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WEIGHTED COMPOSITION OPERATORS ON BERS-TYPE SPACES OF LOO-KENG HUA DOMAINS

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Abstract. Let HE_I, HE_{II}, HE_{III} and HE_{IV} be the first, second, third and fourth type Loo-Keng Hua domain respectively, φ a holomorphic self-map of HE_I, HE_{II}, HE_{III}, or HE_{IV} and $u \in H(\mathcal{M})$ the space of all holomorphic functions on $\mathcal{M} \in \{\mathrm{HE}_I, \mathrm{HE}_{II}, \mathrm{HE}_{III}, \mathrm{HE}_{IV}\}$. In this paper, motivated by the well known Hua's matrix inequality, first some inequalities for the points in the Bers-type spaces of the Loo-Keng Hua domains are obtained, and then the boundedness and compactness of the weighted composition operators $W_{\varphi,u} : f \mapsto u \cdot f \circ \varphi$ on Bers-type spaces of these domains are characterized.

1. Introduction

Let Ω be a domain of $\mathbb{C}^n$ and $H(\Omega)$ the class of all holomorphic functions on Ω. Let φ be a holomorphic self-map of Ω and $u \in H(\Omega)$. The weighted composition operator on some subspaces of $H(\Omega)$ is defined by

$$W_{\varphi,u} f(z) = u(z) f(\varphi(z)),\ z \in \Omega.$$

If $u \equiv 1$, it becomes the composition operator, usually denoted by C_φ. If $\varphi(z) = z$, it becomes the multiplication operator, usually denoted by M_u. A standard problem is to provide function theoretic characterizations when φ and u induce a bounded or compact weighted composition operator. In recent years, there is a great interest in the weighted composition operators on or between spaces of various domains, for example, see [3, 5, 11, 14] for the unit disk, [13, 15, 16, 18] for the unit ball, [12, 19, 20] for the unit polydisk, [1, 2] for the bounded homogeneous domain and [8, 10, 21, 22] for the half-plane.

Now we present some information about the Loo-Keng Hua domains from [24]. It is well-known that the Bergman kernel function plays an important role in several complex variables. But, for which domains can the Bergman kernel function be computed by explicit formulas? This is an important problem. In general, it is difficult to get the domain whose Bergman kernel function can

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be gotten explicitly. In 1998, Yin and Roos constructed a new type of domain called the Cartan-Hartogs domain with explicit Bergman kernel function. Yin generalized continuously them from that time and constructed the four types of domains in 2000, called the Loo-Keng Hua domains. The Loo-Keng Hua domains unify the studies of the symmetric classical domains and Egg domains in the theory of several complex variables. Except some special cases (for example, the unit ball), generally, the Loo-Keng Hua domains are not transitive (see [23]). The first, second, third and fourth type Loo-Keng Hua domains are respectively defined as follows:

$$\begin{aligned}
&\mathrm{HE}_I(N_1,\dots,N_r;m,n;p_1,\dots,p_r)\\
&=\Big\{\xi_j\in\mathbb{C}^{N_j}, Z\in\Re_I(m,n):\sum_{j=1}^r|\xi_j|^{2p_j}<\det(I-Z\overline{Z}^\tau)\Big\},\\
&\mathrm{HE}_{II}(N_1,\dots,N_r;p;p_1,\dots,p_r)\\
&=\Big\{\xi_j\in\mathbb{C}^{N_j}, Z\in\Re_{II}(p):\sum_{j=1}^r|\xi_j|^{2p_j}<\det(I-Z\overline{Z})\Big\},\\
&\mathrm{HE}_{III}(N_1,\dots,N_r;q;p_1,\dots,p_r)\\
&=\Big\{\xi_j\in\mathbb{C}^{N_j}, Z\in\Re_{III}(q):\sum_{j=1}^r|\xi_j|^{2p_j}<\det(I+Z\overline{Z})\Big\},\\
&\mathrm{HE}_{IV}(N_1,\dots,N_r;N;p_1,\dots,p_r)\\
&=\Big\{\xi_j\in\mathbb{C}^{N_j}, z\in\Re_{IV}(N):\sum_{j=1}^r|\xi_j|^{2p_j}<1+|zz^\tau|^2-2z\bar{z}^\tau\Big\},
\end{aligned}$$

where $\xi_j=(\xi_{j1},\dots,\xi_{jN_j})$, $j=1,\dots,r$, $\Re_I(m,n)$, $\Re_{II}(p)$, $\Re_{III}(q)$ and $\Re_{IN}(N)$ denote respectively the Cartan domains of the first type, second type, third type and fourth type in the sense of Loo-Keng Hua, $\overline{Z}$ denotes the conjugate of the matrix Z and Z^τ denotes the transpose of Z, N_1, …, N_r, m, n, p, q, N are positive integers and p_1, …, p_r are positive real numbers. On behalf of the readers, we also present the definitions of the corresponding Cartan domains:

$$\Re_I(m,n)=\Big\{Z\in\mathbb{C}^{m\times n}:I-Z\overline{Z}^\tau>0\Big\},$$

$$\Re_{II}(p)=\Big\{Z\in\mathbb{C}^{p\times p}:I-Z\overline{Z}>0,Z=Z^\tau\Big\},$$

$$\Re_{III}(q)=\Big\{Z\in\mathbb{C}^{q\times q}:I+Z\overline{Z}>0,Z=-Z^\tau\Big\},$$

$$\Re_{IV}(N)=\Big\{z\in\mathbb{C}^N:1+|zz^\tau|^2-2z\bar{z}^\tau>0,1-|zz^\tau|^2>0\Big\}.$$

