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POSITIVELY WEAK MEASURE EXPANSIVE DIFFERENTIABLE MAPS

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Abstract. In this paper, we introduce the new general concept of usual expansiveness which is called "positively weak measure expansiveness" and study the basic properties of positively weak measure expansive C^1-differentiable maps on a compact smooth manifold M. And we prove that the following theorems.

(1) Let $\mathcal{PWE}$ be the set of all positively weak measure expansive differentiable maps of M. Denote by $\mathrm{int}(\mathcal{PWE})$ is a C^1-interior of $\mathcal{PWE}$. $f \in \mathrm{int}(\mathcal{PWE})$ if and only if f is expanding.
(2) For C^1-*generic* $f \in C^1(M)$, f is positively weak measure-expansive if and only if f is expanding.

1. Introduction

Let M be a compact C^∞ Riemannian manifold without boundary and $f : M \to M$ be a diffeomorphism. Denote by d the distance on M induced from the Riemannian metric $\|\cdot\|$ on the tangent bundle TM.

In the middle of 20th century, the notion of expansiveness was introduced by Utz [16]. After that there are many attempts to generalization of concept of expansiveness.

In [16], a diffeomorphism f is *expansive* if there is $e > 0$ such that for any $x, y \in M$ if $d(f^i(x), f^i(y)) < e$ for all $i \in \mathbb{Z}$, then $x = y$. Note that if a diffeomorphism f is expansive, then $\Gamma_e(x) = \{x\}$ for $x \in M$, where $\Gamma_e(x) = \{y \in M : d(f^i(x), f^i(y)) < e \text{ for all } i \in \mathbb{Z}\}$. Morales [11] introduced a general concept of expansiveness which is called *measure expansiveness*. For a Borel probability measure μ on M, we say that f is μ-expansive if there is $e > 0$ such that $\mu(\Gamma_e(x)) = 0$ for all $x \in M$. We say that f is *measure expansive* if it is μ-expansive for every non-atomic Borel probability measure μ on M.

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Very recently, Ahn *et al.* [1] introduce weak measure expansiveness which is a concept generalizing the notion of measure expansiveness. For any $\delta > 0$, a finite collection $\mathrm{P} = \{A_1, A_2, \dots, A_n\}$ of subsets of M is a *finite δ-partition* of M if for all $i = 1, 2, \dots, n$, A_i's are disjoint, $\bigcup_{i=1}^{n} A_i = M$, each A_i is measurable, $\mathrm{int}(A_i) \neq \emptyset$ and $\mathrm{diam} A_i \leq \delta$. Here, $\mathrm{diam} A$ means the diameter of $A \subset M$. However, for any $\delta > 0$, the presence of finite δ-partition is clear by the compactness of M. So we will omit δ except when necessary for convenience (for more details, see [1]).

Let μ be a Borel probability measure. A diffeomorphism $f : M \to M$ is said to be *weak μ-expansive* if there is a finite partition $\mathrm{P} = \{A_1, A_2, \dots, A_n\}$ of M such that $\mu(\Gamma^f_{\mathrm{P}}(x)) = 0$ for all $x \in M$, where

$$\Gamma^f_{\mathrm{P}}(x) = \{y \in M : f^i(y) \in \mathrm{P}(f^i(x)) \text{ for all } i \in \mathbb{Z}\},$$

and $\mathrm{P}(x)$ means that the element of P containing x. The set $\Gamma^f_{\mathrm{P}}(x)$ is called the *dynamic* P-*ball of x with respect to f*. Note that

$$\Gamma^f_{\mathrm{P}}(x) = \bigcap_{i \in \mathbb{Z}} f^{-i}(\mathrm{P}(f^i(x))),$$

and $\Gamma^f_{\mathrm{P}}(x)$ is measurable. And we say that f is *weak measure expansive* if it is μ-expansive for every non-atomic Borel probability measure μ on M.

Various types of expansiveness are useful concepts in studying the stability of dynamics. In fact, C^1-robust properties and C^1-generic properties are used in general work methods.

For C^1-robust cases, Mãñé [10] proved that if a diffeomorphism f belongs to the C^1-interior of the set of all expansive diffeomorphisms, then f is quasi-Anosov. Here, f is quasi-Anosov if for all $v \in TM \setminus \{0\}$, the set $\{\|Df^n(v)\| : n \in \mathbb{Z}\}$ is unbounded.

Sakai *et al.* [14] proved that a diffeomorphism f which is an element of the C^1-interior of the set of all invariant measure expansive diffeomorphisms is quasi-Anosov. From these results, we know that the C^1-interior of the set of all expansive diffeomorphisms is equal to the C^1-interior of the set of all invariant measure expansive diffeomorphisms, even though the set of all expansive diffeomorphisms is a subset of the set of all measure expansive diffeomorphisms. Recently, Ahn and Kim [1] proved that if a diffeomorphism f belongs to the C^1-interior of the set of all weak measure expansive diffeomorphisms, then f satisfies quasi-Anosov.

