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ISOLATION NUMBERS OF INTEGER MATRICES AND THEIR PRESERVERS

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Abstract. Let A be an $m \times n$ matrix over nonnegative integers. The isolation number of A is the maximum number of isolated entries in A. We investigate linear operators that preserve the isolation number of matrices over nonnegative integers. We obtain that T is a linear operator that strongly preserve isolation number k for $1 \leq k \leq \min\{m, n\}$ if and only if T is a (P, Q)-operator, that is, for fixed permutation matrices P and Q, $T(A) = PAQ$ or, $m = n$ and $T(A) = PA^tQ$ for any $m \times n$ matrix A, where A^t is the transpose of A.

1. Introduction

A *semiring* is a set $\mathbb{S}$ equipped with two binary operations $+$ and $\cdot$ such that $(\mathbb{S}, +)$ is a commutative monoid with identity element 0 and $(\mathbb{S}, \cdot)$ is a monoid with identity element 1. In addition, the operations $+$ and $\cdot$ are connected by distributivity and 0 annihilates $\mathbb{S}$.

A semiring $\mathbb{S}$ is called *antinegative* if 0 is the only element to have an additive inverse. The following are some examples of antinegative semirings which occur in combinatorics. Let $\mathbb{B} = \{0, 1\}$. Then $(\mathbb{B}, +, \cdot)$ is an antinegative semiring (the *binary Boolean semiring*) if arithmetic in $\mathbb{B}$ follows the usual rules except that $1 + 1 = 1$. If $\mathbb{F}$ is the real interval $[0, 1]$, then $(\mathbb{F}, +, \cdot) = (\mathbb{F}, max, min)$ is an antinegative semiring (*the fuzzy semiring*). Any nonnegative subsemiring of the real numbers, such as the nonnegative integers, is an antinegative semiring.

Let $\mathbb{Z}_+$ be the semiring of nonnegative integers. There are many papers on linear operators on a matrix space that preserve matrix functions over an algebraic structure; see ([5], [11] and [14]). Integer matrices also have been the subject of research by many authors ([3], [14]).

Finding the factor rank of a Boolean $(0, 1)$-matrix is an NP-complete problem ([13]), and consequently finding bounds on the factor rank of a $(0, 1)$-matrix

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is of interest to those researchers that would use factor rank in their work. If the $(0,1)$-matrix is the reduced adjacency matrix of a bipartite graph, the isolation number of the matrix is the maximum size of a non-competitive matching in the bipartite graph. This is related to the study of such combinatorial problems as the patient-hospital problem, the stable marriage problem, etc. ([4], [9]). An additional reason for studying the isolation number is that it is a lower bound on the factor rank of a Boolean $(0,1)$-matrix ([1], [10]).

Terms not specifically defined here can be found in Brualdi and Ryser [8] for matrix terms, or Bondy and Murty [7] for graph theoretic terms.

Beasley and Pullman ([3]) introduced factor rank of a matrix in $\mathcal{M}_{m,n}(\mathbb{Z}_+)$ and compared it with Boolean rank of its support. Gregory et al. ([10]) introduced set of isolated entries and compared Boolean rank with biclique covering number. Beasley ([1]) introduced isolation number of (0,1)-matrix and compared it with Boolean rank.

In this article, we consider the isolation number of a matrix over $\mathbb{Z}_+$ and characterize the linear operators that preserve sets defined by the isolation number.

2. Preliminaries

Let $\mathcal{M}_{m,n}(\mathbb{Z}_+)$ be the set of all $m \times n$ matrices with entries in the semiring $\mathbb{Z}_+$. The usual definitions for adding and multiplying matrices apply to integer matrices as well. The matrix $A^{(m,n)}$ denotes a matrix in $\mathcal{M}_{m,n}(\mathbb{Z}_+)$, I_n is the $n \times n$ identity matrix, $O^{(m,n)}$ is the $m \times n$ zero matrix, and $J^{(m,n)}$ is the $m \times n$ matrix all of whose entries are 1. Let $E_{i,j}^{(m,n)}$ be the $m \times n$ matrix whose (i,j)th entry is 1 and whose other entries are all 0. We call $E_{i,j}^{(m,n)}$ a *cell* and $\alpha E_{i,j}^{(m,n)}$ a *weighted cell*. We will suppress the superscripts or subscripts on these matrices when the orders are evident from the context and we write A, I, O, J, E_{ij} and $\alpha E_{i,j}$ respectively. For a matrix A, $\#(A)$ denotes the number of nonzero entries in A. Further, we let the set of all cells be denoted $\mathbb{E}$. That is,

$$\mathbb{E} = \{E_{i,j} \in \mathcal{M}_{m,n}(\mathbb{Z}_+) \mid i = 1, \dots, m \text{ and } j = 1, \dots, n\}.$$