Let $\alpha>0$ and $\mathbb{B}=\{z\in\mathbb{C}^n:|z|<1\}$ be the open unit ball of $\mathbb{C}^n$. The well known Bers-type space on $\mathbb{B}$, usually denoted by $\mathcal{A}_\alpha(\mathbb{B})$, consists of all

$f \in H(\mathbb{B})$ such that

$$\|f\| = \sup_{z\in\mathbb{B}}(1-|z|^2)^{\alpha}|f(z)| < +\infty.$$

Under the norm $\|f\|$, $\mathcal{A}_\alpha(\mathbb{B})$ is a Banach space. For the Bers-type spaces and some concrete operators on them, see, for example, [6, 7, 9, 17, 25] and the references therein.

For the sake of convenience, four types of Loo-Keng Hua domains are abbreviated to HE_I, HE_{II}, HE_{III} and HE_{IV} respectively. The Bers-type space on HE_I, denoted by $\mathcal{A}_\alpha(\mathrm{HE}_I)$, consists of all $f \in H(\mathrm{HE}_I)$ such that

$$\|f\| = \sup_{(Z,\xi)\in\mathrm{HE}_I}\big[\det(I - Z\overline{Z}^{\tau}) - \sum_{j=1}^{r}|\xi_j|^{2p_j}\big]^{\alpha}\big|f(Z,\xi)\big| < +\infty.$$

It is easy to see that $\mathcal{A}_\alpha(\mathrm{HE}_I)$ is a Banach space under the norm $\|f\|$.

The Bers-type spaces can be similarly defined on HE_{II}, HE_{III} and HE_{IV} respectively. On HE_{II} the Bers-type space $\mathcal{A}_\alpha(\mathrm{HE}_{II})$ consists of all $f \in H(\mathrm{HE}_{II})$ such that

$$\|f\| = \sup_{(Z,\xi)\in\mathrm{HE}_{II}}\big[\det(I - Z\overline{Z}) - \sum_{j=1}^{r}|\xi_j|^{2p_j}\big]^{\alpha}\big|f(Z,\xi)\big| < +\infty.$$

On HE_{III} the Bers-type space $\mathcal{A}_\alpha(\mathrm{HE}_{III})$ consists of all $f \in H(\mathrm{HE}_{III})$ such that

$$\|f\| = \sup_{(Z,\xi)\in\mathrm{HE}_{III}}\big[\det(I + Z\overline{Z}) - \sum_{j=1}^{r}|\xi_j|^{2p_j}\big]^{\alpha}|f(Z,\xi)| < +\infty.$$

On HE_{IV} the Bers-type space $\mathcal{A}_\alpha(\mathrm{HE}_{IV})$ consists of all $f \in H(\mathrm{HE}_{IV})$ such that

$$\|f\| = \sup_{(z,\xi)\in\mathrm{HE}_{IV}}\big(1 + |zz^{\tau}|^2 - 2z\bar{z}^{\tau} - \sum_{j=1}^{r}|\xi_j|^{2p_j}\big)^{\alpha}|f(z,\xi)| < +\infty.$$

In this paper, motivated by the well known Hua's matrix inequality, we first obtain some inequalities for the points in the Bers-type spaces of the Loo-Keng Hua domains, and then we characterize the boundedness and compactness of the weighted composition operators on the Bers-type spaces for the first, second and third type Loo-Keng Hua domains. For the fourth type Loo-Keng Hua domains, we consider the same problems for the case only when $N = 1$.

Without loss of generality, suppose that $N_j = 1$, that is, $\xi_j \in \mathbb{C}$, $j = 1, 2, \ldots, r$, $\xi = (\xi_1, \ldots, \xi_r)$ and $\|\xi\|^2 = \sum_{j=1}^{r}|\xi_j|^{2p_j}$. Constants are denoted by C, they are positive and may differ from one occurrence to the next.

2. Auxiliary results

First we have the following easy result from the corresponding definitions.