We say that a diffeomorphim f satisfies *Axiom A* if the non-wandering set $\Omega(f)$ is the closure of the periodic set $P(f)$ and it is hyperbolic. Here a closed f-invariant set Λ is called *hyperbolic* if the tangent bundle $T_\Lambda M$ has a Df-invariant splitting $E^s \oplus E^u$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|D_x f^n|_{E^s_x}\| \leq C\lambda^n \text{ and } \|D_x f^{-n}|_{E^u_x}\| \leq C\lambda^n$$

for all $x \in \Lambda$ and $n \geq 0$.

Note that the following properties are mutually equivalent for a diffeomorphism f (see [5]);

- f is Ω-stable,
- f satisfies Axiom A and no cycle condition, and
- $f \in \mathcal{F}^1(M)$,

where $f \in \mathcal{F}^1(M)$ means that there exists a C^1-neighborhood $\mathcal{U}(f)$ such that for all $g \in \mathcal{U}(f)$, every periodic point of g is hyperbolic.

Let Diff(M) be a space of all diffeomorphisms on M with C^1-topology. It is known that Diff(M) is a Baire space. We say that a subset $\mathcal{G} \subset$ Diff(M) is *residual* if $\mathcal{G}$ contains the intersection of a countable family of open and dense subsets of Diff(M); in this case $\mathcal{G}$ is dense in Diff(M). A property "A" is said to be C^1-*generic* if "A" holds for all differentiable maps which belong to some residual subset of Diff(M).

For C^1-generic cases, Arbieto [4] proved that C^1 generically, an expansive diffeomorphism is Axiom A without cycles. Lee [7] proved that C^1 generically, a measure expansive diffeomorphism is Axiom A without cycles and Lee [6] proved that C^1 generically, a continuum-wise expansive diffeomorphism is Axiom A without cycles. Also, Ahn and Kim [1] proved that C^1 generically, a weak measure expansive diffeomorphism is Axiom A without cycles.

From the above results, we consider a general concept of expansive differentiable maps as robust and generic view points.

2. Basic notions and main theorems

Let M be a compact C^∞ Riemannian manifold without boundary and $C^1(M)$ be the space of differentiable maps of M endowed with the C^1-topology. First of all, we recall the definition of positive expansiveness.

Definition 1. A differentiable map $f : M \to M$ is called *positively expansive* if there is $\delta > 0$ such that $d(f^i(x), f^i(y)) \leq \delta$ for all $i \geq 0$ implies $x = y$.

Given $x \in M$ and $\delta > 0$, we define the *dynamic δ-ball of x with respect to f*,

$$\Gamma^f_\delta(x) = \{y \in M : d(f^i(x), f^i(y)) \leq \delta \text{ for all } i \geq 0\}.$$

(We will mark $\Gamma_\delta(x)$ as $\Gamma^f_\delta(x)$ for simplicity, if there is no confusion.) Then we see that f is positively expansive if there is $\delta > 0$ such that $\Gamma_\delta(x) = \{x\}$ for all $x \in M$. Note that

$$\Gamma_\delta(x) = \bigcap_{i \geq 0} f^{-i}(B_\delta[f^i(x)]),$$

where $B_\delta[x] = \{y \in M : d(x, y) \leq \delta\}$.

Let $\mathcal{B}$ be the Borel σ-algebra on M. Denote by $\mathcal{M}(M)$ the set of Borel probability measures on M endowed with the weak* topology. We say that $\mu \in \mathcal{M}(M)$ is *atomic* if there exists a point $x \in M$ such that $\mu(\{x\}) > 0$. Let $\mathcal{M}^*(M) = \{\mu \in \mathcal{M}(M) : \mu \text{ is nonatomic}\}$. For the concept, we introduced positive μ-expansiveness for differentiable maps.

Definition 2. Let $\mu \in \mathcal{M}^*(M)$. A differentiable map $f : M \to M$ is said to be *positively μ-expansive* if there is $\delta > 0$ (called an *expansive constant* of μ with respect to f) such that $\mu(\Gamma_\delta(x)) = 0$ for all $x \in M$.

Definition 3. A differentiable map $f : M \to M$ is said to be *positively measure expansive* if f is positively μ-expansive for all $\mu \in \mathcal{M}^*(M)$.

From now, we introduce a "positively weak measure expansiveness", a new concept that generalizes weak measure expansiveness and check the basic properties of positively weak measure expansiveness for differentiable maps on M by using a finite partition as follow:

Definition 4. Let $\mu \in \mathcal{M}^*(M)$. A differentiable map $f : M \to M$ is said to be *positively weak μ-expansive* if there is a finite partition $\mathrm{P} = \{A_1, A_2, \ldots, A_n\}$ of M such that $\mu(\Gamma^f_{\mathrm{P}}(x)) = 0$ for all $x \in M$, where

$$\Gamma^f_{\mathrm{P}}(x) = \{y \in M : f^i(y) \in \mathrm{P}(f^i(x)) \text{ for all } i \geq 0\}.$$

We will mark $\Gamma_{\mathrm{P}}(x)$ as $\Gamma^f_{\mathrm{P}}(x)$ for simplicity if there is no confusion.

Definition 5. A differentiable $f : M \to M$ is said to be *positively weak measure expansive* if f is positively weak μ-expansive for all $\mu \in \mathcal{M}^*(M)$.

