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THE LOCAL TIME OF THE LINEAR SELF-ATTRACTING DIFFUSION DRIVEN BY WEIGHTED FRACTIONAL BROWNIAN MOTION

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Abstract. In this paper, we introduce the linear self-attracting diffusion driven by a weighted fractional Brownian motion with weighting exponent $a > -1$ and Hurst index $|b| < a+1, 0 < b < 1$, which is analogous to the linear fractional self-attracting diffusion. For the 1-dimensional process we study its convergence and the corresponding weighted local time. As a related problem, we also obtain the renormalized intersection local time exists in L^2 if $\max\{a_1+b_1, a_2+b_2\} < 0$.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\{B_t, \mathcal{F}_t : t \geq 0\}$ be a Brownian motion on $\mathbb{R}^d$ (starting in 0 at time 0), and let $f : \mathbb{R}^d \to \mathbb{R}^d (d \geq 1)$ be a measurable function. Durrett and Rogers [10] introduced the following stochastic differential equation

$$X_t = B_t + \int_0^t \int_0^s f(X_s - X_u) du ds, \tag{1.1}$$

as a model for the shape of a growing polymer, X_t corresponding to the location of the end of the polymer at time t. Without any assumption on the function f, the stochastic differential equation (1.1) defines a self-interacting diffusion, in the sense that the process X evolves in an environment changing with its prior trajectory. Mountford and Tarrès [20] call it self-repelling (resp. self-attracting) if, for all $x \in \mathbb{R}^d, x \cdot f(x) \geq 0$ (resp. ≤ 0), in other words if it is more likely to stay away from (resp. come back to) the places it has already visited before. Note that Equation (1.1) has a pathwise unique strong solution if f is assumed to be Lipschitz continuous (see, for example, Rogers and Williams [21]), it admits

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a unique weak solution if f is locally bounded, using a generalization of Girsanov theorem (see Corollary 3.5.2 in Karatzas and Shreve [19]). Since these processes are self-attracting, it is particularly interesting to describe the asymptotic behaviour of their paths. Some earlier works developed results on the boundedness and convergence of such paths. Cranston and Le Jan [8] extended the model and introduced self-attracting diffusion, where for $d = 1$ two cases are studied: the linear interaction where f is a linear function and the constant interaction in dimension 1, where $f(x) = \sigma sign(x)$ for positive σ, and in both cases the almost sure convergence of X_t is proved. Herrmann and Roynette [13] considered some self-attracting diffusions and studied the behaviour of paths of some self-attracting diffusions when time tends to infinity and generalized those results in Cranston and Le Jan [8]. Herrmann and Scheutzow [14] gave some rate of convergence of the path of some self-attracting diffusions. As an extension of Brownian motion, the fractional Brownian motion exhibits long-range dependence and self similarity, having stationary increments. It is the usual candidate to model phenomena in which the self-similarity property can be observed from the empirical data. Chakravarti and Sebastian [6], Cherayil and Biswas [7] used the statistical properties of fractional Brownian motion (long-range dependence, self similarity and stationary increments) to construct a path integral representation of the conformations of some polymers. Yan, Sun and Lu [28] introduced the linear fractional self-attracting diffusion driven by a fractional Brownian motion with Hurst index $\frac{1}{2} < H < 1$, which is analogous to the linear self-attracting diffusion. On the other hand, many authors have proposed to use more general self-similar Gaussian processes and random fields as stochastic models. Such applications have raised many interesting theoretical questions about self-similar Gaussian processes and fields in general. Therefore, some generalizations of the fractional Brownian motion have been introduced such as bi-fractional Brownian motion, sub-fractional Brownian motion and weighted fractional Brownian motion. However, in contrast to the extensive studies on the fractional Brownian motion, there has been little systematic investigation on other self-similar Gaussian processes. The main reason for this is the complexity of dependence structures for self-similar Gaussian processes which do not have stationary increments.

Motivated by all these results, in this paper, as a natural extension to (1.1) one may consider the path dependent stochastic differential equation of the form (it is not difficult to show that the equation admits a unique strong solution, we will call the solution *the linear weighted fractional self-attracting diffusion*)

$$X_t^{a,b} = B_t^{a,b} - p\int_0^t \int_0^s (X_s^{a,b} - X_u^{a,b})duds + \nu t, \tag{1.2}$$

with $p > 0$, $\nu \in \mathbb{R}^d$ and $0 < b < 1$, where $B^{a,b}$ is a d-dimensional weighted fractional Brownian motion with index $a > -1, |b| < a + 1, 0 < b < 1$ (the precise definition is given below in Section 2). We are interested in the study of the convergence and local time of the processes given by (1.2) with $d = 1$. As a related problem, for the two dimensional process we shall show that the renormalized intersection local time exists in L^2 if $\max\{a_1 + b_1, a_2 + b_2\} < 0$ and obtain the derivative for the renormalized intersection local time.

The rest of this paper is organized as follows. In Section 2 we give details for the weighted fBm and the related Itô type stochastic integral. In Section 3 we investigate convergence of the linear weighted fractional self-attracting diffusion. In Section 4, we define the weighted local time of the process and obtain a Meyer-Tanaka type formula. In Section 5, we show that its renormalized intersection local time exists in L^2 if $\max\{a_1 + b_1, a_2 + b_2\} < 0$ for 2-dimensional process and obtain the derivative for the renormalized intersection local time.

2. Weighted fractional Brownian motion

In this section, we recall some basic results of weighted fractional Brownian motion. The weighted fractional Brownian motion (wfBm for short) $B^{a,b}$ with parameters $a > -1, |b| < 1, |b| < a + 1$ is a centered and self-similar Gaussian process with long/short-range dependence. It admits the relatively simple covariance function

$$\begin{aligned} R^{a,b}(t,s) &:= E\left[B_t^{a,b} B_s^{a,b}\right] \\ &= \frac{1}{2\mathbb{B}(a+1,b+1)} \int_0^{s\wedge t} u^a[(t-u)^b + (s-u)^b]du,\ \ s,t \geq 0, \end{aligned} \tag{2.1}$$

where $\mathbb{B}(\cdot,\cdot)$ is the beta function. Clearly, for $a = 0, b = 0$, $B^{a,b}$ coincides with the standard Brownian motion B. For $a = 0$, (2.1) reduces to

$$E\left[B_t^{a,b} B_s^{a,b}\right] = \frac{1}{2(b+1)\mathbb{B}(a+1,b+1)}[t^{b+1} + s^{b+1} - |s-t|^{b+1}],$$

which corresponds to the covariance of the fractional Brownian motion with Hurst index $\frac{b+1}{2}$ if $-1 < b < 1$. Hence, the wfBm is a family of processes which extend the fractional Brownian motion, perhaps it may be useful in some applications. This process $B^{a,b}$ appeared in Bojdecki *et al.* [4] in a limit of occupation time fluctuations of a system of independent particles moving in $\mathbb{R}^d$ according to a symmetric α-stable Lévy process $(0 < \alpha \leq 2)$, started from an inhomogeneous Poisson configuration with intensity measure $\frac{dx}{1+|x|^\gamma}$ and $0 < \gamma \leq d = 1 < \alpha$, $a = -\gamma/\alpha$, $b = 1 - 1/\alpha$, the ranges of values of a and b being $-1 < a < 0$ and $0 < b \leq 1 + a$. The process $B^{a,b}$ also appeared in Bojdecki *et al.* [5] in a high density limit of occupation time fluctuations of the above mentioned particles system, where the initial Poisson configuration has finite intensity measure, with $d = 1 < \alpha$, $a = -1/\alpha$, $b = 1 - 1/\alpha$. Moreover, the wfBm was first studied by Bojdecki *et al.* [3], and it is neither a semimartingale nor a Markov process unless $a = 0, b = 0$, so many of the powerful techniques from stochastic analysis are not available when dealing with $B^{a,b}$. Recently, Garzón [11] showed that for certain values of the parameters the weighted fractional Brownian sheets are obtained as limits in law of occupation time fluctuations of a stochastic particle model. Shen, Yan and Cui [23] studied Berry-Esséen bounds and almost sure CLT for quadratic variation of the wfBm. Shen, Yin and Yan [24] considered least squares estimation for Ornstein-Uhlenbeck processes driven by the wfBm. Sun, Yan and Zhang [26] studied the quadratic covariation for a wfBm. Sun and Yan [25] considered the asymptotic normality associated with some processes, as an application they study the asymptotic normality of the estimator of parameter. Yan, Wang and Jing [29] gave some path properties of wfBm. The wfBm has properties analogous to those of the fractional Brownian motion (self-similarity, long-range dependence, Hölder paths). However, in comparison with the fractional Brownian motion, the wfBm has non-stationary increments and satisfies the following estimates (see Bojdecki *et al.* [3], Yan *et al.* [29]):