Lemma 2.1. *Let $\alpha > 0$. The following statements hold.*

(i) *For each $(Z,\xi) \in HE_I$ and $f \in \mathcal{A}_\alpha(HE_I)$, it follows that*

$$|f(Z,\xi)| \le \frac{\|f\|}{\left[\det(I - Z\overline{Z}^\tau) - \|\xi\|^2\right]^\alpha}.$$

(ii) *For each $(Z,\xi) \in HE_{II}$ and $f \in \mathcal{A}_\alpha(HE_{II})$, it follows that*

$$|f(Z,\xi)| \le \frac{\|f\|}{\left[\det(I - Z\overline{Z}) - \|\xi\|^2\right]^\alpha}.$$

(iii) *For each $(Z,\xi) \in HE_{III}$ and $f \in \mathcal{A}_\alpha(HE_{III})$, it follows that*

$$|f(Z,\xi)| \le \frac{\|f\|}{\left[\det(I + Z\overline{Z}) - \|\xi\|^2\right]^\alpha}.$$

(iv) *For each $(z,\xi) \in HE_{IV}$ and $f \in \mathcal{A}_\alpha(HE_{IV})$, it follows that*

$$|f(z,\xi)| \le \frac{\|f\|}{\left(1 + |zz^\tau|^2 - 2z\bar{z}^\tau - \|\xi\|^2\right)^\alpha}.$$

In order to characterize the compactness, we need the following result which is similar to Proposition 3.11 in [4].

Lemma 2.2. *Let $\alpha > 0$ and $\mathcal{M} \in \{HE_I, HE_{II}, HE_{III}, HE_{IV}\}$. Then the bounded operator $W_{\varphi,u}$ on $\mathcal{A}_\alpha(\mathcal{M})$ is compact if and only if for every bounded sequence $\{f_k\}$ in $\mathcal{A}_\alpha(\mathcal{M})$ such that $f_k \to 0$ uniformly on every compact subset of $\mathcal{M}$ as $k \to \infty$, it follows that*

$$\lim_{k\to\infty} \|W_{\varphi,u} f_k\| = 0.$$

Proof. We only prove this result for the case HE_I. Suppose that the bounded operator $W_{\varphi,u}$ on $\mathcal{A}_\alpha(\mathrm{HE}_I)$ is compact. Let $\{f_k\}$ be a bounded sequence in $\mathcal{A}_\alpha(\mathrm{HE}_I)$ such that $f_k \to 0$ uniformly on every compact subset of HE_I as $k \to \infty$. If $\|W_{\varphi,u} f_k\| \nrightarrow 0$ as $k \to \infty$, then there exists a subsequence $\{f_{k_j}\}$ of $\{f_k\}$ such that

$$\inf_{j\in\mathbb{N}} \|W_{\varphi,u} f_{k_j}\| > 0. \tag{1}$$

Since $W_{\varphi,u}$ is compact on $\mathcal{A}_\alpha(\mathrm{HE}_I)$, there exist a function $g \in \mathcal{A}_\alpha(\mathrm{HE}_I)$ and a subsequence of $\{f_{k_j}\}$ (without loss of generality, still written by $\{f_{k_j}\}$), such that

$$\lim_{j\to\infty} \|W_{\varphi,u} f_{k_j} - g\| = 0.$$

Let K be a compact subspace of HE_I. From Lemma 2.1 it follows that

$$\left|(W_{\varphi,u} f_{k_j} - g)(Z,\xi)\right| \le \frac{\|W_{\varphi,u} f_{k_j} - g\|}{\left[\det(I - Z\overline{Z}^\tau) - \|\xi\|^2\right]^\alpha} \tag{2}$$

for $(Z,\xi) \in K$. From (2), we see that $W_{\varphi,u} f_{k_j} - g \to 0$ uniformly on K as $j \to \infty$. From this, for arbitrary $\varepsilon > 0$, there exists a positive integer N_1 such that

$$\big|u(Z,\xi) f_{k_j}(\varphi(Z,\xi)) - g(Z,\xi)\big| < \varepsilon \tag{3}$$

for all $(Z,\xi) \in \mathrm{HE}_I$, whenever $j > N_1$. Since $f_{k_j} \to 0$ uniformly on K as $j \to \infty$, also there exists a positive integer N_2 such that $|f_{k_j}(Z,\xi)| < \varepsilon$ for all $(Z,\xi) \in K$, whenever $j > N_2$. Let $N = \max\{N_1, N_2\}$ and $M = \max_{(Z,\xi)\in K} |u(Z,\xi)|$. From (3), we have

$$\begin{aligned} |g(Z,\xi)| &\le |f_{k_j}(\varphi(Z,\xi))| \max_{(Z,\xi)\in K} |u(Z,\xi)| + \varepsilon \\ &= (1+M)\varepsilon \end{aligned} \tag{4}$$

for all $(Z,\xi) \in K$, whenever $j > N$. From (4) and the arbitrariness of ε, we obtain $g(Z,\xi) \equiv 0$ on K, which leads to $g \equiv 0$ on HE_I. This shows that

$$\lim_{j\to\infty} \|W_{\varphi,u} f_{k_j}\| = 0,$$

which contradicts (1).

Now suppose that $\{f_k\}$ is a bounded sequence in $\mathcal{A}_\alpha(\mathrm{HE}_I)$. Then it is locally uniform bounded on HE_I, which shows that there exists a subsequence $\{f_{k_j}\}$ of $\{f_k\}$ such that $f_{k_j} \to f$ uniformly on every compact subset of HE_I as $j \to \infty$. From this we have $f_{k_j} - f \to 0$ uniformly on every compact subset of HE_I as $j \to \infty$. Consequently, we obtain

$$\lim_{j\to\infty} \|W_{\varphi,u}(f_{k_j} - f)\| = \lim_{j\to\infty} \|W_{\varphi,u} f_{k_j} - W_{\varphi,u} f\| = 0,$$

which shows that $W_{\varphi,u}$ is compact on $\mathcal{A}_\alpha(\mathrm{HE}_I)$. □

The Hermitian or positive definite matrix is well known in linear algebra, so here we omit their definitions.