For differentiable maps, Aoki *et al.* [3] showed that the C^1-interior of the set of maps satisfying the two following conditions

(i) periodic points are hyperbolic, and

(ii) singular points belonging to the nonwandering set are sinks,

coincides with the set of Axiom A maps having the no cycle property.

For $f \in C^1(M)$ and $p \in P(f)$, denote by $\pi(p) > 0$ the period, that is, $f^{\pi(p)}(p) = p$ and $P(f)$ is the set of all periodic points of f. We say that p is *hyperbolic* if $D_pf : T_pM \to T_pM$ has no eigenvalues with modulus equal to 0 or 1. Thus T_pM splits into the direct sum $E^s_p \oplus E^u_p$ of subspaces such that $D_pf^{\pi(p)}(E^s_p) = E^s_p$ and $D_pf^{\pi(p)}(E^u_p) = E^u_p$, and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

- $\|D_pf^n(v)\| \leq C\lambda^n\|v\|$ for $v \in E^s_p$ and
- $\|D_xf^n(v)\| \leq C\lambda^n\|v\|$ for $v \in E^u_p$.

We say that a differentiable map f is *expanding* if there are constants $C > 0$ and $\lambda > 1$ such that for any $v \in T_xM (x \in M)$,

$$\|D_xf^n(v)\| \geq C\lambda^n\|v\|$$

for any $n \geq 0$. It is known that every expanding map is positively measure expansive, but the converse is not true. Since every expanding map f is structurally stable, there is a C^1-neighborhood $\mathcal{U}(f)$ of f such that any $g \in \mathcal{U}(f)$ is positively measure expansive.

Sakai [13] showed that the C^1-interior of the set of positively expansive differentiable maps coincides with the set of expanding maps. Recently, Lee *et al.* [9] proved that if a differentiable map which contains singularities is

C^1 robustly positively measure expansive, then it is expanding and Lee [6] proved that the C^1-interior of the set of positively continuum-wise expansive differentiable maps coincides with the set of expanding maps. In this paper, we study the space of positively weak measure expansive differentiable maps of M.

From the above facts, the following theorem is one of main theorem of this paper.

Theorem A. *Let $\mathcal{PWE}$ be the set of all positively weak measure expansive differentiable maps of M. Denote by $\mathrm{int}(\mathcal{PWE})$ is a C^1-interior of $\mathcal{PWE}$. $f \in \mathrm{int}(\mathcal{PWE})$ if and only if f is expanding.*

Sakai [13] showed that C^1 generically, a differentiable map is positively expansive if and only if it is expanding, Lee *et al.* [9] proved that for positively measure expansive differentiable map the above result is expanding and Lee [8] proved that C^1 generically, a differentiable map is positively expansive if and only if it is also hold. In [2], Ahn *et al.* showed that C^1 generically, a differentiable map is positively measure expansive if and only if it is expanding without considering of singularities. Recently, Lee *et al.* [9] proved the above statement by considering of existence of singularities.

In this direction we prove the following theorem which is a main result of this paper.

Theorem B. *For C^1-generic $f \in C^1(M)$, f is positively weak measure-expansive if and only if f is expanding.*

In Section 3, we check and prepare some basics of positively weak measure expansive differentiable maps, even though some theorems do not use to prove main theorems. Because this paper is the first which presents the positively weak measure expansive differentiable maps, Section 3 is significant implications. Next, we give a proof of Theorem A in Section 4 and finally the proof of Theorem B is constructed in Section 5.

3. Basic properties of positively weak measure expansiveness

In this section we explain positively weak measure expansive differentiable maps. Let M be as before, and let $f \in C^1(M)$.

Theorem 3.1. *If a differentiable map f is positively μ-expansive, then f is positively weak μ-expansive for any $\mu \in \mathcal{M}^*(M)$.*

Proof. Since f is positively μ-expansive, there exists $\delta > 0$ such that $\mu(\Gamma_\delta(x)) = 0$ for all $x \in M$. Let P be a finite δ-partition of M. Let $y \in \Gamma_{\mathrm{P}}(x)$, then

$$f^i(y) \in \mathrm{P}(f^i(x))$$

for all $i \geq 0$. Since $\mathrm{diam}\mathrm{P}(f^i(x)) \leq \delta$, we have $d(f^i(x), f^i(y)) \leq \delta$ for all $i \geq 0$. Therefore $y \in \Gamma_\delta(x)$. That is, for any $x \in M$, we get $\Gamma_{\mathrm{P}}(x) \subset \Gamma_\delta(x)$. Since $\mu(\Gamma_{\mathrm{P}}(x)) \leq \mu(\Gamma_\delta(x))$, we have $\mu(\Gamma_{\mathrm{P}}(x)) = 0$. Hence f is positively weak μ-expansive. □

The following lemma is a property of positively weak measure expansiveness for $f \in C^1(M)$. Simply, if $f : M \to M$ is the identity map, then M is not positively weak measure expansive.