$$c_{a,b}(t\vee s)^a|t-s|^{b+1} \le E\left[\left(B^{a,b}_t - B^{a,b}_s\right)^2\right] \le C_{a,b}(t\vee s)^a|t-s|^{b+1}$$

for $s,t\ge 0$. Thus, Kolmogorov's continuity criterion implies that the wfBm is Hölder continuous of order δ for any $\delta < \frac{1}{2}(1+b)$. For simplicity throughout this paper, we let $c_{a,b}, C_{a,b}, C_{a,b,\theta}$ stand for positive constants depending only on the subscripts and their value may be different in different appearances. We can rewrite its covariance as

$$R^{a,b}(t,s) = \frac{1}{2\mathbb{B}(a+1,b+1)}\int_0^{t\wedge s} u^a(t\vee s-u)^b du + \frac{1}{2}(t\wedge s)^{a+b+1},$$

which gives

$$\frac{\partial^2}{\partial t\partial s}R^{a,b}(t,s) = \frac{b}{2\mathbb{B}(a+1,b+1)}(t\wedge s)^a|t-s|^{b-1}$$

for $b>0$.

As a Gaussian process, it is possible to construct a stochastic calculus of variations with respect to $B^{a,b}$. We refer to Alós *et al* [1] and Yan [27] for the complete descriptions of stochastic calculus with respect to Gaussian processes. Here we recall only the basic elements of this theory. The crucial ingredient is the canonical Hilbert space $\mathcal{H}$ (is also said to be reproducing kernel Hilbert space) associated to the wfBm $B^{a,b}$ which is defined as the closure of the linear space $\mathscr{E}$ generated by the indicator functions $\{\mathbf{1}_{[0,t]}, t\in[0,T]\}$ with respect to the scalar product $\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]}\rangle_{\mathcal{H}} = R^{a,b}(t,s)$. The application $\mathscr{E}\ni\varphi\mapsto B^{a,b}(\varphi)$ ($B^{a,b}(\varphi)$ is a Gaussian process on $\mathcal{H}$, which can be extended to all of $\mathcal{H}$, such that $E[B^{a,b}(\varphi)B^{a,b}(\psi)] = \langle\varphi,\psi\rangle_{\mathcal{H}}$ for all $\varphi,\psi\in\mathcal{H}$) is an isometry from $\mathscr{E}$ to the Gaussian space generated by $B^{a,b}$. The Hilbert space $\mathcal{H}$ can be written as

$$\mathcal{H} = \left\{\varphi : [0,T]\to\mathbb{R}\,|\,\|\varphi\|_{\mathcal{H}} < \infty\right\},$$

where

$$\|\varphi\|^2_{\mathcal{H}} := \int_0^T\int_0^T \varphi(t)\varphi(s)\phi(t,s)dtds$$

with $\phi(t,s) = \frac{b}{2\mathbb{B}(a+1,b+1)}(t \wedge s)^a |t-s|^{b-1}$, which is just the second partial derivative of $R^{a,b}(t,s)$ as calculated above. We can use the subspace $|\mathcal{H}|$ of $\mathcal{H}$ which is defined as the set of measurable function φ on $[0,T]$ such that

$$\|\varphi\|^2_{|\mathcal{H}|} := \int_0^T \int_0^T |\varphi(s)||\varphi(r)|\phi(s,r)dsdr < \infty. \tag{2.2}$$

It has been shown that $|\mathcal{H}|$ is a Banach space with the norm $\|\varphi\|_{|\mathcal{H}|}$ and $\mathcal{E}$ is dense in $|\mathcal{H}|$.

For $b>0$, we denote by $\mathcal{S}$ the set of smooth functionals of the form

$$F = f(B^{a,b}(\varphi_1), \dots, B^{a,b}(\varphi_n)),$$

where $f \in C_b^\infty(\mathbb{R}^n)$ (f and all its derivatives are bounded) and $\varphi_i \in \mathcal{H}, i = 1,2,\dots,n$. The Malliavin derivative of a function $F \in \mathcal{S}$ is given by

$$D^{a,b}F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(B^{a,b}(\varphi_1), \dots, B^{a,b}(\varphi_n))\varphi_i.$$

The derivative operator $D^{a,b}$ is then a closable operator from $L^2(\Omega)$ into $L^2(\Omega;\mathcal{H})$. We denote by $\mathbb{D}^{1,2}$ the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,2} := \sqrt{E|F|^2 + E\|D^{a,b}F\|^2_{\mathcal{H}}}.$$

The divergence integral $\delta^{a,b}$ is the adjoint operator of $D^{a,b}$. That is, we say that a random variable u in $L^2(\Omega;\mathcal{H})$ belongs to the domain of the divergence operator $\delta^{a,b}$, denoted by $\mathrm{Dom}(\delta^{a,b})$, if $E\left|\langle D^{a,b}F, u\rangle_{\mathcal{H}}\right| \le c\|F\|_{L^2(\Omega)}$ for every $F \in \mathcal{S}$. In this case $\delta^{a,b}(u)$ is defined by the duality relationship

$$E\left[F\delta^{a,b}(u)\right] = E\langle D^{a,b}F, u\rangle_{\mathcal{H}} \tag{2.3}$$

for any $u \in \mathbb{D}^{1,2}$. We have $\mathbb{D}^{1,2} \subset \mathrm{Dom}(\delta^{a,b})$ and for any $u \in \mathbb{D}^{1,2}$

$$\begin{aligned} E\left[\delta^{a,b}(u)^2\right] &= E\|u\|^2_{\mathcal{H}} + E\int_{[0,T]^4} D^{a,b}_\xi u_r D^{a,b}_\eta u_s \phi(\eta,r)\phi(\xi,s)dsdrd\xi d\eta \\ &\le E\|u\|^2_{|\mathcal{H}|} + E\int_{[0,T]^4} |D^{a,b}_\xi u_r||D^{a,b}_\eta u_s|\phi(\eta,r)\phi(\xi,s)dsdrd\xi d\eta. \end{aligned}$$

We will use the notation

$$\delta^{a,b}(u) = \int_0^T u_s \delta^{a,b} B^{a,b}_s$$

to express the Skorohod integral of a process u, and the indefinite Skorohod integral is defined as $\int_0^t u_s \delta^{a,b} B^{a,b}_s = \delta^{a,b}(u\mathbf{1}_{[0,t]})$.

Recall that the Malliavin ϕ-derivative of the function $U : \Omega \to \mathbb{R}$ is defined in Duncan *et al.* [9] as follows:

$$D^\phi_s U = \int_0^\infty \phi(r,s) D_r U dr,$$

where $D_r U$ is the fractional Malliavin derivative at r. Define the space $\mathbb{L}_\phi^{1,2}$ to be the set of measurable processes u such that $D_t^\phi u_s$ exists for a.s., $t \geq 0$ and

$$\|u\|_{\mathbb{L}_\phi^{1,2}}^2 := E\Big[\int_0^\infty \int_0^\infty D_s^\phi u_t D_t^\phi u_s ds dt + \int_0^\infty \int_0^\infty u_s u_t \phi(s,t) ds dt\Big] < \infty. \tag{2.4}$$

Thus, the integral $\int_0^\infty u_s dB_s^{a,b}$ can be well defined as an element of $L^2(\mu)$ if u satisfies (2.4). For the integral process $\eta_t = \int_0^t u_s dB_s^{a,b}$, we have

$$D_s^\phi \eta_t = \int_0^t D_s^\phi u_r dB_r^{a,b} + \int_0^t u_r \phi(s,r) dr.$$

In particular, if u is deterministic, then $D_s^\phi \eta_t = \int_0^t u_r \phi(s,r) dr$. An Itô formula in the analogous form with respect to wfBms is given. This generalization of the Itô formula is useful in applications.