Lemma 2.3. *The following statements hold.*
(i) *If $Z \in \Re_I(m,n)$, then $I - Z\overline{Z}^\tau$ is Hermitian and positive definite.*
(ii) *If $Z \in \Re_{II}(p)$, then $I - Z\overline{Z}$ is Hermitian and positive definite.*
(iii) *If $Z \in \Re_{III}(q)$, then $I + Z\overline{Z}$ is Hermitian and positive definite.*

Proof. Here we only prove (i). It is obvious that $I - Z\overline{Z}^\tau$ is positive definite. Since

$$\overline{I - Z\overline{Z}^\tau}^\tau = (I - \overline{Z} Z^\tau)^\tau = I - Z\overline{Z}^\tau,$$

$I - Z\overline{Z}^\tau$ is Hermitian. □

In the studies of the several complex variables, Loo-Keng Hua found the following so-called Hua's matrix inequality in 1955.

Hua's matrix inequality:

$$\det(I - A\overline{A}^\tau)\det(I - B\overline{B}^\tau) \le \big|\det(I - A\overline{B}^\tau)\big|^2, \tag{5}$$

where $I-A\overline{A}^{\tau}$ and $I-B\overline{B}^{\tau}$ are Hermitian and positive definite.

By using the Hua's matrix inequality (5), we obtain the following result.

Lemma 2.4. *The following statements hold.*

(i) *If (Z,ξ), $(S,t)\in HE_I$, then*

$$\big[\det(I-Z\overline{Z}^{\tau})-\|\xi\|^2\big]\big[\det(I-S\overline{S}^{\tau})-\|t\|^2\big]\le\big[|\det(I-Z\overline{S}^{\tau})|-\|\xi\|\|t\|\big]^2.$$

(ii) *If (Z,ξ), $(S,t)\in HE_{II}$, then*

$$\big[\det(I-Z\overline{Z})-\|\xi\|^2\big]\big[\det(I-S\overline{S})-\|t\|^2\big]\le\big[|\det(I-Z\overline{S})|-\|\xi\|\|t\|\big]^2.$$

(iii) *If (Z,ξ), $(S,t)\in HE_{III}$, then*

$$\big[\det(I+Z\overline{Z})-\|\xi\|^2\big]\big[\det(I+S\overline{S})-\|t\|^2\big]\le\big[|\det(I+Z\overline{S})|-\|\xi\|\|t\|\big]^2.$$

Proof. Here we only prove (i), because the rest can be proved similarly. Suppose that a, b, c, d are nonnegative real numbers with $b\le a$ and $d\le c$. Then it is obvious that

$$(a^2-b^2)(c^2-d^2)\le(ac-bd)^2. \tag{6}$$

From (6) and Hua's matrix inequality (5), we obtain

$$\begin{aligned}&\big[\det(I-Z\overline{Z}^{\tau})-\|\xi\|^2\big]\big[\det(I-S\overline{S}^{\tau})-\|t\|^2\big]\\ \le&\Big\{\big[\det(I-Z\overline{Z}^{\tau})\big]^{\frac12}\big[\det(I-S\overline{S}^{\tau})\big]^{\frac12}-\|\xi\|\|t\|\Big\}^2\\ \le&\big[|\det(I-Z\overline{S}^{\tau})|-\|\xi\|\|t\|\big]^2.\end{aligned}$$

This finishes the proof. □

Remark 2.1. In the proof of Lemma 2.4, we have seen that, if (Z,ξ), $(S,t)\in$ HE$_I$, then $\|\xi\|\|t\|\le\big|\det(I-Z\overline{S}^{\tau})\big|$. By the way, from an easy calculation we also have

$$\Big|\sum_{j=1}^{r}\xi_j^{p_j}\bar{t}_j^{\,p_j}\Big|\le\|\xi\|\|t\|.$$

As an application of Lemma 2.4, we obtain some special functions in $\mathcal{A}_\alpha(\mathcal{M})$ where $\mathcal{M}\in\{\text{HE}_I,\text{HE}_{II},\text{HE}_{III}\}$.

Lemma 2.5. *Let $\alpha>0$. Then the following statements hold.*

(i) *If $(S,t)\in HE_I$, then the function*

$$f_{(S,t)}(Z,\xi)=\frac{\big[\det(I-S\overline{S}^{\tau})-\|t\|^2\big]^{\alpha}}{\big[\det(I-Z\overline{S}^{\tau})-\sum\limits_{j=1}^{r}\xi_j^{p_j}\bar{t}_j^{\,p_j}\big]^{2\alpha}}$$

belongs to $\mathcal{A}_\alpha(HE_I)$, and $\|f_{(S,t)}\|\le1$ for all $(S,t)\in HE_I$.