Lemma 3.2. *The identity map on M is not positively weak measure expansive.*

Proof. Put $\mathtt{Id}$ is the identity map. For a finite partition $\mathrm{P} = \{A_1, A_2, \dots, A_n\}$ of M, $\Gamma^{\mathtt{Id}}_{\mathrm{P}}(x) = A_i$ for all $x \in A_i$ and $i = 1, \dots, n$. Choose $A_i \in \mathrm{P}$ such that $\mu(A_i) > 0$. Then $\mu(\Gamma^{\mathtt{Id}}_{\mathrm{P}}(x)) > 0$ for all $x \in A_i$. Therefore $\mathtt{Id}$ is not positively weak measure expansive. □

Remark 3.3. A differentiable map f is positively weak measure expansive if and only if f^n is positively weak measure expansive for all $n \in \mathbb{N}$.

Proof. The proof of this remark is similar to the proof of Lemma 2.5 in [1]. But we will provide a detailed proof for convenience. First, we prove the necessary part. Let f^n be positively weak measure expansive for $n \in \mathbb{N}$. This means that there exists a finite partition P of M such that $\mu(\Gamma^{f^n}_{\mathrm{P}}(x)) = 0$ for all $x \in M$ and $\mu \in \mathcal{M}^*(M)$. And it is easy to check $\Gamma^{f}_{\mathrm{P}}(x) \subset \Gamma^{f^n}_{\mathrm{P}}(x)$. This fact implies $\mu(\Gamma^{f}_{\mathrm{P}}(x)) \le \mu(\Gamma^{f^n}_{\mathrm{P}}(x)) = 0$. Therefore f is also positively weak measure expansive.

Conversely, suppose that f is positively weak measure expansive with a finite partition P of M, that is, $\mu(\Gamma^{f}_{\mathrm{P}}(x)) = 0$ for all $x \in M$ and $\mu \in \mathcal{M}^*(M)$. We consider $\mathrm{Q} = \bigvee_{i=0}^{n} f^{-i}(\mathrm{P})$, then Q is a finite partition of M satisfying

$$\mathrm{Q}(x) = \bigcap_{i=0}^{n} f^{-i}(\mathrm{P}(f^i(x))).$$

Here, $\bigvee_{i=0}^{n} f^{-i}(\mathrm{P})$ means the set $\{\bigcap_{i=0}^{n} \zeta_i : \zeta_i \in f^{-i}(\mathrm{P}) \text{ for all } 0 \le i \le n\}$ and it is called the *join* of the partition P. Now, take $y \in \Gamma^{f^n}_{\mathrm{Q}}(x)$, then clearly $y \in \mathrm{Q}(x)$. From this, we can know that

$$f^i(y) \in \mathrm{P}(f^i(x)) \text{ for every } 0 \le i \le n.$$

Take $k > n$, so $k = pn + i$ for some $p \in \mathbb{N}$ and $0 \le i < n$. Since $y \in \Gamma^{f^n}_{\mathrm{Q}}(x)$, we have $f^{pn}(y) \in \mathrm{Q}(f^{pn}(x))$ and then

$$f^k(y) = f^{pn+i}(y) = f^i(f^{pn}(y)) \in \mathrm{P}(f^i(f^{pn}(x))) = \mathrm{P}(f^k(x))$$

for all $k \in \mathbb{N}$, i.e., $y \in \Gamma^{f}_{\mathrm{P}}(x)$. Therefore we get $\Gamma^{f^n}_{\mathrm{Q}}(x) \subset \Gamma^{f}_{\mathrm{P}}(x)$ and so $\mu(\Gamma^{f^n}_{\mathrm{Q}}(x)) = 0$ for all $x \in M$ and $\mu \in \mathcal{M}^*(M)$. It follows that f^n is positively weak measure expansive with a finite partition Q of M. □

The support of a measure μ is denoted by $Supp(\mu)$. Given an f-invariant Borel set $Y(\subset M)$ of some $f \in C^1(M)$, set

$$\mathcal{M}^*(Y) = \{\mu : \mu \text{ is an } f\text{-invariant Borel probability on } M \text{ such that } \mathrm{Supp}(\mu) \subset Y\},$$

endowed with the weak* topology. It can be seen that this is clearly a subset of $\mathcal{M}^*(M)$.

Theorem 3.4. *If a differentiable map $f : M \to M$ is positively weak measure expansive, then $f|_Y : Y \to Y$ is positively weak measure-expansive for all f-invariant Borel set $Y(\subset M)$ of f.*

Proof. Since $f : M \to M$ is positively weak measure expansive, there is a finite partition $\mathrm{P} = \{A_1, \dots, A_n\}$ of M such that $\mu(\Gamma^f_{\mathrm{P}}(x)) = 0$ for all $x \in M$ and for all $\mu \in \mathcal{M}^*(M)$. Let $Y \subset M$. We defined $\nu \in \mathcal{M}^*(Y)$ by $\nu(A) = \mu(A)$ for a Borel set $A \subset Y$. Put