Theorem 2.1. *Let $F \in C^2(\mathbb{R})$ have polynomial growth and let the process X be given as follows*:

$$dX_t = v_t dt + u_t dB_t^{a,b}, X_0 = x \in \mathbb{R},$$

where $u \in L_\phi^{1,2}$ and measurable process v satisfies $\int_0^t |v_s| ds < \infty$ a.s. Then we have, for all $t \geq 0$

$$F(X_t) = F(x) + \int_0^t \frac{\partial}{\partial x} F(s, X_s) dX_s + \int_0^t \frac{\partial^2}{\partial x^2} F(s, X_s) u_s D_s^\phi X_s ds. \tag{2.5}$$

3. Convergence of the linear weighted fractional self-attracting diffusion

In this section, we investigate convergence of the linear weighted fractional self-attracting diffusion which is the solution of the (1.2). Using the method of Cranston and Le Jan [8], the solution to (1.2) can be expressed as

$$X_t^{a,b} = X_0^{a,b} + \int_0^t h(t,s) dB_s^{a,b} + \nu \int_0^t h(t,s) ds, \tag{3.1}$$

where

$$h(t,s) = \begin{cases} 1 - pse^{\frac{1}{2}ps^2} \int_s^t e^{-\frac{1}{2}pu^2} du, & t \geq s; \\ 0, & t < s \end{cases} \tag{3.2}$$

for $s, t \geq 0$.

It is easy to obtain that

$$\lim_{t \uparrow \infty} h(t,s) = 1 - pse^{\frac{p}{2}s^2} \int_s^\infty e^{-\frac{p}{2}u^2} du := h(s), \tag{3.3}$$

which is continuous on $[0, \infty)$. It follows from the Itô type formula that

$$F(X_t^{a,b}) = F(0) + \int_0^t F'(X_s^{a,b}) dX_s^{a,b} + \int_0^t F''(X_s^{a,b}) D_s^\phi X_s^{a,b} ds$$

$$
\begin{aligned}
&= F(0) + \int_0^t F'(X_s^{a,b})dX_s^{a,b} \\
&\quad + \frac{b}{\mathbb{B}(a+1,b+1)} \int_0^t F''(X_s^{a,b})ds \int_0^s h(s,m)m^a(s-m)^{b-1}dm
\end{aligned}
$$

for $F \in C^2(\mathbb{R})$ having polynomial growth.

Theorem 3.1. *For the solution $X_t^{a,b}$ of (1.2), we have*

$$
X_t^{a,b} \xrightarrow{\mathrm{L}^2} X_\infty^{a,b} \equiv \int_0^\infty h(s)dB_s^{a,b} + \nu \int_0^\infty h(s)ds, \ t \to \infty.
$$

Proof. We have

$$
\begin{aligned}
&|h(t,s_1) - h(s_1)||h(t,s_2) - h(s_2)| \\
&= |ps_1 e^{\frac{p}{2}s_1^2} \int_t^\infty e^{-\frac{p}{2}u^2}du||ps_2 e^{\frac{p}{2}s_2^2} \int_t^\infty e^{-\frac{p}{2}u^2}du| \\
&= p^2 s_1 s_2 e^{\frac{p}{2}(s_1^2+s_1^2)}(\int_t^\infty e^{-\frac{p}{2}u^2}du)^2 \\
&\le p^2 s_1 s_2 e^{\frac{p}{2}(s_1^2+s_2^2)}(\int_t^\infty \frac{u}{t}e^{-\frac{p}{2}u^2}du)^2 \\
&= \frac{1}{t^2} s_1 s_2 e^{\frac{p}{2}(s_1^2+s_2^2)} e^{-pt^2}
\end{aligned}
$$

for $s_1, s_2 \le t$ and

$$
\begin{aligned}
|\int_0^t [h(t,s) - h(s)]ds| &= \int_0^t pse^{\frac{p}{2}s^2} \int_t^\infty e^{-\frac{p}{2}u^2}duds \\
&\le p\int_0^t se^{\frac{p}{2}s^2} \int_t^\infty \frac{u}{t}e^{-\frac{p}{2}u^2}duds \\
&= \frac{1}{t}\int_0^t se^{\frac{p}{2}(s^2-t^2)}ds \\
&= \frac{1}{pt}(1 - e^{-\frac{p}{2}t^2}) \le \frac{1}{pt} \to 0, \ t \to \infty.
\end{aligned}
$$

It follows from (2.2) that

$$
\begin{aligned}
(3.4) \quad & E|\int_0^t [h(t,s) - h(s)]dB_s^{a,b}|^2 \\
&= \frac{b}{2\mathbb{B}(a+1,b+1)} \int_0^t \int_0^t [h(t,s)-h(s)][h(t,r)-h(r)](s\wedge r)^a|s-r|^{b-1}drds \\
&\le \frac{b}{2\mathbb{B}(a+1,b+1)} \int_0^t \int_0^t \frac{sr}{t^2}e^{\frac{p}{2}(s^2+r^2)}e^{-pt^2}(s\wedge r)^a|s-r|^{b-1}drds \\
&= \frac{be^{-pt^2}}{\mathbb{B}(a+1,b+1)t^2} \int_0^t \int_0^s sre^{\frac{p}{2}(s^2+r^2)}(s\wedge r)^a(s-r)^{b-1}drds
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{be^{-pt^2}}{\mathbb{B}(a+1,b+1)t^2}\int_0^t s^{a+2}e^{ps^2}ds\int_0^s (s-r)^{b-1}dr\\
&= \frac{e^{-pt^2}}{\mathbb{B}(a+1,b+1)t^2}\int_0^t s^{a+b+2}e^{ps^2}ds\\
&\leq \frac{e^{-pt^2}}{\mathbb{B}(a+1,b+1)t^2}t^{a+b+1}\int_0^t se^{ps^2}ds\\
&= \frac{e^{-pt^2}}{2\mathbb{B}(a+1,b+1)}t^{a+b-1}(\frac{1}{p}e^{pt^2}-1)\\
&\leq \frac{t^{a+b-1}}{2p\mathbb{B}(a+1,b+1)} \to 0,\ t\to\infty.
\end{aligned}
$$

This proves

$$
\begin{aligned}
E|X_t^{a,b}-X_\infty^{a,b}|^2 &\leq 2E|\int_0^t [h(t,s)-h(s)]dB_s^{a,b}|^2\\
&\quad + 2|\int_0^t [h(t,s)-h(s)]ds|^2 \to 0,\ t\to\infty.
\end{aligned}
$$

This completes the proof. □

Theorem 3.2. *The solution $X_t^{a,b}$ of (1.2) converges to $X_\infty^{a,b}$ almost surely as $t\to\infty$.*

Proof. Note that

$$
\begin{aligned}
&X_t^{a,b}-X_\infty^{a,b}\\
&= \int_0^t [h(t,s)-h(s)]dB_s^{a,b} - \int_t^\infty h(s)dB_s^{a,b} + \nu(\int_0^t h(t,s)ds - \int_0^\infty h(s)ds)\\
&\equiv Y_t^{a,b} - \int_t^\infty h(s)dB_s^{a,b} + \nu(\int_0^t h(t,s)ds - \int_0^\infty h(s)ds),\ t\geq 0.
\end{aligned}
$$

In order to complete the proof of Theorem, it only needs to be proved that $Y_t^{a,b}$ converges to 0 almost surely as $t\to\infty$.

Let $Z_{n,k}^{a,b} = Y_{n+\frac{k}{n}}^{a,b}$, $0\leq k<n$. Then $Z_{n,k}^{a,b}$ is Gaussian, and by (3.4) we have

$$
\begin{aligned}
E[(Z_{n,k}^{a,b})^2] &= E[|\int_0^{n+\frac{k}{n}} [h(n+\frac{k}{n},s)-h(s)]dB_s^{a,b}|^2]\\
&\leq \frac{1}{2p\mathbb{B}(a+1,b+1)}n^{a+b-1}
\end{aligned}
$$

and for any $\varepsilon>0$

$$
P(|Z_{n,k}^{a,b}|>\varepsilon) = \frac{2}{\sqrt{2\pi}\sigma}\int_\varepsilon^\infty e^{-\frac{t^2}{2\sigma^2}}dt \leq \frac{2}{\sqrt{2\pi}\sigma}\int_\varepsilon^\infty \frac{t}{\varepsilon}e^{-\frac{t^2}{2\sigma^2}}dt
$$

$$= \frac{2\sigma}{\sqrt{2\pi}\varepsilon} e^{-\frac{\varepsilon^2}{2\sigma^2}} \le \frac{n^{\frac{1}{2}(a+b-1)} e^{-\frac{\varepsilon^2 p\mathbb{B}(a+1,b+1)}{n^{a+b-1}}}}{\varepsilon\sqrt{\pi p\mathbb{B}(a+1,b+1)}}.$$