(ii) *If* $(S,t) \in HE_{II}$*, then the function*

$$g_{(S,t)}(Z,\xi) = \frac{\big[\det(I - S\overline{S}) - \|t\|^2\big]^{\alpha}}{\big[\det(I - Z\overline{S}) - \sum_{j=1}^{r} \xi_j^{p_j} \bar{t}_j^{\,p_j}\big]^{2\alpha}}$$

belongs to $\mathcal{A}_\alpha(HE_{II})$*, and* $\|g_{(S,t)}\| \le 1$ *for all* $(S,t) \in HE_{II}$*.*

(iii) *If* $(S,t) \in HE_{III}$*, then the function*

$$h_{(S,t)}(Z,\xi) = \frac{\big[\det(I + S\overline{S}) - \|t\|^2\big]^{\alpha}}{\big[\det(I + Z\overline{S}) - \sum_{j=1}^{r} \xi_j^{p_j} \bar{t}_j^{\,p_j}\big]^{2\alpha}}$$

belongs to $\mathcal{A}_\alpha(HE_{III})$*, and* $\|h_{(S,t)}\| \le 1$ *for all* $(S,t) \in HE_{III}$*.*

Proof. Here we only prove (i). From a direct calculation and Lemma 2.4, we have

$$\begin{aligned}
&\big[\det(I - Z\overline{Z}^{\tau}) - \|\xi\|^2\big]^{\alpha} |f_{(S,t)}(Z,\xi)| \\
&= \big[\det(I - Z\overline{Z}^{\tau}) - \|\xi\|^2\big]^{\alpha} \frac{\big[\det(I - S\overline{S}^{\tau}) - \|t\|^2\big]^{\alpha}}{\big|\det(I - Z\overline{S}^{\tau}) - \sum_{j=1}^{r} \xi_j^{p_j} \bar{t}_j^{\,p_j}\big|^{2\alpha}} \\
&\le \big[\det(I - Z\overline{Z}^{\tau}) - \|\xi\|^2\big]^{\alpha} \frac{\big[\det(I - S\overline{S}^{\tau}) - \|t\|^2\big]^{\alpha}}{\big[|\det(I - Z\overline{S}^{\tau})| - |\sum_{j=1}^{r} \xi_j^{p_j} \bar{t}_j^{\,p_j}|\big]^{2\alpha}} \\
&\le \big[\det(I - Z\overline{Z}^{\tau}) - \|\xi\|^2\big]^{\alpha} \frac{\big[\det(I - S\overline{S}^{\tau}) - \|t\|^2\big]^{\alpha}}{\big[\big|\det(I - Z\overline{S}^{\tau})\big| - \|\xi\|\|t\|\big]^{2\alpha}} \\
&\le 1,
\end{aligned}$$

from which the desired result follows. □

3. Main results and proofs

Suppose that φ is a holomorphic self-map of $\mathcal{M} \in \{\mathrm{HE}_I, \mathrm{HE}_{II}, \mathrm{HE}_{III}\}$. Let us write $(W,\eta) = \varphi(Z,\xi)$ for $(Z,\xi) \in \mathcal{M}$.

Theorem 3.1. *Let* $\alpha > 0$*,* φ *the holomorphic self-map of* $\mathcal{M}$ *and* $u \in H(\mathcal{M})$*. Then the following statements hold.*

(i) *The operator* $W_{\varphi,u}$ *is bounded on* $\mathcal{A}_\alpha(HE_I)$ *if and only if*

$$M_I := \sup_{(Z,\xi)\in HE_I} |u(Z,\xi)| \frac{\big[\det(I - Z\overline{Z}^{\tau}) - \|\xi\|^2\big]^{\alpha}}{\big[\det(I - W\overline{W}^{\tau}) - \|\eta\|^2\big]^{\alpha}} < +\infty.$$

(ii) *The operator* $W_{\varphi,u}$ *is bounded on* $\mathcal{A}_\alpha(HE_{II})$ *if and only if*

$$M_{II} := \sup_{(Z,\xi)\in HE_{II}} |u(Z,\xi)| \frac{\big[\det(I - Z\overline{Z}) - \|\xi\|^2\big]^{\alpha}}{\big[\det(I - W\overline{W}) - \|\eta\|^2\big]^{\alpha}} < +\infty.$$

(iii) *The operator $W_{\varphi,u}$ is bounded on $\mathcal{A}_\alpha(HE_{III})$ if and only if*

$$M_{III} := \sup_{(Z,\xi)\in HE_{III}} |u(Z,\xi)| \frac{\big[\det(I+Z\overline{Z})-\|\xi\|^2\big]^\alpha}{\big[\det(I+W\overline{W})-\|\eta\|^2\big]^\alpha} < +\infty.$$

