int_{a_n^q}^{\infty} t^{-2/q} E\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}\left(Y_{ni}^t - EY_{ni}^t\right)\right|^2\right) dt \\
\le\ & C\sum_{n=1}^{\infty}\sum_{i=1}^{n} a_n^{-q}\int_{a_n^q}^{\infty} E\left|Y_{ni}^t\right|^2 t^{-2/q} dt \\
=\ & C\sum_{n=1}^{\infty}\sum_{i=1}^{n} a_n^{-q}\int_{a_n^q}^{\infty} EX_{ni}^2 I\left(|X_{ni}| \le d_n\right) t^{-2/q} dt \\
& + C\sum_{n=1}^{\infty}\sum_{i=1}^{n} a_n^{-q}\int_{a_n^q}^{\infty} EX_{ni}^2 I\left(d_n < |X_{ni}| \le t^{1/q}\right) t^{-2/q} dt \\
& + C\sum_{n=1}^{\infty}\sum_{i=1}^{n} a_n^{-q}\int_{a_n^q}^{\infty} P\left(|X_{ni}| > t^{1/q}\right) dt \\
\doteq\ & I_{51} + I_{52} + I_{53}.
\end{aligned} \tag{3.16}$$

For I_{51}, one has

$$\begin{aligned}
I_{51} =\ & C\sum_{n=1}^{\infty}\sum_{i=1}^{n} a_n^{-q} EX_{ni}^2 I\left(|X_{ni}| \le d_n\right)\int_{a_n^q}^{\infty} t^{-2/q} dt \\
\le\ & C\sum_{n=1}^{\infty}\sum_{i=1}^{n} \frac{EX_{ni}^2 I\left(|X_{ni}| \le d_n\right)}{a_n^2} \\
=\ & C\sum_{n=1}^{\infty}\sum_{i=1}^{n} \frac{EX_{ni}^2 I\left(|X_{ni}| \le a_n\right)}{a_n^2} \\
& + C\sum_{n=1}^{\infty}\sum_{i=1}^{n} \frac{EX_{ni}^2 I\left(a_n < |X_{ni}| \le d_n\right)}{a_n^2} \\
\le\ & C\sum_{n=1}^{\infty}\sum_{i=1}^{n} \frac{E|X_{ni}|^p I\left(|X_{ni}| \le a_n\right)}{a_n^p}
\end{aligned}$$

$$+C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\left(\frac{a_n+1}{a_n}\right)^{2-q}\frac{E|X_{ni}|^qI\left(a_n<|X_{ni}|\le d_n\right)}{a_n^q}$$

$$\le C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E\psi_i\left(X_{ni}\right)}{\psi_i\left(a_n\right)}+C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E|X_{ni}|^qI\left(|X_{ni}|>a_n\right)}{a_n^q}$$

$$\le 2C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E\psi_i\left(X_{ni}\right)}{\psi_i\left(a_n\right)}<\infty. \tag{3.17}$$

For I_{52}, note that

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n}a_n^{-q}\int_{a_n^q}^{d_n^q}EX_{ni}^2I\left(d_n<|X_{ni}|\le t^{1/q}\right)t^{-2/q}dt=0.$$

Taking $t=x^q$. By the condition (1.4) and $1\le q<2$,

$$\begin{aligned}
I_{52}&=C\sum_{n=1}^{\infty}\sum_{i=1}^{n}a_n^{-q}\sum_{m=d_n}^{\infty}\int_m^{m+1}EX_{ni}^2I\left(d_n<|X_{ni}|\le x\right)x^{q-3}dx\\
&\le C\sum_{n=1}^{\infty}\sum_{i=1}^{n}a_n^{-q}\sum_{m=d_n}^{\infty}EX_{ni}^2I\left(d_n<|X_{ni}|\le m+1\right)m^{q-3}\\
&\le C\sum_{n=1}^{\infty}\sum_{i=1}^{n}a_n^{-q}\sum_{m=d_n}^{\infty}m^{q-3}\sum_{j=d_n}^{m}EX_{ni}^2I\left(j<|X_{ni}|\le j+1\right)\\
&\le C\sum_{n=1}^{\infty}\sum_{i=1}^{n}a_n^{-q}\sum_{j=d_n}^{\infty}j^{q-2}EX_{ni}^2I\left(j<|X_{ni}|\le j+1\right)\\
&\le C\sum_{n=1}^{\infty}\sum_{i=1}^{n}a_n^{-q}E|X_{ni}|^qI\left(|X_{ni}|>d_n\right)\\
&\le C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E|X_{ni}|^qI\left(|X_{ni}|>a_n\right)}{a_n^q}
\end{aligned}$$

$$\le C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E\psi_i\left(X_{ni}\right)}{\psi_i\left(a_n\right)}<\infty. \tag{3.18}$$

By an argument similar to the proof of $I_6<\infty$, one has that $I_{53}<\infty$. Hence, the desired result $I_5<\infty$ is proved for $1\le q<p\le 2$. The proof of Theorem 2.3 is completed. □

Proof of Theorem 2.4. Similarly, by the proofs of Theorems 2.2 and 2.3, the corresponding results of $I_3<\infty$, $I_6<\infty$, (3.14) and (3.15) hold. To prove the desired result (2.2), it needs only to show that $I_5<\infty$ for $1\le q<p$ and $p>2$.