On the other hand, let $R_s^{n,k} = Y^{a,b}_{n+\frac{k+s}{n}} - Y^{a,b}_{n+\frac{k}{n}}, s \in (0,1)$. Then $R_s^{n,k}, 0 \le s \le 1$ is Gaussian and

$$E[(R_s^{n,k} - R_{s'}^{n,k})^2] = E[(Y^{a,b}_{n+\frac{k+s}{n}} - Y^{a,b}_{n+\frac{k+s'}{n}})^2].$$

We first calculate

$$\begin{aligned}
&E[(Y^{a,b}_{t_1} - Y^{a,b}_{t_2})^2] \\
&= E\left[\left(\int_0^{t_1}[h(t_1,s) - h(t_2,s)]dB_s^{a,b} + \int_{t_1}^{t_2}[h(s) - h(t_2,s)]dB_s^{a,b}\right)^2\right] \\
&\le 2\left(E(\int_0^{t_1}[h(t_1,s) - h(t_2,s)]dB_s^{a,b})^2 + E(\int_{t_1}^{t_2}[h(t_2,s) - h(s)]dB_s^{a,b})^2\right) \\
&:= 2(A_1 + A_2),
\end{aligned}$$

where

$$\begin{aligned}
A_1 &= \int_0^{t_1}\int_0^{t_1}[h(t_1,u) - h(t_2,u)][h(t_1,v) - h(t_2,v)]\frac{b(u\wedge v)^a|u-v|^{b+1}}{2\mathbb{B}(a+1,b+1)}dudv \\
&\le \frac{b}{2\mathbb{B}(a+1,b+1)}e^{-pt_1^2}(t_2-t_1)^2\int_0^{t_1}\int_0^{t_1} p^2uve^{\frac{1}{2}p(u^2+v^2)}(u\wedge v)^a|u-v|^{b-1}dudv \\
&= \frac{b}{2\mathbb{B}(a+1,b+1)}e^{-pt_1^2}(t_2-t_1)^2[\int_0^{t_1}\int_0^{v} p^2uve^{\frac{1}{2}p(u^2+v^2)}u^a(v-u)^{b-1}dudv \\
&\quad + \int_0^{t_1}\int_0^{u} p^2uv^{a+1}e^{\frac{1}{2}p(u^2+v^2)}(u-v)^{b-1}dudv] \\
&\le \frac{b}{2\mathbb{B}(a+1,b+1)}e^{-pt_1^2}(t_2-t_1)^2[\int_0^{t_1} p^2ve^{pv^2}dv\int_0^{v} u^{a+1}(v-u)^{b-1}du \\
&\quad + \int_0^{t_1} p^2ue^{pu^2}du\int_0^{u} v^{a+1}(u-v)^{b-1}dv] \\
&\le \frac{b}{2\mathbb{B}(a+1,b+1)}e^{-pt_1^2}(t_2-t_1)^2[\int_0^{t_1} p^2v^{a+2}e^{pv^2}dv\int_0^{v} (v-u)^{b-1}du \\
&\quad + \int_0^{t_1} p^2u^{a+2}e^{pu^2}du\int_0^{u} (u-v)^{b-1}dv] \\
&= \frac{b}{2\mathbb{B}(a+1,b+1)}e^{-pt_1^2}(t_2-t_1)^2[\frac{p}{b}\int_0^{t_1} pv^{a+b+2}e^{pv^2}dv + \frac{p}{b}\int_0^{t_1} pu^{a+b+2}e^{pu^2}du] \\
&\le \frac{b}{2\mathbb{B}(a+1,b+1)}e^{-pt_1^2}(t_2-t_1)^2\frac{p}{b}t_1^{a+b+1}e^{pt_1^2} \\
&= \frac{p}{2\mathbb{B}(a+1,b+1)}t_1^{a+b+1}(t_2-t_1)^2,
\end{aligned}$$

and

$$\begin{aligned}A_2 &= \int_{t_1}^{t_2}\int_{t_1}^{t_2} p^2 uve^{\frac{1}{2}p(u^2+v^2)}(\int_{t_2}^{\infty} e^{-\frac{1}{2}p\xi^2}d\xi)^2\phi(u,v)dudv\\ &\le \frac{e^{-pt_2^2}}{(pt_2)^2}p^2t_2^2e^{at_2^2}\int_{t_1}^{t_2}\int_{t_1}^{t_2}\phi(u,v)dudv\\ &\le C_{a,b}(t_2\vee t_1)^a|t_2-t_1|^{b+1}.\end{aligned}$$

Let $t_1 = n+\frac{k+s'}{n}, t_2 = n+\frac{k+s}{n}$, we can easily obtain

$$\begin{aligned}&E[(R_s^{n,k}-R_{s'}^{n,k})^2]\\ &\le C_{a,b,p}[(n+\frac{k+s}{n})^{a+b+1}(\frac{s'-s}{n})^2+(n+\frac{k+s\vee s'}{n})^a|\frac{s'-s}{n}|^{b+1}]\\ &= C_{a,b,p}|\frac{s'-s}{n}|^{b+1}(s'\vee s)^a(\frac{1}{(s'\vee s)^a}(n+\frac{k+s}{n})^{a+b+1}|\frac{s'-s}{n}|^{1-b}\\ &\quad +(\frac{n}{s'\vee s}+\frac{k+s\vee s'}{n(s'\vee s)})^a)\\ &\le C_{a,b,p}(n^{a+b-1}+n^{a-b-1})E(B_s^{a,b}-B_{s'}^{a,b})^2.\end{aligned}$$

For any $\varepsilon > 0$, it follows from Slepian's Lemma and Markov's inequality that

$$\begin{aligned}P(\sup_{0\le s\le 1}|R_s^{n,k}|>\varepsilon) &\le P(\sqrt{C_{a,b,p}n^{a+b-1}+n^{a-b-1}}\sup_{0\le s\le 1}|B_s^{a,b}|>\varepsilon)\\ &\le \frac{C_{a,b,p}(n^{a+b-1}+n^{a-b-1})E[\sup_{0\le s\le 1}|B_s^{a,b}|^2]}{\varepsilon^2}\\ &\le \frac{C_{a,b,p}(n^{a+b-1}+n^{a-b-1})}{\varepsilon^2}.\end{aligned}$$

Thus, the result holds due to the Borel-Cantelli Lemma and

$$\{\sup_{n+\frac{k}{n}<t<n+\frac{k+1}{n}}|Y_t|>\varepsilon\}\subseteq\{|Z_{n,k}^{a,b}|>\varepsilon/2\}\cup\{\sup_{0\le s\le 1}|R_s^{n,k}|>\varepsilon/2\}$$

for all $k, n \ge 0$. This completes the proof. □

4. Local time and Meyer-Tanaka type formula

In this section, we consider the linear weighted fractional self-attracting diffusion $X^{a,b} = \{X_t^{a,b}, 0 \le t \le T\}$ with $\nu = 0$. The main goals are to study the local time and weighted local time of the process and get the Meyer-Tanaka type formula.

For $0 \le s \le t \le T$, let

$$\sigma_t^2 \equiv E[(X_t^{a,b})^2] = \int_0^t\int_0^t h(t,u)h(t,v)\phi(u,v)dudv.$$

Then

$$e^{-\frac{p}{2}t^2}t^{a+b+1} \le \sigma_t^2 \le t^{a+b+1}, \tag{4.1}$$

since

$$\int_0^t\int_0^t \phi(u,v)dudv = t^{a+b+1}, \qquad e^{-\frac{p}{2}(t^2-s^2)} \le h(t,s) \le 1.$$

Let

$$\begin{aligned}\sigma_{t,s}^2 &\equiv E[(X_t^{a,b}-X_s^{a,b})^2] \\ &= \int_0^t\int_0^t [h(t,u)-h(s,u)][h(t,v)-h(s,v)]\phi(u,v)dudv.\end{aligned} \tag{4.2}$$

Lemma 4.1. *For any* $0 \le s \le t$, *we have*

$$c_{p,a,b,T}t^a(t-s)^{b+1} \le \sigma_{t,s}^2 \le C_{p,a,b,T}t^a(t-s)^{b+1}, \tag{4.3}$$

where $C_{p,a,b,T}, c_{p,a,b,T} > 0$ *are two constants depending on* p, a, b, T.