$$\mathrm{Q} = \{A_i \cap Y \mid A_i \in \mathrm{P},\ i = 1, \dots, n\}$$

for all f-invariant set Y in M. Then Q is a finite partition of Y, and we can easily check $\Gamma^{f|_Y}_{\mathrm{Q}}(x) \subset \Gamma^f_{\mathrm{P}}(x)$ for all $x \in Y$. Indeed, if $y \in \Gamma^{f|_Y}_{\mathrm{Q}}(x)$, then $f^i|_Y(y)$ and $f^i|_Y(x)$ are contained in the same element $A_k \cap Y$ of Q for some $k \in \{1, 2, \dots, n\}$ and all $i \in \mathbb{N}$. This means that $f^i(y)$ and $f^i(x)$ are contained in the same element A_k of P for all $i \in \mathbb{N}$. So, $y \in \Gamma^f_{\mathrm{P}}(x)$ by the definition. Thus, from the following inequality $\nu(\Gamma^{f|_Y}_{\mathrm{Q}}(x)) = \mu(\Gamma^{f|_Y}_{\mathrm{Q}}(x)) \le \mu(\Gamma^f_{\mathrm{P}}(x)) = 0$ for all $\nu \in \mathcal{M}^*(Y)$, we get the conclusion, $f|_Y : Y \to Y$ is positively weak measure expansive. □

The following statement is the contraposition of the above theorem.

Corollary 3.5. *If $f|_Y : Y \to Y$ is not positively weak measure-expansive for some f-invariant Borel set Y of f, then $f : M \to M$ is not positively weak measure-expansive.*

4. Proof of Theorem A

To prove Theorem A, we need some lemmas. Let M be as before and let $f \in C^1(M)$.

The following lemma is called Franks lemma which is a version of differentiable maps (see [4]).

Lemma 4.1. *Let $f \in C^1(M)$ and $\mathcal{U}(f)$ be any given C^1-neighborhood of f. Then there exists $\delta > 0$ such that for a finite set $\{x_1, x_2, \dots, x_n\}$, a neighborhood U of $\{x_1, x_2, \dots, x_n\}$ and linear maps $L_i : T_{x_i}M \to T_{f(x_i)}M$ satisfying $\|L_i - D_{x_i}f\| < \delta$ for $1 \le i \le n$, there are $\varepsilon_0 > 0$ and $g \in \mathcal{U}(f)$ such that*

(a) *$g(x) = f(x)$ if $x \in \{x_1, x_2, \dots, x_N\}$ and*
(b) *$g(x) = \exp_{f(x_i)} \circ L_i \circ \exp^{-1}_{x_i}(x)$ if $x \in B_{\varepsilon_0}(x_i)$ for all $1 \le i \le n$.*

Observe that the assertion (b) implies that

$$g(x) = f(x) \text{ if } x \in \{x_1, x_2, \dots, x_n\}$$

and that $D_{x_i}g = L_i$ for all $1 \le i \le n$.

Let $\mathcal{U}(f)$ be a C^1 neighborhood of $f \in C^1(M)$. We are going to show that if a differentiable map f belongs to the C^1 interior of the set of $\mathcal{PWE}$, then for any $g \in \mathcal{U}(f)$, every $p \in P(g)$ is hyperbolic, where $P(g)$ is the set of all periodic point of g.

Lemma 4.2. *If $f \in \text{int}(\mathcal{PWE})$, then every periodic point of g C^1-nearby f is hyperbolic, that is, $f \in \mathcal{F}^1(M)$.*

Proof. Suppose that there is a C^1-neighborhood $\mathcal{U}(f)$ of f such that every $g \in \mathcal{U}(f)$ satisfies positively weak measure expansive. To derive a contradiction, we assume that there exists a non-hyperbolic periodic point $p \in P(g)$ for some $g \in \mathcal{U}(f)$. By Lemma 4.1, we can assume that $D_p g^{\pi(p)}$ has either only one eigenvalue λ with $|\lambda| = 1$, or only one pair of complex conjugated eigenvalues. Denote by E^c_p the eigenspace corresponding to λ.

At first, we consider the case $\dim E^c_p = 1$. In this case, suppose that $\lambda = 1$ for simplicity (other case is similar). Then by Lemma 4.1, there are $\varepsilon_0 > 0$ and $h \in \mathcal{U}(g) (\subset \mathcal{U}(f))$ such that

(i) $h^{\pi(p)}(p) = g^{\pi(p)} = p$,
(ii) $h(x) = \exp_{g^{i+1}(p)} \circ D_{g^i(p)} g \circ \exp^{-1}_{g^i(p)}(x)$, if $x \in B_{\varepsilon_0}(g^i(p))$ for all $0 \le i \le \pi(p) - 2$, and
(iii) $h(x) = \exp_p \circ D_{g^{\pi(p)-1}(p)} g \circ \exp^{-1}_{g^{\pi(p)-1}(p)}(x)$, if $x \in B_{\varepsilon_0}(g^{\pi(p)-1}(p))$.

Since $\lambda = 1$, there is a small arc $\mathcal{I}_p \subset B_{\varepsilon_0}(p) \cap \exp_p(E^c_p(\varepsilon_0))$ with its center at p such that

- $h^i(\mathcal{I}_p) \cap h^j(\mathcal{I}_p) = \emptyset$ if $i \ne j$ for $0 \le i, j \le \pi(p) - 1$,
- $h^{\pi(p)}(\mathcal{I}_p) = \mathcal{I}_p$ and
- $h^{\pi(p)}|_{\mathcal{I}_p}$ is the identity map.

Here, $E^c_p(\varepsilon_0)$ is the ε_0-ball in E^c_p centered at the origin O_p.