2.1. Factor rank and isolation numbers

The *factor rank* ([3]), $r(A)$, of a nonzero matrix A in $\mathcal{M}_{m,n}(\mathbb{Z}_+)$ is the minimal number k such that there exist matrices $B \in \mathcal{M}_{m,k}(\mathbb{Z}_+)$ and $C \in \mathcal{M}_{k,n}(\mathbb{Z}_+)$ such that $A = BC$. The factor rank of the zero matrix is 0. The factor rank of matrices over fields is the same as usual rank. If $r(A) = k$, then $A = [\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k] \times [\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_k]^t = \mathbf{a}_1\mathbf{b}_1{}^t + \mathbf{a}_2\mathbf{b}_2{}^t + \cdots + \mathbf{a}_k\mathbf{b}_k{}^t$, which is a sum of k $m \times n$ matrices of factor rank 1. Therefore $r(A)$ is the least k such that A is the sum of k matrices of factor rank 1.

Let $A \in \mathcal{M}_{m,n}(\mathbb{B})$ be the set of all $m \times n$ matrices with entries in the binary Boolean semiring $\mathbb{B}$. The *Boolean rank* ([5], [12]), $\beta(A)$, of a nonzero Boolean matrix A in $\mathcal{M}_{m,n}(\mathbb{B})$ is the minimal number r such that there exist Boolean matrices $B \in \mathcal{M}_{m,r}(\mathbb{B})$ and $C \in \mathcal{M}_{r,n}(\mathbb{B})$ such that $A = BC$. That is, the Boolean rank is also factor rank over $\mathbb{B}$. Some authors call this the Schein rank [13].

From now on we will assume that $2 \le m \le n$. It follows that $0 \le r(A) \le m$ for all nonzero $A \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$.

By considering a minimal sum of factor rank one matrices for A and B such as $A = A_1 + \cdots + A_k$, and $B = B_1 + \cdots + B_l$, we have that $A + B = A_1 + \cdots + A_k + B_1 + \cdots + B_l$, so that $A + B$ has factor rank at most $k + l$. This establishes the following lemma.

Lemma 2.1. *For matrices A and B in $\mathcal{M}_{m,n}(\mathbb{Z}_+)$, we have $r(A + B) \le r(A) + r(B)$.*

For $A = [a_{i,j}] \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$, we define $\overline{A} \in \mathcal{M}_{m,n}(\mathbb{B})$ to be the matrix $[\overline{a_{i,j}}]$ where $\overline{a_{i,j}} = 1$ if and only if $a_{i,j} \ne 0$. $\overline{A}$ is called the *pattern* of A.

If A and B are matrices in $\mathcal{M}_{m,n}(\mathbb{Z}_+)$, we say that B *dominates* A (written $A \sqsubseteq B$ or $B \sqsupseteq A$) if $b_{i,j} = 0$ implies $a_{i,j} = 0$ for all i and j. Equivalently, $A \sqsubseteq B$ if and only if $\overline{A + B} = \overline{B}$. This provides a reflexive and transitive relation on $\mathcal{M}_{m,n}(\mathbb{Z}_+)$.

Let $A \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$. A set, $I(A)$, of nonzero entries is called a set of *independent entries* ([4]) of A if any two of them are neither in the same row nor in the same column. A set, $S(A)$, of independent entries of A is called a set of *isolated entries* ([4], [10]) of A if for any pair, $a_{i,j}, a_{k,l} \in S(A)$, the submatrix $\left[\begin{smallmatrix} a_{i,j} & a_{i.l} \\ a_{k,j} & a_{k,l} \end{smallmatrix}\right]$ of A on rows i and k and on columns j and l has $a_{i,l} = 0$ or $a_{k,j} = 0$. The *isolation number of* A ([1], [4]), $\iota(A)$, is the maximum cardinality of any set of isolated entries in A. Hence we have:

Lemma 2.2. *If there are k isolated entries in A, then $\iota(A) \ge k$.*

In [1] it was shown that the set of matrices of Boolean rank one and the set of matrices whose isolation number is one are the same set. It was also shown that the set of matrices of Boolean rank two and the set of matrices whose isolation number is two are the same set.

Since no two isolated entries can lie in any single factor rank one submatrix, we have:

Lemma 2.3. *Let $A \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$. Then $\iota(A) \le r(A)$.*

However, as the following example shows, the factor rank of a matrix may be greater than the isolation number of the matrix.