Proof. Obviously, we have

$$\begin{aligned}\sigma_{t,s}^2 &= \int_0^t\int_0^t [h(t,u)-h(s,u)][h(t,v)-h(s,v)]\phi(u,v)dudv \\ &= \int_s^t\int_s^t h(t,u)h(t,v)\phi(u,v)dudv \\ &\quad + \int_0^s\int_s^t h(t,u)[h(t,v)-h(s,v)]\phi(u,v)dudv \\ &\quad + \int_s^t\int_0^s [h(t,u)-h(s,u)]h(t,v)\phi(u,v)dudv \\ &\quad + \int_0^s\int_0^s [h(t,u)-h(s,u)][h(t,v)-h(s,v)]\phi(u,v)dudv \\ &= \Delta_{[s,t]^2} + \Delta_{[s,t]\times[0,s]} + \Delta_{[0,s]\times[s,t]} + \Delta_{[0,s]^2}.\end{aligned}$$

For any $0 \le s \le t$, we have

$$\begin{aligned}\Delta_{[s,t]\times[0,s]} &= \Delta_{[0,s]\times[s,t]} \\ &= -(\int_s^t e^{-\frac{p}{2}w^2}dw)\int_0^s pue^{\frac{p}{2}u^2}du\int_s^t h(t,v)\phi(u,v)dv,\end{aligned}$$

and

$$\Delta_{[0,s]^2} = 2(\int_s^t e^{-\frac{p}{2}w^2}dw)^2\int_0^s p^2ue^{\frac{p}{2}u^2}du\int_0^u ve^{\frac{p}{2}v^2}\phi(u,v)dv.$$

On the one hand,

$$\begin{aligned}\Delta_{[0,s]^2} &= \frac{b}{\mathbb{B}(a+1,b+1)}p^2(\int_s^t e^{-\frac{p}{2}w^2}dw)^2\int_0^s ue^{\frac{p}{2}u^2}du\int_0^u ve^{\frac{p}{2}v^2}v^a(u-v)^{b-1}dv \\ &\le \frac{b}{\mathbb{B}(a+1,b+1)}p^2e^{-ps^2}(t-s)^2\int_0^s u^{a+2}e^{pu^2}du\int_0^u (u-v)^{b-1}dv \\ &= \frac{1}{\mathbb{B}(a+1,b+1)}p^2e^{-ps^2}(t-s)^2\int_0^s u^{a+b+2}e^{pu^2}du\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\mathbb{B}(a+1,b+1)} p^2 e^{-ps^2}(t-s)^2 e^{ps^2} s^2 \int_0^s u^{a+b} du \\
&= \frac{1}{(a+b+1)\mathbb{B}(a+1,b+1)} p^2 (t-s)^2 s^{a+b+3}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\Delta_{[0,s]^2} &\geq \frac{b}{\mathbb{B}(a+1,b+1)} p^2 e^{-pT^2}(t-s)^2 \int_0^s u e^{\frac{p}{2}u^2} du \int_0^u v e^{\frac{p}{2}v^2}(u-v)^{b-1} v^a dv \\
&\geq \frac{b}{\mathbb{B}(a+1,b+1)} p^2 e^{-pT^2}(t-s)^2 \int_0^s \int_0^u v^{a+2}(u-v)^{b-1} dv du \\
&= \frac{b}{\mathbb{B}(a+1,b+1)} p^2 e^{-pT^2}(t-s)^2 \mathbb{B}(a+3,b) \int_0^s u^{a+b+2} du \\
&= \frac{b\mathbb{B}(a+3,b)}{(a+b+3)\mathbb{B}(a+1,b+1)} p^2 e^{-pT^2}(t-s)^2 s^{a+b+3},
\end{aligned}
$$

which gives

$$
c_{a,b} p^2 e^{-pT^2}(t-s)^2 s^{a+b+3} \leq \Delta_{[0,s]^2} \leq C_{a,b} p^2 (t-s)^2 s^{a+b+3}.
$$

Similarly, we have

$$
\begin{aligned}
\Delta_{[s,t]^2} &= \int_s^t \int_s^t h(t,u)h(t,v)\phi(u,v) du dv \\
&\leq \int_s^t \int_s^t \phi(u,v) du dv \\
&= \int_s^t \int_s^t b(u \wedge v)^a |u-v|^{b-1} du dv \\
&= \int_s^t \int_s^v b u^a (v-u)^{b-1} du dv + \int_s^t \int_v^t b v^a (u-v)^{b-1} du dv \\
&= \int_s^t \int_u^t b u^a (v-u)^{b-1} dv du + \int_s^t \int_v^t b v^a (u-v)^{b-1} du dv \\
&= 2 \int_s^t u^a (t-u)^b du \\
&\leq C_2 (s^a \vee t^a)(t-s)^{b+1},
\end{aligned}
$$

and

$$
\begin{aligned}
\Delta_{[s,t]^2} &\geq \int_s^t \int_s^t e^{-\frac{p}{2}(t^2-u^2)} e^{-\frac{p}{2}(t^2-v^2)} b(u \wedge v)^a |u-v|^{b-1} du dv \\
&\geq b e^{-p(t^2-s^2)} \int_s^t \int_s^t (u \wedge v)^a |u-v|^{b-1} du dv \\
&= 2 e^{-p(t^2-s^2)} \int_s^t u^a (t-u)^b du
\end{aligned}
$$

$$\geq 2(s^a \wedge t^a) e^{-p(t^2-s^2)} \int_s^t (t-u)^b du$$
$$= C_1 (s^a \wedge t^a)(t-s)^{b+1} e^{-p(t^2-s^2)},$$

which gives

$$C_1 (s^a \wedge t^a)(t-s)^{b+1} e^{-p(t^2-s^2)} \leq \Delta_{[s,t]^2} \leq C_2 (s^a \vee t^a)(t-s)^{b+1}.$$

It is easy to obtain that

$$\lim_{s \uparrow t} \frac{\Delta_{[s,t]\times[0,s]}}{t^a (t-s)^{b+1}} = 0, \qquad \lim_{s \downarrow 0} \frac{\Delta_{[s,t]\times[0,s]}}{t^a (t-s)^{b+1}} = 0.$$

Hence

$$\lim_{s \uparrow t} \frac{\sigma_{t,s}^2}{t^a (t-s)^{b+1}} = C_{a,b,T}, \quad c_{a,b,T} \leq \lim_{s \downarrow 0} \frac{\sigma_{t,s}^2}{t^a (t-s)^{b+1}} \leq C_{p,a,b,T}.$$

So we have

$$c_{p,a,b,T} t^a (t-s)^{b+1} \leq \sigma_{t,s}^2 \leq C_{p,a,b,T} t^a (t-s)^{b+1}.$$

This completes the proof. □

From the Lemma above, we see that

$$\int_0^t \int_0^t [E(X_u^{a,b} - X_v^{a,b})^2]^{-\frac{1}{2}} du dv < \infty$$

holds for all $t \geq 0$, and combining this with Berman [2, 18], the solution $X^{a,b}$ of (1.2) has continuous *local time* $\mathfrak{L}_t^x,\ t \geq 0,\ x \in \mathbb{R}$ such that

$$\mathfrak{L}_t^x = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t 1_{[x-\varepsilon, x+\varepsilon]}(X_s^{a,b}) ds = \int_0^t \delta(X_s^{a,b} - x) ds,$$

where $\delta(X_s^{a,b} - \cdot)$ denotes the delta function of $X_s^{a,b}$.

For $t \geq 0, x \in \mathbb{R}$, put

$$\mathcal{L}_t^x = \frac{b}{\mathbb{B}(a+1, b+1)} \int_0^t \delta(X_s^{a,b} - x) ds \int_0^s h(s,m) s^a (s-m)^{b+1} dm.$$

Then $\mathcal{L}_t^x$ is well-defined and $\mathcal{L}_t^x = \int_0^t \delta(X_s^{a,b} - x) D_s^\phi X_s^{a,b} ds$. The process $(\mathcal{L}_t^x)_{t \geq 0}$ is called *the weighted local time* of $X^{a,b}$ at $x \in \mathbb{R}$.