Put $h_1 = h^{\pi(p)}|_{\mathcal{I}_p}$. Since h_1 is the identity map, by Lemma 3.2 h_1 is not positively weak measure expansive. By Remark 3.3, h is not positively weak measure expansive on $\mathcal{I}_p$. To conclude, h is not positively weak measure expansive on M by Corollary 3.5. This contradicts the fact that $h \in \mathcal{U}(f)$.

Finally we consider that $\dim E^c_p = 2$. For the sake of symbolic convenience, this case only $g(p) = p$ will be covered. By Lemma 4.1, we can find $\varepsilon_0 > 0$ and $h \in \mathcal{U}(g) (\subset \mathcal{U}(f))$ such that

(i) $h(p) = g(p) = p$ and
(ii) $h(p) = \exp_p \circ D_p g \circ \exp^{-1}_p(x)$ if $x \in B_{\varepsilon_0}(p)$.

Then we can choose $m > 0$ such that $D_p g^m(\rho) = \rho$ for any $\rho \in E^c_p(\varepsilon_0)$. Take nonzero vector $\rho_p \in E^c_p(\varepsilon_0/4)$ such that $\|\rho_p\| = \varepsilon_0/8$ and establish $\mathcal{C}_p = \exp_p\{\alpha \cdot \rho_p : 1 \le \alpha \le 1 + \varepsilon_0/8\}$. Then $\mathcal{C}_p$ is a disc such that

- $h^i(\mathcal{C}_p) \cap h^j(\mathcal{C}_p) = \emptyset$ if $i \ne j$ for $0 \le i, j \le m - 1$,
- $h^m(\mathcal{C}_p) = \mathcal{C}_p$ and
- $h^m|_{\mathcal{C}_p}$ is the identity map.

Then we can make the same conclusion for the same reason as the case $\dim E^c_p = 1$. So this is complete for proof. □

Next lemma means that there are no sinks and saddles for $f \in \mathrm{int}(\mathcal{PWE})$.

Lemma 4.3. *Let $P_i(f)$ $(0 \leq i \leq \dim M)$ be the set of periodic points of f with $\dim E^s_p = i$. If $f \in \mathrm{int}(\mathcal{PWE})$, then $P_i(f) = \emptyset$ for $1 \leq i \leq \dim M$.*

Proof. Since $f \in \mathrm{int}(\mathcal{PWE})$, there are δ and a finite δ-partition $\mathrm{P} = \{A_1, \ldots, A_n\}$ of M. Since $f \in \mathrm{int}(\mathcal{PWE})$ by Lemma 4.2, we may assume that there is a (hyperbolic) saddle periodic point $p \in P_i(f)$ for some $1 \leq i \leq \dim M$ and we consider that p is a fixed point for convenience. Since the above finite partition P covers M, we choose $A_k \in \mathrm{P}$ containing p. Then the dynamic P-ball of p with respect to f is

$$\begin{aligned}\Gamma_{\mathrm{P}}(x) &= \{y \in M : f^n(y) \in \mathrm{P}(f^n(p)) \ \text{ for all } n \geq 0\}\\ &= \{y \in M : f^n(y) \in \mathrm{P}(p) = A_k \ \text{ for all } n \geq 0\}\\ &\subset A_k.\end{aligned}$$

Due to $p \in P_i(f)$ for some $1 \leq i \leq \dim M$, there exists the stable manifold $W^s(p)$ of p such that $\dim W^s(p) = i$, where $W^s(p) = \{x \in M : f^n(x) \to p$ as $n \to \infty\}$. Let $\epsilon > 0$ be the one that satisfies $\epsilon < \delta$. We set

$$C = B_\epsilon(p) \cap W^s(p) \cap A_k.$$

Let $\mathfrak{M}_C$ be the normalized Lebesgue measure on C. Define $\tilde{\mu} \in \mathcal{M}^*(M)$ by

$$\tilde{\mu}(B) = \mathfrak{M}_C\,(B \cap C)$$

for any Borel set B of M. Take $c = \epsilon/2$ and let

$$\begin{aligned}\Phi_{\mathrm{P}}(p) &= \{y \in A_k : f^n(y) \in \mathrm{P}(f^n p) \ \text{ and } \ d(f^n(y), f^n(p)) \leq c \ \text{ for all } \ n \geq 0\}\\ &= \{y \in A_k : f^n(y) \in \mathrm{P}(p) \ \text{ and } \ d(f^n(y), p) \leq c \ \text{ for all } \ n \geq 0\}\\ &= \{y \in A_k : f^n(y) \in A_k \ \text{ and } \ d(f^n(y), p) \leq c \ \text{ for all } \ n \geq 0\}.\end{aligned}$$

It is clear that $\Phi_{\mathrm{P}}(p) \subset \Gamma_P(p)$ (see Figure 1).