By the Markov inequality, (3.2) and the c_r inequality,

$$I_5\le\sum_{n=1}^{\infty}a_n^{-q}\int_{a_n^q}^{\infty}P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}\left(Y_{ni}^t-EY_{ni}^t\right)\right|>\frac{t^{1/q}}{2}\right)dt$$

$$
\begin{aligned}
&\leq C\sum_{n=1}^{\infty} a_n^{-q}\int_{a_n^q}^{\infty} t^{-p/q} E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}\left(Y_{ni}^t - EY_{ni}^t\right)\right|^p\right) dt \\
&\leq C\sum_{n=1}^{\infty} a_n^{-q}\int_{a_n^q}^{\infty} t^{-p/q}\left(\sum_{i=1}^{n} E\left|Y_{ni}^t\right|^p + \left(\sum_{i=1}^{n} E\left|Y_{ni}^t\right|^2\right)^{p/2}\right) dt \\
&\leq C\sum_{n=1}^{\infty} a_n^{-q}\sum_{i=1}^{n}\int_{a_n^q}^{\infty} E\left|Y_{ni}^t\right|^p t^{-p/q} dt \\
&\quad + C\sum_{n=1}^{\infty} a_n^{-q}\int_{a_n^q}^{\infty} t^{-p/q}\left(\sum_{i=1}^{n} E\left(Y_{ni}^t\right)^2\right)^{p/2} dt \\
&\doteq I_7 + I_8. \qquad (3.19)
\end{aligned}
$$

For I_7, by $1 \leq q < p$ and $p > 2$, let $d_n = [a_n] + 1$,

$$
\begin{aligned}
I_7 &= C\sum_{n=1}^{\infty} a_n^{-q}\sum_{i=1}^{n}\int_{a_n^q}^{\infty} E|X_{ni}|^p t^{-p/q} I\left(|X_{ni}| \leq d_n\right) dt \\
&\quad + C\sum_{n=1}^{\infty} a_n^{-q}\sum_{i=1}^{n}\int_{a_n^q}^{\infty} E|X_{ni}|^p t^{-p/q} I\left(d_n < |X_{ni}| \leq t^{1/q}\right) dt \\
&\quad + C\sum_{n=1}^{\infty} a_n^{-q}\sum_{i=1}^{n}\int_{a_n^q}^{\infty} P\left(|X_{ni}| > t^{1/q}\right) dt \\
&\doteq I_{71} + I_{72} + I_{73}. \qquad (3.20)
\end{aligned}
$$

By the argument similar to the proofs of $I_{51} < \infty$, $I_{52} < \infty$ and $I_{53} < \infty$ (replacing the exponent 2 by p), $I_7 < \infty$ can be also proved.

For I_8, since $p > 2$,

$$
\begin{aligned}
I_8 &= C\sum_{n=1}^{\infty} a_n^{-q}\int_{a_n^q}^{\infty} t^{-p/q}\left(\sum_{i=1}^{n} E\left(Y_{ni}^t\right)^2\right)^{p/2} dt \\
&\leq C\sum_{n=1}^{\infty} a_n^{-q}\int_{a_n^q}^{\infty} t^{-p/q}\left(\sum_{i=1}^{n} EX_{ni}^2 I\left(|X_{ni}| \leq a_n\right)\right)^{p/2} dt \\
&\quad + C\sum_{n=1}^{\infty} a_n^{-q}\int_{a_n^q}^{\infty} t^{-p/q}\left(\sum_{i=1}^{n} EX_{ni}^2 I\left(a_n < |X_{ni}| \leq t^{1/q}\right)\right)^{p/2} dt \\
&\quad + C\sum_{n=1}^{\infty} a_n^{-q}\int_{a_n^q}^{\infty}\left(\sum_{i=1}^{n} P\left(|X_{ni}| > t^{1/q}\right)\right)^{p/2} dt \\
&\doteq I_{81} + I_{82} + I_{83}. \qquad (3.21)
\end{aligned}
$$

For I_{81}, by $1 \le q < p$, $p > 2$, $p > 2s$ and (1.5),

$$I_{81} \le C \sum_{n=1}^{\infty} \left(\sum_{i=1}^{n} \frac{E X_{ni}^2 I\left(|X_{ni}| \le a_n\right)}{a_n^2} \right)^{p/2}$$
$$\le C \left(\sum_{n=1}^{\infty} \left(\sum_{i=1}^{n} \frac{E|X_{ni}|^r I\left(|X_{ni}| \le a_n\right)}{a_n^r} \right)^s \right)^{p/2s} < \infty.$$

The proof of $I_{82} < \infty$ is proceeded with the following two cases of $1 \le q \le 2$ and $2 < q < p$.