Lemma 4.2 (Hu and Øksendal [15]). *Let Y be normally distributed with mean 0 and variance $\sigma^2 (\sigma > 0)$. Then the delta function $\delta(Y - \cdot)$ of Y exists uniquely and we have*

$$\delta(Y - x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi(Y-x)} d\xi,\ x \in \mathbb{R}. \tag{4.4}$$

Proposition 4.3. *Assume that $t \in [0,T]$. Then $\mathcal{L}_t^x$ and $\mathfrak{L}_t^x$ are square integrable for all $x \in \mathbb{R}$ and*

$$E[\mathfrak{L}_t^x]^2 \le C_{p,a,b,T} t^{1-(a+b)}, \tag{4.5}$$

$$E[\mathcal{L}_t^x]^2 \le C_{p,a,b,T} t^{1-(a+b)}. \tag{4.6}$$

Proof. It follows from Lemma 4.2 that

$$\begin{aligned} E[(\mathfrak{L}_t^x)^2] &\le \frac{1}{(2\pi)^2}\int_0^t\int_0^t dudv \int_{\mathbb{R}^2} E[e^{i(\xi X_u^{a,b}+\eta X_v^{a,b})}]d\xi d\eta \\ &\le \frac{2}{(2\pi)^2}\int_0^t du\int_0^u dv\int_{\mathbb{R}^2} e^{-\frac{1}{2}Var(\xi X_u^{a,b}+\eta X_v^{a,b})}d\xi d\eta. \end{aligned}$$

By local nondeterminacy of the process $X^{a,b}$, we have

$$\mathrm{Var}(\xi X_u^{a,b}+\eta X_v^{a,b}) \ge k[\xi^2\sigma_{u,v}^2 + (\eta+\xi)^2\sigma_v^2]$$

for a positive $k>0$. Hence, by Lemma 4.1,

$$\begin{aligned} E[(\mathfrak{L}_t^x)^2] &\le \frac{2}{(2\pi)^2}\int_0^t du\int_0^u dv\int_{\mathbb{R}^2} e^{-\frac{k}{2}(\xi^2\sigma_{u,v}^2+(\eta+\xi)^2\sigma_v^2)}d\xi d\eta \\ &\le \frac{1}{k\pi}\int_0^t du\int_0^u \frac{1}{\sigma_{u,v}\sigma_v}dv \\ &\le \frac{1}{k\pi c_{p,a,b,T}}\int_0^t du\int_0^u \frac{1}{u^{\frac{a}{2}}(u-v)^{\frac{b+1}{2}}v^{\frac{a+b+1}{2}}}dv \\ &\le C_{p,a,b,T}t^{1-(a+b)}. \end{aligned}$$

In fact, when $a>0$, we have

$$\begin{aligned} \int_0^t du\int_0^u \frac{dv}{u^{\frac{a}{2}}(u-v)^{\frac{b+1}{2}}v^{\frac{a+b+1}{2}}} &\le \int_0^t du\int_0^u \frac{dv}{(u-v)^{\frac{a+b+1}{2}}v^{\frac{a+b+1}{2}}} \\ &= \int_0^t \frac{1}{v^{\frac{a+b+1}{2}}}dv\int_v^t \frac{du}{(u-v)^{\frac{a+b+1}{2}}} \\ &= \frac{2}{2-(a+b+1)}\int_0^t \frac{1}{v^{\frac{a+b+1}{2}}}(t-v)^{1-\frac{a+b+1}{2}}dv \\ &\le C_{a,b}t^{2-(a+b+1)}. \end{aligned}$$

When $a<0$, we have

$$\begin{aligned} \int_0^t du\int_0^u \frac{dv}{u^{\frac{a}{2}}(u-v)^{\frac{b+1}{2}}v^{\frac{a+b+1}{2}}} &\le \int_0^t du\int_0^u \frac{dv}{u^{\frac{a+b+1}{2}}v^{\frac{a+b+1}{2}}} \\ &\le C_{a,b}t^{2-(a+b+1)}. \end{aligned}$$

We obtain (4.5). Similarly, we can show that the inequality (4.6) holds as follows

$$\begin{aligned}
E[\mathcal{L}_t^x]^2 &= \frac{b^2}{\mathbb{B}^2(a+1,b+1)} E\Big[\int_0^t \delta(X_s^{a,b}-x)ds \int_0^s h(s,m)s^a(s-m)^{b+1}dm\Big]^2 \\
&\le \frac{b^2}{(2\pi)^2\mathbb{B}^2(a+1,b+1)} \int_0^t\int_0^t dudv \int_{\mathbb{R}^2} Ee^{i(\xi X_u+\eta X_v)}d\xi d\eta \\
&\quad \cdot \int_0^u\int_0^v h(u,m_1)h(v,m_2)u^a v^a(u-m_1)^{b+1}(v-m_2)^{b+1}dm_1dm_2 \\
&\le C_{p,a,b,T}\int_0^t\int_0^t dudv \int_{\mathbb{R}^2} Ee^{i(\xi X_u+\eta X_v)}d\xi d\eta \\
&\le C_{p,a,b,T}t^{1-(a+b)}. \qquad \square
\end{aligned}$$

Theorem 4.4. *Suppose that $\Phi:\mathbb{R}^+\to\mathbb{R}$ is a convex function having polynomial growth. Then*

$$\Phi(X_t^{a,b}) = \Phi(z) + \int_0^t D^-\Phi(X_s^{a,b})dX_s^{a,b} + \int_{\mathbb{R}} \mathcal{L}_t^x \mu_\Phi(dx), \tag{4.7}$$

where $X^{a,b}$ is the solution of (1.2) *with $a>-1, |b|<a+1, 0<b<1, X_0^{a,b}=z, \nu=0$ and $\mathcal{L}$ is the weighted local time of $X^{a,b}$, $D^-\Phi$ is the left derivative of Φ and the signed measure μ_Φ is defined by*

$$\mu_\Phi([a,b]) = D^-\Phi(b)-D^-\Phi(a), a<b, a,b\in\mathbb{R}.$$

Proof. Let

$$\Phi_\varepsilon(x) = \int_{\mathbb{R}} p_\varepsilon(x-y)\Phi(y)dy,\ \varepsilon>0,\ x\in\mathbb{R},$$

where $p_\varepsilon(x)=\frac{1}{\sqrt{2\pi\varepsilon}}e^{-\frac{x^2}{2\varepsilon}}$. Then $\Phi_\varepsilon\in\mathbb{C}^2$ and

$$\lim_{\varepsilon\downarrow 0}\Phi_\varepsilon(x)=\Phi(x),\quad \lim_{\varepsilon\downarrow 0}\Phi'_\varepsilon(x)=D^-\Phi(x)$$

for all $x\in\mathbb{R}$. Hence for any $\varepsilon>0$

$$\Phi_\varepsilon(X_t^{a,b}) = \Phi_\varepsilon(z)+\int_0^t \Phi'_\varepsilon(X_s^{a,b})dX_s^{a,b} + \frac{b}{\mathbb{B}(a+1,b+1)}\int_0^t \Phi''_\varepsilon(X_s^{a,b})\tilde{h}(s)ds,$$

where $\tilde{h}(s)=\int_0^s h(s,m)m^a(s-m)^{b-1}dm$. On the other hand,

$$\Phi_\varepsilon(X_t^{a,b}) \xrightarrow{\text{a.s.}} \Phi(X_t^{a,b}),\quad \varepsilon\to 0,$$

and

$$\int_0^t \Phi'_\varepsilon(X_s^{a,b})dX_s^{a,b} \xrightarrow{\text{a.s.}} \int_0^t D^-\Phi(X_s^{a,b})dX_s^{a,b},\quad \varepsilon\to 0.$$

Finally, we have

$$\int_0^t \Phi''_\varepsilon(X_s^{a,b})\tilde{h}(s)ds = \int_0^t ds\tilde{h}(s)\int_{\mathbb{R}}\Phi''_\varepsilon(x)\delta(X_s^{a,b}-x)dx$$

$$\to \frac{\mathbb{B}(a+1,b+1)}{b} \int_{\mathbb{R}} \mathcal{L}_t^x \mu_\Phi(dx), \ \varepsilon \to 0.$$

This completes the proof. □

Corollary 4.5. *Let $X^{a,b}$ be the solution to* (1.2) *with parameter $a > -1, |b| < a+1, 0 < b < 1, X_0^{a,b} = z, \nu = 0$ and let $\mathcal{L}$ be the weighted local time of $X^{a,b}$. Then the Tanaka formula*

$$|X_t^{a,b} - x| = |X_0^{a,b} - x| + \int_0^t sign(X_s^{a,b} - x) dX_s^{a,b} + \mathcal{L}_t^x \tag{4.8}$$

holds for all $x \in \mathbb{R}$.