Proof. Here we only prove the statement (i). Suppose that $W_{\varphi,u}$ is bounded on $\mathcal{A}_\alpha(\mathrm{HE}_I)$. Then for each $f \in \mathcal{A}_\alpha(\mathrm{HE}_I)$, there exists a positive constant C such that

$$\|W_{\varphi,u} f\| \le C\|f\|. \tag{7}$$

For the fixed point $(S,t) \in \mathrm{HE}_I$, choose the function

$$f_{(S,t)}(Z,\xi) = \frac{\big[\det(I - A\overline{A}^\tau) - \|\zeta\|^2\big]^\alpha}{\big[\det(I - Z\overline{A}^\tau) - \sum_{j=1}^{r} \xi_j^{p_j}\bar{\zeta}_j^{\,p_j}\big]^{2\alpha}},$$

where $(A,\zeta) = \varphi(S,t)$. From Lemma 2.5, it follows that $f_{(S,t)} \in \mathcal{A}_\alpha(\mathrm{HE}_I)$ and $\|f_{(S,t)}\| \le 1$. Applying the boundedness of $W_{\varphi,u}$ on $\mathcal{A}_\alpha(\mathrm{HE}_I)$ to $f_{(S,t)}$ and (7), we have

$$\begin{aligned}
&\big[\det(I - S\overline{S}^\tau) - \|t\|^2\big]^\alpha \big|W_{\varphi,u} f_{(S,t)}(S,t)\big| \\
&= \big[\det(I - S\overline{S}^\tau) - \|t\|^2\big]^\alpha \big|u(S,t) f_{(S,t)}(\varphi(S,t))\big| \\
&= \frac{\big[\det(I - S\overline{S}^\tau) - \|t\|^2\big]^\alpha}{\big[\det(I - A\overline{A}^\tau) - \|\zeta\|^2\big]^\alpha} |u(S,t)| \\
&\le \|W_{\varphi,u} f_{(S,t)}\| \le C\|f_{(S,t)}\| \le C,
\end{aligned}$$

which shows that

$$\sup_{(S,t)\in \mathrm{HE}_I} |u(S,t)| \frac{\big[\det(I - S\overline{S}^\tau) - \|t\|^2\big]^\alpha}{\big[\det(I - A\overline{A}^\tau) - \|\zeta\|^2\big]^\alpha} < +\infty,$$

that is, $M_I < +\infty$.

Conversely, by Lemma 2.1, for all $f \in \mathcal{A}_\alpha(\mathrm{HE}_I)$ we have

$$\begin{aligned}
&\sup_{(Z,\xi)\in \mathrm{HE}_I} \big[\det(I - Z\overline{Z}^\tau) - \|\xi\|^2\big]^\alpha \big|W_{\varphi,u} f(Z,\xi)\big| \\
&= \sup_{(Z,\xi)\in \mathrm{HE}_I} \big[\det(I - Z\overline{Z}^\tau) - \|\xi\|^2\big]^\alpha \big|u(Z,\xi) f(\varphi(Z,\xi))\big| \\
&\le \sup_{(Z,\xi)\in \mathrm{HE}_I} |u(Z,\xi)| \frac{\big[\det(I - Z\overline{Z}^\tau) - \|\xi\|^2\big]^\alpha}{\big[\det(I - W\overline{W}^\tau) - \|\eta\|^2\big]^\alpha} \|f\| \\
&= M_I \|f\|. \qquad (8)
\end{aligned}$$

From (8), we see that $W_{\varphi,u}$ is bounded on $\mathcal{A}_\alpha(\mathrm{HE}_I)$. □

Next we characterize the compactness of the operator $W_{\varphi,u}$ on $\mathcal{A}_\alpha(\mathcal{M})$.

Theorem 3.2. *Let $\alpha > 0$, φ the holomorphic self-map of $\mathcal{M}$ and $u \in H(\mathcal{M})$. Then the following statements hold.*

(i) *The operator $W_{\varphi,u}$ is compact on $\mathcal{A}_\alpha(HE_I)$ if and only if*

$$\lim_{\varphi(Z,\xi)\to\partial HE_I} |u(Z,\xi)| \frac{\big[\det(I - Z\overline{Z}^\tau) - \|\xi\|^2\big]^\alpha}{\big[\det(I - W\overline{W}^\tau) - \|\eta\|^2\big]^\alpha} = 0. \tag{9}$$

(ii) *The operator $W_{\varphi,u}$ is compact on $\mathcal{A}_\alpha(HE_{II})$ if and only if*

$$\lim_{\varphi(Z,\xi)\to\partial HE_{II}} |u(Z,\xi)| \frac{\big[\det(I - Z\overline{Z}) - \|\xi\|^2\big]^\alpha}{\big[\det(I - W\overline{W}) - \|\eta\|^2\big]^\alpha} = 0.$$

(iii) *The operator $W_{\varphi,u}$ is compact on $\mathcal{A}_\alpha(HE_{III})$ if and only if*

$$\lim_{\varphi(Z,\xi)\to\partial HE_{III}} |u(Z,\xi)| \frac{\big[\det(I + Z\overline{Z}) - \|\xi\|^2\big]^\alpha}{\big[\det(I + W\overline{W}) - \|\eta\|^2\big]^\alpha} = 0.$$