(a) For $1 \le q \le 2$ and $p > 2$. By (1.4) and the c_r inequality,

$$I_{82} \le C \sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} \left(t^{-1} \sum_{i=1}^{n} E|X_{ni}|^q I\left(a_n < |X_{ni}| \le t^{1/q}\right) \right)^{p/2} dt$$
$$\le C \sum_{n=1}^{\infty} a_n^{-q} \left(\sum_{i=1}^{n} E|X_{ni}|^q I\left(|X_{ni}| > a_n\right) \right)^{p/2} \int_{a_n^q}^{\infty} t^{-p/2} dt$$
$$\le C \sum_{n=1}^{\infty} \left(\sum_{i=1}^{n} \frac{E|X_{ni}|^q I\left(|X_{ni}| > a_n\right)}{a_n^q} \right)^{p/2}$$
$$\le C \left(\sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{E\psi_i\left(X_{ni}\right)}{\psi_i\left(a_n\right)} \right)^{p/2} < \infty. \tag{3.22}$$

(b) For $2 < q < p$. By (1.4) and the c_r inequality again,

$$I_{82} \le C \sum_{n=1}^{\infty} a_n^{-q} \left(\sum_{i=1}^{n} E|X_{ni}|^2 I\left(|X_{ni}| > a_n\right) \right)^{p/2} \int_{a_n^q}^{\infty} t^{-p/q} dt$$
$$\le C \sum_{n=1}^{\infty} \left(\sum_{i=1}^{n} \frac{E|X_{ni}|^2 I\left(|X_{ni}| > a_n\right)}{a_n^2} \right)^{p/2}$$
$$\le C \sum_{n=1}^{\infty} \left(\sum_{i=1}^{n} \frac{E|X_{ni}|^q I\left(|X_{ni}| > a_n\right)}{a_n^q} \right)^{p/2}$$
$$\le C \left(\sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{E\psi_i\left(X_{ni}\right)}{\psi_i\left(a_n\right)} \right)^{p/2} < \infty. \tag{3.23}$$

For I_{83}, it follows from (2.1) that $\psi_i(|t|) \uparrow$ as $|t| \uparrow$. By (1.3),

$$\sup_{t \ge a_n^q} \sum_{i=1}^{n} P\left(|X_{ni}| > t^{1/q}\right) \le \sum_{i=1}^{n} P\left(|X_{ni}| > a_n\right)$$
$$\le \sum_{i=1}^{n} \frac{E\psi_i\left(|X_{ni}|\right)}{\psi_i\left(a_n\right)} \to 0 \quad \text{as } n \to \infty. \tag{3.24}$$

Hence for sufficiently large n, $\sum_{i=1}^{n} P\left(|X_{ni}| > t^{1/q}\right) < 1$ holds uniformly for all $t \geq a_n^q$. Similarly to the proof of (3.13), $I_{83} < \infty$ holds. The proof of Theorem 2.4 is completed. □

Proof of Theorem 2.5. Following those notations in the proof of Theorem 2.3. Firstly, one will prove (2.4) for $1 < p \leq 2$. Noting that

$$
\begin{aligned}
E\left(\frac{1}{a_n}\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_{ni}\right|\right)^q &= \frac{1}{a_n^q}\int_0^\infty P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_{ni}\right| > t^{1/q}\right)dt \\
&= \frac{1}{a_n^q}\int_0^{\varepsilon a_n^q} P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_{ni}\right| > t^{1/q}\right)dt \\
&\quad + \frac{1}{a_n^q}\int_{\varepsilon a_n^q}^\infty P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_{ni}\right| > t^{1/q}\right)dt \\
&\leq \varepsilon + \frac{1}{a_n^q}\int_{\varepsilon a_n^q}^\infty P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} Y_{ni}^t\right| > t^{1/q}\right)dt \\
&\quad + \frac{1}{a_n^q}\int_{\varepsilon a_n^q}^\infty \sum_{i=1}^{n} P\left(|X_{ni}| > t^{1/q}\right)dt \\
&\doteq \varepsilon + I_9 + I_{10}. \qquad (3.25)
\end{aligned}
$$