5. Derivative for the intersection local time

In this section, let $B_t^{a,b,j}, j = 1,2$ are two independent wfBms. We will study the renormalized intersection local time of the linear weighted fractional self-attracting diffusion $X^{a,b} = (X^{a,b,1}, X^{a,b,2})$ on $\mathbb{R}^2$, where $X^{a,b,j} (j = 1,2)$ is the solution of the equation

$$X_t^{a,b,j} = B_t^{a,b,j} - p \int_0^t \int_0^u (X_u^{a,b,j} - X_v^{a,b,j}) dv du, \qquad 0 \le t \le T,$$

with $p > 0$. The renormalized intersection local time l_T of the process

$$X_t^{a,b} = (X_t^{a,b,1}, X_t^{a,b,2}), \ 0 \le t \le T$$

is formally defined as

$$\begin{aligned} l_T - E[l_T] = & \int_0^T \int_0^T \delta(X_t^{a_1,b_1} - X_s^{a_2,b_2}) ds dt \\ & - E\Big[\int_0^T \int_0^T \delta(X_t^{a_1,b_1} - X_s^{a_2,b_2}) ds dt\Big], \end{aligned}$$

where δ is the delta function. For $\varepsilon > 0$, we define

$$l_{\varepsilon,T} = \int_0^T \int_0^T p_\varepsilon(X_t^{a_1,b_1} - X_s^{a_2,b_2}) ds dt,$$

where

$$p_\varepsilon(x) = \frac{1}{\sqrt{2\pi\varepsilon}} e^{-\frac{|x|^2}{2\varepsilon}}, \ x \in \mathbb{R}^2$$

is the heat kernel.

In order to obtain the convergence of $l_{\varepsilon,T} - E[l_{\varepsilon,T}]$, we need some lemmas.

For $0 \le s \le t, 0 \le s' \le t'$. Denote

$$\begin{aligned} \lambda_{t,s} &= E(X_t^{a_1,b_1,1} - X_s^{a_2,b_2,1})^2 = t^{a_1+b_1+1} + s^{a_2+b_2+1}, \\ \mu &= E(X_t^{a_1,b_1,1} - X_s^{a_2,b_2,1})(X_{t'}^{a_1,b_1,1} - X_{s'}^{a_2,b_2,1}), \end{aligned}$$

and

$$d(s,t,s',t') = \lambda_{s,t}\lambda_{s',t'} - \mu^2.$$

Then, from Shen and Chen [22] we can establish the following lemmas.

Lemma 5.1. (1) *For* $0 < s < s' < t < t' < T$ *or* $0 < s < t < s' < t' < T$, *we have*

$$d(s,t,s',t') \geq C(s^{a_2+b_2+1} + t^{a_1+b_1+1})(s^{a_2}(s'-s)^{1+b_2} + (t')^{a_1}(t'-t)^{1+b_1}). \tag{5.1}$$

(2) *For* $0 < s' < s < t < t' < T$, *we have*

$$d(s,t,s',t') \geq C((s')^{a_2+b_2+1} + t^{a_1+b_1+1})(s^{a_2}(s-s')^{1+b_2} + (t')^{a_1}(t'-t)^{1+b_1}). \tag{5.2}$$

Lemma 5.2. *Let* $\lambda_{t,s}$ *and* μ *be as above. Then*

$$\int_{\mathbb{T}} \frac{\mu^2 ds dt ds' dt'}{d(s,t,s',t')(\lambda_{t,s}\lambda_{t',s'})} < \infty,$$

if $\max\{a_1+b_1, a_2+b_2\} < 0$.

Proof. In fact,

$$\begin{aligned}
\mu &= EX_t^{a_1,b_1} X_{t'}^{a_1,b_1} + EX_s^{a_2,b_2} X_{s'}^{a_2,b_2} \\
&\leq \int_0^t \int_0^{t'} \phi(u,v) dv du + \int_0^s \int_0^{s'} \phi(u,v) dv du \\
&\leq \frac{1}{2}(t^{a_1+b_1+1} + (t')^{a_1+b_1+1} + s^{a_2+b_2+1} + (s')^{a_2+b_2+1}).
\end{aligned}$$

For $0 < s < s' < t < t' < T$, we have

$$\begin{aligned}
&\int_{\mathbb{T}} \frac{\mu^2 ds dt ds' dt'}{d(s,t,s',t')(\lambda_{t,s}\lambda_{t',s'})} \\
&\leq \int_{\mathbb{T}} \frac{\mu^2 ds dt ds' dt'}{t^{a_1+\frac{1}{2}(b_1+1)}(t')^{a_1+b_1+1} s^{a_2+\frac{1}{2}(b_2+1)}(s')^{a_2+b_2+1}|t-t'|^{\frac{1}{2}(b_1+1)}|s-s'|^{\frac{1}{2}(b_2+1)}} \\
&< \infty
\end{aligned}$$

holds for $\max\{a_1+b_1, a_2+b_2\} < 0$. Similarly, we can estimate the inequality holds for $0 < s < t < s' < t' < T$ and $0 < s' < s < t < t' < T$. Thus, the Lemma follows. □

Theorem 5.3. *The random variable* $l_{\varepsilon,T} - E[l_{\varepsilon,T}]$ *converges in* L^2 *as* $\varepsilon \to 0$ *if* $\max\{a_1+b_1, a_2+b_2\} < 0$.

Proof. Clearly, as $\varepsilon \to 0$, $l_{\varepsilon,T} - E[l_{\varepsilon,T}]$ converges in L^2 if and only if

$$\mathrm{V}ar(l_{\varepsilon,T}) = E[(l_{\varepsilon,T})^2] - [E(l_{\varepsilon,T})]^2 \tag{5.3}$$

tends to a constant. Next, we will prove $\mathrm{V}ar(l_{\varepsilon,T})$ converges as $\varepsilon \to 0$. Note that

$$l_{\varepsilon,T} = \frac{1}{2\pi^2} \int_0^T \int_0^T \int_{\mathbb{R}^2} e^{i\langle \xi, X_t^{a_1,b_1} - X_s^{a_2,b_2}\rangle} e^{-\varepsilon\frac{|\xi|^2}{2}} d\xi ds dt, \tag{5.4}$$

since

$$p_\varepsilon(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i\langle \xi, x\rangle} e^{-\varepsilon\frac{|\xi|^2}{2}} d\xi.$$

Combining this with the facts $\langle\xi, X_t^{a_1,b_1} - X_s^{a_2,b_2}\rangle \sim N(0, |\xi|^2\lambda_{t,s})$ and

$$E[e^{i\langle\xi, X_t^{a_1,b_1} - X_s^{a_2,b_2}\rangle}] = e^{-\frac{1}{2}|\xi|^2\lambda_{t,s}},$$
$$\int_{\mathbb{R}^2} e^{-\frac{1}{2}|\xi|^2(\lambda_{t,s}+\varepsilon)}d\xi = \frac{2\pi}{\lambda_{t,s}+\varepsilon},$$

we have

$$\begin{aligned} E(l_{\varepsilon,T}) &= 2\int_0^T\int_0^t E[p_\varepsilon(X_t^{a_1,b_1} - X_s^{a_2,b_2})]dsdt \\ &= \frac{1}{\pi}\int_0^T\int_0^t(\lambda_{t,s}+\varepsilon)^{-1}dsdt. \end{aligned} \tag{5.5}$$

Denote $\mathbb{T} = \{(s,t,s',t') : 0 < s < t < T, 0 < s' < t' < T\}$, then

$$\begin{aligned} &E[(l_{\varepsilon,T})^2] \\ &= \frac{1}{4\pi^4}\int_{\mathbb{T}}\int_{\mathbb{R}^4} Ee^{i\langle\xi, X_t^{a_1,b_1} - X_s^{a_2,b_2}\rangle + i\langle\eta, X_{t'}^{a_1,b_1} - X_{s'}^{a_2,b_2}\rangle}e^{-\varepsilon\frac{|\xi|^2+|\eta|^2}{2}}d\xi d\eta dsdtds'dt'. \end{aligned}$$