Proof. We also only prove the statement (i). Suppose that the operator $W_{\varphi,u}$ is compact on $\mathcal{A}_\alpha(\mathrm{HE}_I)$. Then it is clear that the operator $W_{\varphi,u}$ is bounded on $\mathcal{A}_\alpha(\mathrm{HE}_I)$. Consider a sequence $\{(A_i,\zeta_i)\} = \{\varphi(S_i,t_i)\}$ in HE_I such that $\varphi(S_i,t_i) \to \partial\mathrm{HE}_I$ as $i \to \infty$. If such a sequence does not exist, then condition (9) obviously holds. Using this sequence, we define the function sequence $f_i(Z,\xi) = f_{(S_i,t_i)}(Z,\xi)$, where $f_{(S_i,t_i)}$ is the function $f_{(S,t)}$ replaced (S,t) by (S_i,t_i) in the proof of Theorem 3.1. By Lemma 2.5, we see that the sequence $\{f_i\}$ is uniformly bounded in $\mathcal{A}_\alpha(\mathrm{HE}_I)$, and $f_i \to 0$ uniformly on any compact subset of HE_I as $i \to \infty$. So by Lemma 2.2,

$$\lim_{i\to\infty} \|W_{\varphi,u} f_i\| = 0.$$

From this and a direct calculation, we have

$$\lim_{i\to\infty} |u(S_i,t_i)| \frac{\big[\det(I - S_i\overline{S_i}^\tau) - \|t_i\|^2\big]^\alpha}{\big[\det(I - A_i\overline{A_i}^\tau) - \|\zeta_i\|^2\big]^\alpha} = 0.$$

Conversely, in order to prove that the operator $W_{\varphi,u}$ is compact on $\mathcal{A}_\alpha(\mathrm{HE}_I)$, by Lemma 2.2 we only need to prove that, if $\{f_i\}$ is a sequence in $\mathcal{A}_\alpha(\mathrm{HE}_I)$ such that $\sup_{i\in\mathbb{N}} \|f_i\| \le M$ and $f_i \to 0$ uniformly on any compact subset of HE_I as $i \to \infty$, then

$$\lim_{i\to\infty} \|W_{\varphi,u} f_i\| = 0.$$

We first observe that the condition (9) implies that for every $\varepsilon > 0$, there exists $\sigma > 0$ such that for any $(Z,\xi) \in K = \{(Z,\xi) \in \mathrm{HE}_I : \mathrm{dist}(\varphi(Z,\xi), \partial\mathrm{HE}_I) < \sigma\}$ it follows that

$$|u(Z,\xi)| \frac{\big[\det(I - Z\overline{Z}^\tau) - \|\xi\|^2\big]^\alpha}{\big[\det(I - W\overline{W}^\tau) - \|\eta\|^2\big]^\alpha} < \varepsilon. \tag{10}$$

For such ε and σ, by using (10) and Lemma 2.1, we have

$$\|W_{\varphi,u} f_i\| = \sup_{(Z,\xi)\in\mathrm{HE}_I} \big[\det(I - Z\overline{Z}^\tau) - \|\xi\|^2\big]^\alpha \big|u(Z,\xi) f_i(\varphi(Z,\xi))\big|$$

$$\le \Big(\sup_{(Z,\xi)\in K} + \sup_{(Z,\xi)\in \mathrm{HE}_I\setminus K}\Big)\big[\det(I-Z\overline{Z}^{\tau})-\|\xi\|^2\big]^{\alpha}\big|u(Z,\xi)f_i(\varphi(Z,\xi))\big|$$

$$\le M\varepsilon + \sup_{(Z,\xi)\in \mathrm{HE}_I\setminus K}\big[\det(I-Z\overline{Z}^{\tau})-\|\xi\|^2\big]^{\alpha}\big|u(Z,\xi)f_i(\varphi(Z,\xi))\big|$$

$$\le M\varepsilon + \sup_{(Z,\xi)\in \mathrm{HE}_I\setminus K}\big[\det(I-Z\overline{Z}^{\tau})-\|\xi\|^2\big]^{\alpha}\big|u(Z,\xi)\big|$$

$$\sup_{(Z,\xi)\in \mathrm{HE}_I\setminus K}\big|f_i(\varphi(Z,\xi))\big|. \tag{11}$$

Since $\{(Z,\xi)\in \mathrm{HE}_I\setminus K\}$ is a compact subset of HE_I, $f_i\to 0$ uniformly on this set as $i\to\infty$. From this and (11) we get

$$\lim_{i\to\infty}\|W_{\varphi,u}f_i\|=0,$$

which shows that the operator $W_{\varphi,u}$ is compact on $\mathcal{A}_\alpha(\mathrm{HE}_I)$. □

Remark 3.1. From the above studies, we have seen that Hua's matrix inequality plays an important role in constructing special functions of the Bers-type spaces for HE_I, HE_{II} and HE_{III}. So, in order to obtain the similar functions in $\mathcal{A}_\alpha(\mathrm{HE}_{IV})$, we need to look for a similar inequality in $\Re_{IV}(N)$. Unfortunately, although we have spent lots of time in looking for such inequality, we still don't obtain any useful result. This is why we consider the case only when $N=1$ for the fourth type Loo-Keng Hua domains. We will continue to consider this problem in the next research project, and we also hope that it can attract people's attention.