Without loss of generality, assume that $0 < \varepsilon < 1$. For I_{10}, by the c_r inequality and (2.3),

$$
\begin{aligned}
I_{10} &\leq C\sum_{i=1}^{n}\frac{1}{a_n^q}\int_{\varepsilon a_n^q}^\infty P\left(|X_{ni}|\, I\left(\varepsilon a_n^q < |X_{ni}| \leq a_n\right) > t^{1/q}\right)dt \\
&\quad + C\sum_{i=1}^{n}\frac{1}{a_n^q}\int_{\varepsilon a_n^q}^\infty P\left(|X_{ni}|\, I\left(|X_{ni}| > a_n\right) > t^{1/q}\right)dt \\
&\leq C\sum_{i=1}^{n}\frac{1}{a_n^q}E|X_{ni}|^p I\left(\varepsilon a_n^q < |X_{ni}| \leq a_n\right)\int_{\varepsilon a_n^q}^\infty t^{-p/q}dt \\
&\quad + C\sum_{i=1}^{n}\frac{1}{a_n^q}\int_0^\infty P\left(|X_{ni}|\, I\left(|X_{ni}| > a_n\right) > t^{1/q}\right)dt \\
&\leq C\varepsilon^{1-(p/q)}\sum_{i=1}^{n}\frac{E|X_{ni}|^p I\left(|X_{ni}| \leq a_n\right)}{a_n^p} \\
&\quad + C\sum_{i=1}^{n}\frac{1}{a_n^q}E|X_{ni}|^q I\left(|X_{ni}| > a_n\right) \\
&\leq C\sum_{i=1}^{n}\frac{E\psi_i\left(X_{ni}\right)}{\psi_i\left(a_n\right)} \to 0 \quad \text{as } n\to\infty. \qquad (3.26)
\end{aligned}
$$

By $EX_{ni}=0$, (2.3) and (1.2), one has that

$$\begin{aligned}
&\max_{t\geq\varepsilon a_n^q}\frac{1}{t^{1/q}}\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}EY_{ni}^{t}\right|\\
&=\max_{t\geq\varepsilon a_n^q}\frac{1}{t^{1/q}}\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}EZ_{ni}^{t}\right|\\
&\leq C\max_{t\geq a_n^q}\frac{1}{t^{1/q}}\sum_{i=1}^{n}E\left|X_{ni}\right|I\left(\left|X_{ni}\right|>t^{1/q}\right)\\
&\leq C\varepsilon^{-1/q}\sum_{i=1}^{n}\frac{E|X_{ni}|^{q}I\left(|X_{ni}|>a_n\right)}{a_n^q}\\
&\quad+C\varepsilon^{-p/q}\sum_{i=1}^{n}\frac{E|X_{ni}|^{p}I\left(\varepsilon^{1/q}a_n<|X_{ni}|\leq a_n\right)}{a_n^p}\\
&\leq C\left(\varepsilon^{-1/q}+\varepsilon^{-p/q}\right)\sum_{i=1}^{n}\frac{E\psi_i\left(X_{ni}\right)}{\psi_i\left(a_n\right)}\to 0\quad\text{as } n\to\infty. \qquad (3.27)
\end{aligned}$$

Hence for sufficiently large n , $\frac{1}{t^{1/q}}\max\limits_{1\leq j\leq n}\left|\sum\limits_{i=1}^{j}EY_{ni}^{t}\right|\to 0$ holds uniformly for all $t\geq\varepsilon a_n^q$.

Let $d_n=[a_n]+1$, by the Markov inequality, (3.1) and the c_r inequality,

$$\begin{aligned}
I_9&\leq C\sum_{i=1}^{n}\frac{1}{a_n^q}\int_{\varepsilon a_n^q}^{\infty}t^{-2/q}E\big(Y_{ni}^{t}-EY_{ni}^{t}\big)^2dt\\
&\leq C\sum_{i=1}^{n}\frac{1}{a_n^q}\int_{\varepsilon a_n^q}^{\infty}t^{-2/q}EX_{ni}^{2}I\left(|X_{ni}|\leq d_n\right)dt\\
&\quad+C\sum_{i=1}^{n}\frac{1}{a_n^q}\int_{\varepsilon a_n^q}^{\infty}t^{-2/q}EX_{ni}^{2}I\left(d_n<|X_{ni}|\leq t^{1/q}\right)dt\\
&\quad+C\sum_{i=1}^{n}\frac{1}{a_n^q}\int_{\varepsilon a_n^q}^{\infty}P\left(|X_{ni}|>t^{1/q}\right)dt\\
&\doteq I_{91}+I_{92}+I_{93}. \qquad (3.28)
\end{aligned}$$

By the similar argument as those in the proofs of $I_{51}<\infty$, $I_{52}<\infty$ and $I_{10}\to 0$, one can have that $I_{91}\to 0$, $I_{92}\to 0$ and $I_{93}\to 0$.

The proof of (2.4) for $1\leq q<p$ and $p>2$ is similar to that of Theorem 2.4, here omits the detail. The proof of Theorem 2.5 is completed. □

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