Note that

$$\langle\xi, X_t^{a_1,b_1} - X_s^{a_2,b_2}\rangle + \langle\eta, X_{t'}^{a_1,b_1} - X_{s'}^{a_2,b_2}\rangle \sim N(0, |\xi|^2\lambda_{t,s} + 2\mu\langle\xi,\eta\rangle + |\eta|^2\lambda_{t',s'})$$

for any $\xi, \eta \in \mathbb{R}^2$, so

$$\begin{aligned} E[(l_{\varepsilon,T})^2] &= \frac{1}{4\pi^4}\int_{\mathbb{T}}\int_{\mathbb{R}^4} e^{-\frac{1}{2}((\lambda_{t,s}+\varepsilon)|\xi|^2 + 2\mu\langle\xi,\eta\rangle + (\lambda_{t',s'}+\varepsilon)|\eta|^2)}d\xi d\eta dsdtds'dt' \\ &= \frac{1}{4\pi^2}\int_{\mathbb{T}}((\lambda_{t,s}+\varepsilon)(\lambda_{t',s'}+\varepsilon) - \mu^2)^{-1}dsdtds'dt' \end{aligned}$$

for all $\varepsilon > 0$. It follows from (5.5) that

$$\begin{aligned} &E[(l_{\varepsilon,T})^2] - [E(l_{\varepsilon,T})]^2 \\ &= \frac{1}{\pi^2}\int_{\mathbb{T}}[((\lambda_{t,s}+\varepsilon)(\lambda_{t',s'}+\varepsilon) - \mu^2)^{-1} - ((\lambda_{t,s}+\varepsilon)(\lambda_{t',s'}+\varepsilon))^{-1}]dsdtds'dt' \\ &= \frac{1}{\pi^2}\int_{\mathbb{T}}\frac{\mu^2 dsdtds'dt'}{((\lambda_{t,s}+\varepsilon)(\lambda_{t',s'}+\varepsilon) - \mu^2)(\lambda_{t,s}+\varepsilon)(\lambda_{t',s'}+\varepsilon)}. \end{aligned}$$

Hence, this completes the proof by Lemma 5.2. □

Last, we will study the derivative of the renormalized intersection local time of the linear weighted self-attracting diffusion X^{a_1,b_1}, X^{a_2,b_2} on $\mathbb{R}^2$, where $X^{a_j,b_j}(j = 1,2)$ is the solution of the equation

$$X_t^{a_j,b_j} = B_t^{a_j,b_j} - p\int_0^t\int_0^u(X_u^{a_j,b_j} - X_v^{a_j,b_j})dvdu,\ 0 \le t \le T,$$

with $p > 0$ and $B_t^{a_j,b_j}, j = 1,2$ are two independent wfBms. Then we have

$$X_t^{a_j,b_j} = \int_0^t h(t,s)dB_s^{a_j,b_j},\ j = 1,2$$

from Section 3, and for any $s, t \geq 0$

$$h(t,s) = \begin{cases} 1 - pse^{\frac{1}{2}ps^2} \int_s^t e^{-\frac{1}{2}pu^2} du, & t \geq s, \\ 0, & t < s. \end{cases}$$

Denote

$$l'_{\varepsilon,T} = \int_0^T \int_0^t p'_\varepsilon(X_t^{a_1,b_1} - X_s^{a_2,b_2}) dsdt.$$

Recall that the process

$$\begin{aligned} l'_T - E[l'_T] := & \int_0^T \int_0^t \delta_0(X_t^{a_1,b_1} - X_s^{a_2,b_2}) dsdt \\ & - E\Big[\int_0^T \int_0^t \delta_0(X_t^{a_1,b_1} - X_s^{a_2,b_2}) dsdt\Big] \end{aligned}$$

is called the derivative for the renormalized intersection local time of the processes $X_t^{a_j,b_j}$, $j = 1, 2$, $0 \leq t \leq T$. For the derivative of self-intersection local time of fractional Brownian motion, we can see Yan *et al.* [2], Yan [27], Yan and Yu [30], Jung and Markowsky [17, 18], Jaramillo and Nualart [16], Guo *et al.* [12]. Now, the main object of this section is to explain and prove Theorem 5.4.

Theorem 5.4. *The random variable $l'_{\varepsilon,T} - E[l'_{\varepsilon,T}]$ converges in L^2 as ε tends to zero if* $\max\{b_1, b_2, a_1 + b_1, a_2 + b_2\} < \frac{1}{3}$.

Proof. Clearly, as ε tends to zero, $l'_T - E[l'_T]$ converges in L^2 if and only if

$$\mathrm{V}ar(l'_{\varepsilon,T}) = E[(l'_{\varepsilon,T})^2] - (E(l'_{\varepsilon,T}))^2 \tag{5.6}$$

tends to constant. Now let us show that $\mathrm{V}ar(l'_{\varepsilon,T})$ converges as ε tends to zero. Using the classical equality

$$p_\varepsilon(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi x} e^{-\varepsilon \frac{\xi^2}{2}} d\xi,$$

one can obtain

$$l'_{\varepsilon,T} = \frac{1}{\pi} \int_0^T \int_0^t \int_{\mathbb{R}} e^{i\xi(X_t^{a_1,b_1} - X_s^{a_2,b_2})} e^{-\varepsilon \frac{\xi^2}{2}} d\xi dsdt. \tag{5.7}$$

Combining this with the facts $\xi(X_t^{a_1,b_1} - X_s^{a_2,b_2}) \sim N(0, \xi^2 \lambda_{t,s})$ and

$$E[e^{i\xi(X_t^{a_1,b_1} - X_s^{a_2,b_2})}] = e^{-\frac{1}{2}\xi^2 \lambda_{t,s}},$$
$$\int_{\mathbb{R}} \xi e^{-\frac{1}{2}\xi^2(\lambda_{t,s} + \varepsilon)} d\xi = 0,$$

we get

$$\begin{aligned} E[l'_{\varepsilon,T}] &= -2 \int_0^T \int_0^t E(p'_\varepsilon(X_t^{a_1,b_1} - X_s^{a_2,b_2})) dsdt \\ &= -\frac{i}{\pi} \int_0^T \int_0^t \int_{\mathbb{R}} \xi e^{i\xi(X_t^{a_1,b_1} - X_s^{a_2,b_2})} e^{-\varepsilon \frac{\xi^2}{2}} d\xi dsdt = 0. \end{aligned} \tag{5.8}$$

Denote $\mathbb{T} = \{(s,t,s',t') : 0 < s < t < T, 0 < s' < t' < T\}$. According to the representation (5.7) we get

$$
\begin{aligned}
&E[(l'_{\varepsilon,T})^2]\\
&= \frac{1}{\pi^2}\int_{\mathbb{T}}\int_{\mathbb{R}^2} \xi\eta E e^{i\xi(X_t^{a_1,b_1}-X_s^{a_2,b_2})+i\eta(X_{t'}^{a_1,b_1}-X_{s'}^{a_2,b_2})} e^{-\varepsilon\frac{\xi^2+\eta^2}{2}} d\xi d\eta ds dt ds' dt'.
\end{aligned}
$$

Noting that

$$
\xi(X_t^{a_1,b_1}-X_s^{a_2,b_2})+\eta(X_{t'}^{a_1,b_1}-X_{s'}^{a_2,b_2}) \sim N(0,\xi^2\lambda_{t,s}+2\xi\eta\mu+\eta^2\lambda_{t',s'})
$$

for any $\xi,\eta\in\mathbb{R}$, we can write

$$
\begin{aligned}
E[(l'_{\varepsilon,T})^2] &= \frac{1}{(2\pi)^2}\int_{\mathbb{T}}\int_{\mathbb{R}^2} \xi\eta e^{-\frac{1}{2}(\xi^2\lambda_{t,s}+2\xi\eta\mu+\eta^2\lambda_{t',s'})} e^{-\varepsilon\frac{\xi^2+\eta^2}{2}} d\xi d\eta ds dt ds' dt'\\
&= C\int_{\mathbb{T}} \frac{\mu ds dt ds' dt'}{((\lambda_{t,s}+\varepsilon)(\lambda_{t',s'}+\varepsilon)-\mu^2)^{\frac{3}{2}}}
\end{aligned}
$$

for all $\varepsilon > 0$. It follows from (5.8) that

$$
E[(l'_{\varepsilon,T})^2] - [E(l'_{\varepsilon,T})]^2 = C\int_{\mathbb{T}} \frac{\mu ds dt ds' dt'}{((\lambda_{t,s}+\varepsilon)(\lambda_{t',s'}+\varepsilon)-\mu^2)^{\frac{3}{2}}}.
$$

So, the Theorem follows from Shen and Chen [22]. □

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