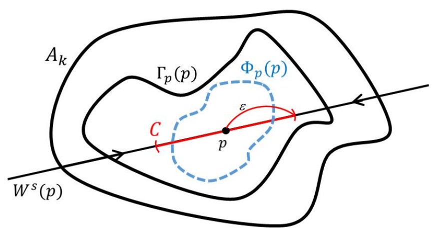

FIGURE 1. $\Phi_{\mathrm{P}}(p) \subset \Gamma_P(p)$

Since $f \in \mathrm{int}(\mathcal{PWE})$, we know that

$$\tilde{\mu}(\Gamma_{\mathrm{P}}(p)) = 0$$

for all $\tilde{\mu} \in \mathcal{M}^*(M)$. Since $\Phi_{\mathrm{P}}(p) \subset \Gamma_P(p)$, we have

$$\tilde{\mu}(\Gamma_{\mathrm{P}}(p)) > \tilde{\mu}(\Phi_{\mathrm{P}}(p)) > 0.$$

This is a contradiction since $f \in \mathrm{int}(\mathcal{PWE})$. □

For $f \in C^1(M)$, if $D_x f : T_x M \to T_{f(x)}M$ is not injective, then x is called a *singular point* for f. Denote by $S(f)$ the set of singularities of f. The following lemma shows that if a differentiable map $f \in \mathrm{int}(\mathcal{PWE})$, then M does not contains any singularities.

Lemma 4.4. *If $f \in \mathrm{int}(\mathcal{PWE})$, then $S(f) = \emptyset$.*

Proof. Let $\mathcal{U}(f)$ be a C^1-neighborhood of f such that any $g \in \mathcal{U}(f)$ is positively weak measure expansive, that is, there is a finite partition $P = \{A_1, \ldots, A_n\}$ of M such that $\mu(\Gamma_P(x)) = 0$ for all $x \in M$ and all $\mu \in \mathcal{M}^*(M)$. Suppose that there exists $x \in S(f)$. Then by Lemma 4.1, we can construct $g \in \mathcal{U}(f)$ possessing a small disk $B_{r_0}(x)$ centered at x with radius $r_0 > 0$ such that

- $\mathrm{dim} B_{r_0}(x) = 1$,
- $B_{r_0}(x) \subset A_i$ for some $i \in \{1, \ldots, n\}$, and
- $g(B_{r_0}(x)) = \{g(x)\}$.

Let $\mathfrak{M}_{B_{r_0}(x)}$ be the normalized Lebesgue measure on $B_{r_0}(x)$, and define $\nu \in \mathcal{M}^*(M)$ by

$$\nu(B) = \mathfrak{M}_{B_{r_0}(x)}(B \cap B_{r_0}(x))$$

for any Borel set B of M. Since we can check $B_{r_0}(x) \subset \Gamma_P^g(y)$ for all $y \in B_{r_0}(x)$, it is clear that $\nu(\Gamma_P^g(x)) = 1$ and this is a contradiction to the fact $g \in \mathcal{U}(f)$. □

End of Proof of Theorem A. Let $f \in \mathrm{int}(\mathcal{PWE})$. By Lemma 4.2 and Lemma 4.4, $f \in \mathcal{F}^1(M)$ and $S(f) = \emptyset$. Then by [3, Proposition 1, Proposition 2] and Lemma 4.3, $\Omega(f) = \overline{P(f)} = \overline{P_0(f)}$ is hyperbolic and so it is expanding. Then as in the proof of [9, Lemma 2.8], we can obtain $M = \overline{P_0(f)}$. □

5. Proof of Theorem B

In this section, we introduce a C^1-generic differentiable map and prove Theorem B. A subset $\mathcal{G} \subset C^1(M)$ is *residual* if $\mathcal{G}$ contains the intersection of a countable family of open and dense subsets of $C^1(M)$; in this case $\mathcal{G}$ is dense in $C^1(M)$. A property "P" is said to be C^1-*generic* if "P" holds for all differentiable maps which belong to some residual subset of $C^1(M)$. We use the terminology for C^1-*generic* f to express "there is a residual subset of $\mathcal{G} \subset C^1(M)$ such that for any $f \in \mathcal{G} \ldots$".

To prove Theorem B, we need some definitions and lemmas. Denote by $P_h(f)$ the set of hyperbolic periodic points of f. We say a hyperbolic periodic point p of f with period $\pi(p)$ said to have a *θ-weak eigenvalue* if there is an eigenvalue λ of $Df^{\pi(p)}(p)$ such that

$$(1-\theta)^{\pi(p)} < |\lambda| < (1+\theta)^{\pi(p)}.$$

Moreover, we say that the periodic point p has *simple real spectrum* if all of its eigenvalues are real and have multiplicity one.

We say that a differentiable map f is *Kupka-Smale* if f satisfies the following three conditions.

- The periodic points of f are all hyperbolic.
- If p is a periodic point of f, $W^s(p)$ is a 1-1 immersed submanifold.
- If p and q are periodic points of f, $W^s(p) \pitchfork W^s(q)$.

Denote by $\mathcal{KS}(M)$ the set of all Kupka-Smale differentiable maps (see [15]). Observe that by Kupka-Smale's theorem for differentiable maps, for C^1-generic $f \in C^1(M)$, every $p \in P(f)$ is hyperbolic, and thus, such p is source if f is positively weak measure expansive.