Example 2.4. Consider a matrix $A \in \mathcal{M}_{5,5}(\mathbb{Z}_+)$:

$$A = \begin{bmatrix} 0 & \mathbf{1} & 1 & 1 & 1 \\ 1 & 0 & \mathbf{1} & 1 & 1 \\ \mathbf{1} & 1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Then $\iota(A) \geq 3$ since the three bold **1**'s in A constitute a set of isolated entries, but $r(A) \leq 4$ with given factorization of A. An exhaustive search will show that $\iota(A) = 3$ and $r(A) = 4$ so that $i(A) < r(A)$.

Lemma 2.5. *Let $A \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$. Then $\beta(\overline{A}) \leq r(A)$.*

Proof. Let $r(A) = k$. Then $A = BC$ with $B \in \mathcal{M}_{m,k}(\mathbb{Z}_+)$ and $C \in \mathcal{M}_{k,n}(\mathbb{Z}_+)$. Then $\overline{A} = \overline{BC}$ with $\overline{B} \in \mathcal{M}_{m,k}(\mathbb{B})$ and $\overline{C} \in \mathcal{M}_{k,n}(\mathbb{B})$. Therefore $\beta(\overline{A}) \leq k$. $\square$

Lemma 2.6. *For matrices A and B in $\mathcal{M}_{m,n}(\mathbb{Z}_+)$, we have $\iota(A+B) \leq \iota(A) + \iota(B)$.*

Proof. Let $A + B = C = [c_{i,j}]$ and $\iota(A + B) = k$. Then there is a set $\{c_{i_\alpha,j_\alpha} : \alpha = 1, \ldots, k\}$ of isolated entries for $A + B = C$. Therefore those corresponding entries a_{i_α,j_α} in A (or b_{i_α,j_α} in B) are contained in a set of isolation numbers for A (or B, resp.). This implies that $\iota(A) + \iota(B) \geq k$. $\square$

2.2. Upper ideals

A subset, $\mathbb{U}$, of $\mathcal{M}_{m,n}(\mathbb{Z}_+)$ is called an *upper ideal* if $X \in \mathbb{U}$ implies $X+Y \in \mathbb{U}$ for every $Y \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$. Upper ideals and linear operators preserving upper ideals was first introduced by Beasley and Pullman in 1992 [6]. See also [2].

A subset, $G \subset \mathcal{M}_{m,n}(\mathbb{Z}_+)$, is *bigraphical* if $X \in G$ implies $PXQ \in G$ for all permutation matrices P and Q of appropriate orders. The reason for this name is that the bipartite graph associated with a matrix has the same properties as any other bipartite that has only relabeling of the weighted bipartite sets.

An upper ideal $\mathbb{U}$ is said to *separate cells* if for any two distinct cells E and F in $\mathbb{E}$, there is some $X \notin \mathbb{U}$ such that $X + E \in \mathbb{U}$ while $X + F \notin \mathbb{U}$.

Let $\mathbb{F}$ be a subset of $\mathcal{M}_{m,n}(\mathbb{Z}_+)$. The *upper ideal generated by* $\mathbb{F}$, $\mathbb{U}(\mathbb{F})$, is the set of matrices not dominated by any element of $\mathbb{F}$, i.e., $\mathbb{U}(\mathbb{F}) = \{A \in \mathcal{M}_{m,n}(\mathbb{Z}_+) \mid \text{for all } B \in \mathcal{M}_{m,n}(\mathbb{Z}_+), A + B \notin \mathbb{F}\}$.

Lemma 2.7. *Let $\mathbb{F}$ be the set of $(0,1)$-matrices in $\mathcal{M}_{m,n}(\mathbb{Z}_+)$ whose isolation number is k, for some $2 \leq k \leq m$. Then $\mathbb{U}(\mathbb{F})$ separates cells.*

Proof. Let C and D be cells in $\mathcal{M}_{m,n}(\mathbb{Z}_+)$. Then, by permuting we can assume that $C = E_{1,1}$ and $D = E_{i,j}$ for some $(i,j) \neq (1,1)$.

If $j \neq 1$, interchange the jth column with the nth column, so that $D = E_{i,n}$.

If $j = 1$, then $i \neq 1$. In this case, we may interchange the ith row with the mth row, so that $D = E_{m,1}$.