Since it follows that $1+|zz^{\tau}|^2-2z\bar{z}^{\tau}=(1-|z|^2)^2$ when $N=1$, the function f belongs to $\mathcal{A}_\alpha(\mathrm{HE}_{IV}(1))$ if and only if

$$\sup_{(z,\xi)\in \mathrm{HE}_{IV}}\big[(1-|z|^2)^2-\|\xi\|^2\big]^{\alpha}|f(z,\xi)|<+\infty. \tag{12}$$

Motivated by (12), we have the following result to obtain some special functions.

Lemma 3.1. *If $z,s\in\Re_{IV}(1)$, then*

$$(1-|z|^2)(1-|s|^2)\le|1-\bar{s}z|^2. \tag{13}$$

Proof. Letting $z=r_1e^{i\theta_1}$, $s=r_2e^{i\theta_2}$, we have

$$1-|z|^2=1-r_1^2,\ 1-|s|^2=1-r_2^2, \tag{14}$$

and

$$|1-\bar{s}z|^2=\big|1-r_1r_2e^{i(\theta_1-\theta_2)}\big|^2$$

$$=\big[1-r_1r_2\cos(\theta_1-\theta_2)\big]^2+r_1^2r_2^2\sin^2(\theta_1-\theta_2)$$

$$=1+r_1^2r_2^2-2r_1r_2\cos(\theta_1-\theta_2). \tag{15}$$

Since it is obvious that

$$r_1^2+r_2^2\ge 2r_1r_2\cos(\theta_1-\theta_2),$$

from (14) and (15) it follows that

$$(1-r_1^2)(1-r_2^2) = 1 - r_2^2 - r_1^2 + r_1^2 r_2^2 \le 1 + r_1^2 r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2),$$

from which (13) follows. □

Remark 3.2. In fact, Lemma 3.1 holds, if z, $s \in \mathbb{C}$ and $|z|$, $|s| < 1$.

By using (13), we have the following result for the points in $\mathrm{HE}_{IV}(1)$.

Lemma 3.2. *If (z,ξ), $(s,t) \in HE_{IV}(1)$, then*

$$\left[(1-|z|^2)^2 - \|\xi\|^2\right]\left[(1-|s|^2)^2 - \|t\|^2\right] \le \left(\left|1-\bar{s}z\right|^2 - \|\xi\|\|t\|\right)^2.$$

Proof. Suppose that a, b, c, d are nonnegative real numbers with $b \le a$ and $d \le c$. Then $(a^2-b^2)(c^2-d^2) \le (ac-bd)^2$, from which and the inequality (13) we obtain

$$\begin{aligned}[(1-|z|^2)^2 - \|\xi\|^2][(1-|s|^2)^2 - \|t\|^2] &\le \left[(1-|z|^2)(1-|s|^2) - \|\xi\|\|t\|\right]^2 \\ &\le \left(|1-\bar{s}z|^2 - \|\xi\|\|t\|\right)^2.\end{aligned}$$

This finishes the proof. □

The following three results can be obtained similarly, so the proofs are omitted.

Lemma 3.3. *Let $\alpha > 0$ and $(s,t) \in HE_{IV}(1)$. Then the function*

$$k_{(s,t)}(z,\xi) = \frac{\left[(1-|s|^2)^2 - \|t\|^2\right]^\alpha}{\left[(1-\bar{s}z)^2 - \sum_{j=1}^{r} \xi_j^{p_j} \bar{t}_j^{\,p_j}\right]^{2\alpha}}$$

belongs to $\mathcal{A}_\alpha(HE_{IV}(1))$ and $\|k_{(s,t)}\| \le 1$ for all $(s,t) \in HE_{IV}(1)$.

For φ the holomorphic self-map of $\mathrm{HE}_{IV}(1)$, we write $(w,\eta) = \varphi(z,\xi)$ for $(z,\xi) \in \mathrm{HE}_{IV}(1)$.

Theorem 3.3. *Let $\alpha > 0$, φ the holomorphic self-map of $HE_{IV}(1)$ and $u \in H(HE_{IV}(1))$. Then the operator $W_{\varphi,u}$ is bounded on $\mathcal{A}_\alpha(HE_{IV}(1))$ if and only if*

$$M_{IV} := \sup_{(z,\xi)\in HE_{IV}(1)} |u(z,\xi)| \frac{\left[(1-|z|^2)^2 - \|\xi\|^2\right]^\alpha}{\left[(1-|w|^2)^2 - \|\eta\|^2\right]^\alpha} < +\infty.$$

Theorem 3.4. *Let $\alpha > 0$, φ the holomorphic self-map of $HE_{IV}(1)$ and $u \in H(HE_{IV}(1))$. Then the operator $W_{\varphi,u}$ is compact on $\mathcal{A}_\alpha(HE_{IV}(1))$ if and only if*

$$\lim_{\varphi(z,\xi)\to\partial HE_{IV}(1)} |u(z,\xi)| \frac{\left[(1-|z|^2)^2 - \|\xi\|^2\right]^\alpha}{\left[(1-|w|^2)^2 - \|\eta\|^2\right]^\alpha} = 0.$$

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