Lemma 5.1. *If $f \in C^1(M)$ is Kupka-Smale positively weak measure expansive, then $P_i(f) = \emptyset$ for $(1 \le i \le \dim M)$, that is, $P(f) = P_0(f)$.*

Proof. Suppose that f is a Kupka-Smale positively weak measure expansive differentiable map. If there exists a hyperbolic saddle periodic point p of f, then we can construct a local stable manifold of p. Using the same way in the proof of Lemma 4.3, we can define the normalized Lebesgue measure $\tilde{\mu} \in \mathcal{M}^*(M)$ on the local stable manifold satisfying $\tilde{\mu}(\Gamma_P(p)) > 0$, where P is a finite partition of M. Therefore we derive a contradiction, because f is positively weak measure expansive. □

We introduce a C^1 generic properties for differentiable maps which need to proof Theorem B.

Lemma 5.2. *There is a residual set $\mathcal{R}_1 \subset C^1(M)$ such that for any $f \in \mathcal{R}_1$,*

(a) *for any $\theta > 0$, if for any sufficiently small C^1-neighborhood $\mathcal{U}(f)$ of f there exists $g \in \mathcal{U}(f)$ such that g has a periodic point q of g with a θ-weak eigenvalue, then there exists $p \in P(f)$ with 2θ-weak eigenvalue.*
(b) *if any sufficiently small C^1-neighborhood $\mathcal{U}(f)$ of f, there is $g \in \mathcal{U}(f)$ having $q \in P(g)$ with $\dim E^s_q = i$ $(0 \le i \le \dim M)$, then f also has $p \in P(f)$ with $\dim E^s_p = i$.*

Proof. See [9, Lemma 3.1] and [9, Lemma 3.3]. □

Lemma 5.3. *There is a residual set $\mathcal{R}_2 \subset C^1(M)$ such that if $f \in \mathcal{R}_2$ is positively weak measure expansive, then there exists $\theta > 0$ such that every hyperbolic periodic point p of f has no θ-weak eigenvalue.*

Proof. Let $\mathcal{R}_2 = \mathcal{R}_1 \cap \mathcal{KS}(M)$ and let $f \in \mathcal{R}_2$. Assume that f is a positively weak measure expansive differentiable map. To derive a contradiction, suppose that for any $\theta > 0$, there exists $p \in P_h(f)$ satisfying $D_pf^{\pi(p)}$ has a θ-weak eigenvalue λ. By Lemma 4.1, there are g (C^1-close to f) and a saddle point $q \in P_h(g)$ with $\dim(E^s_q) = i$ $(1 \leq i \leq \dim M)$. Suppose that this $\dim E^s_q = i$ is a constant. Then there exists $r \in P_h(f)$ with $\dim E^s_r = i \geq 1$ by Lemma 5.2 (b) and this is a contradiction by Lemma 5.1. □

Proposition 5.4. *For any $f \in \mathcal{R}_2$, if f is a positively weak measure expansive differentiable map, then $P_0(f) \cap S(f) = \emptyset$.*

Proof. Let $f \in \mathcal{R}_2$ be a positively weak measure expansive differentiable map. To prove this proposition, we suppose that there exists a point p in $P_0(f) \cap S(f)$. Then there exists a sequence of periodic points $\{p_n\} \subset P_0(f)$ with period $\pi(p_n)$ such that $p_n \to p$ as $n \to \infty$. By Lemma 4.1, we can choose g (C^1-close to f) $g^{\pi(p_n)}(p_n) = p_n$ satisfying $p_n \in S(g)$ with $1 \leq \dim E^s_{p_n} = i \leq \dim M$. Then f also has a hyperbolic saddle periodic point q with $\dim E^s_q = i$ by Lemma 5.2 (b). Since f is Kupka-Smale positively weak measure expansive, we can get a contradiction by Lemma 5.1. □

Lemma 5.5. *For any $f \in \mathcal{R}_2$, if f is a positively weak measure expansive differentiable map, then $f \in \mathcal{F}^1(M)$.*

Proof. Since $f \in \mathcal{R}_2$ is a positively weak measure expansive differentiable map, we can obtain a neighborhood U of $\overline{P_0(f)}$ satisfying $U \cap S(f) = \emptyset$ by the above proposition. On the contrary, if $f \in \mathcal{R}_2$ is a positively weak measure expansive differentiable map, then there exists a C^1-neighborhood $\mathcal{U}(f)$ of f such that every $p \in P(g)$ is hyperbolic for any $g \in \mathcal{U}(f)$ by Lemma 5.2 and Lemma 5.3, that is, $f \in \mathcal{F}^1(M)$. So we can get $\Omega(f) = \overline{P(f)}$. □

End of Proof of Theorem B. From the above Lemma 5.1 and Lemma 5.5, we can draw a conclusion that there is a residual set $\mathcal{R}_2 \subset C^1(M)$ such that for any $f \in \mathcal{R}_2$, if f is positively weak measure expansive, then $f \in \mathcal{F}(M)$ and $\Omega(f) = \overline{P(f)} = \overline{P_0(f)}$. By [3, Proposition 2], $\overline{P_0(f)}$ is hyperbolic that is expanding. Then by [12, Corollary 2] if for g C^1-close to f such that $g = f$ in U, then $P_i(f) = P_i(g)$ $(0 \leq i \leq \dim M)$, where U is a neighborhood of $\overline{P_i(f)}$. Thus by [9, Lemma 3.8], $M = \overline{P_0(f)}$, that is, f is an expanding map. □

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