In either case, since $k \leq \min\{m,n\}$, we have $D = E_{r,s}$ with $r + s \geq k + 1$. Let $A = (a_{i,j})$ where $a_{i,j} = 0$ if $i + j \leq k$ and $a_{i,j} = 1$ otherwise. Then $A \in \mathbb{F}$ and $\{a_{i,j} \mid i + j = k + 1\}$ is a set of isolated entries of size k. Further, $A \geq D$ so $\iota(A + D) = k$ but $\iota(A + C) = k - 1$. So $A + D \notin \mathbb{U}(\mathbb{F})$ while $A + C \in \mathbb{U}(\mathbb{F})$. That is, $\mathbb{U}(\mathbb{F})$ separates cells. $\square$

2.3. Linear operators

A mapping $T : \mathcal{M}_{m,n}(\mathbb{Z}_+) \to \mathcal{M}_{m,n}(\mathbb{Z}_+)$ is called a *linear operator* if for any $X, Y \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$, $T(X+Y) = T(X) + T(Y)$, and $T(\alpha X) = \alpha T(X)$ for any $\alpha \in \mathbb{Z}_+$.

Let $f : \mathcal{M}_{m,n}(\mathbb{Z}_+) \to \mathbb{Z}_+$ be a mapping. Let $\mathbb{W}$ be a subset of $\mathcal{M}_{m,n}(\mathbb{Z}_+)$. For a linear operator $T : \mathcal{M}_{m,n}(\mathbb{Z}_+) \to \mathcal{M}_{m,n}(\mathbb{Z}_+)$, we say that T

(1) *preserves* f if for any $k \in \mathbb{Z}_+$, $f(T(X)) = k$ whenever $f(X) = k$ for all $X \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$;
(2) *preserves* $\mathbb{W}$ if $T(X) \in \mathbb{W}$ whenever $X \in \mathbb{W}$ for all $X \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$;
(3) *strongly preserves* $\mathbb{W}$ if $T(X) \in \mathbb{W}$ if and only if $X \in \mathbb{W}$ for all $X \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$;

A linear operator $T : \mathcal{M}_{m,n}(\mathbb{Z}_+) \to \mathcal{M}_{m,n}(\mathbb{Z}_+)$ is called a (P,Q)-*operator* if there are permutation matrices P and Q such that $T(X) = PXQ$ for all $X \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$, or when $m = n$, $T(X) = PX^tQ$ for all $X \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$, where X^t is the transpose of X.

A linear operator $T : \mathcal{M}_{m,n}(\mathbb{Z}_+) \to \mathcal{M}_{m,n}(\mathbb{Z}_+)$ is said to be *nonsingular* if $T(X) = O$ only if $X = O$. Unlike linear operators over a field, nonsingular operators over $\mathbb{Z}_+$ need not be invertible. For example, $T(X) = 2X$ for any $X \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$ is nonsingular, but it is not invertible.

Theorem 2.8. *Let $T : \mathcal{M}_{m,n}(\mathbb{Z}_+) \to \mathcal{M}_{m,n}(\mathbb{Z}_+)$ be a linear operator. Then the following are equivalent*:

(1) *T is bijective on $\mathbb{E}$;*
(2) *T is surjective on $\mathbb{E}$;*
(3) *there exists a permutation σ on $\{(i,j) \mid i = 1,2,\dots,m; j = 1,2,\dots,n\}$ such that $T(E_{i,j}) = E_{\sigma(i,j)}$ for all $1 \le i \le n; 1 \le j \le m$.*

Proof. That (1) implies (2) and (3) implies (1) is straight forward. We now show that (2) implies (3).

We assume that T is surjective. Then, for any pair (i,j), there exists some $X \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$ such that $T(X) = E_{i,j}$. Clearly $X \neq O$ by the linearity of T. Thus there is a pair of indices (r,s) such that $X = x_{r,s}E_{r,s} + X'$ where the (r,s) entry of X' is zero and the following two conditions are satisfied: $x_{r,s} \neq 0$ and $T(E_{r,s}) \neq O$. Indeed, if in the contrary for all pairs (r,s) either $x_{r,s} = 0$ or $T(E_{r,s}) = O$, then $T(X) = 0$ which contradicts with the assumption $T(X) = E_{i,j}$. For $x_{r,s} \in \mathbb{Z}_+$, it follows that

$$T(x_{r,s}E_{r,s}) \sqsubseteq T(x_{r,s}E_{r,s}) + T(X \setminus (x_{r,s}E_{r,s})) = T(X) = E_{i,j}.$$

Hence, $x_{r,s}T(E_{r,s}) = T(x_{r,s}E_{r,s}) \sqsubseteq E_{i,j}$ and $T(E_{r,s}) \neq O$ by the above. Therefore, $T(E_{r,s}) \sqsubseteq E_{i,j}$. Indeed, if on the contrary, $T(E_{r,s})$ is a sum of certain multiples of cells, then $x_{r,s}T(E_{r,s})$ is.

Let $P_{i,j} = \{E_{r,s} \mid T(E_{r,s}) \sqsubseteq E_{i,j}\}$. By the above $P_{i,j} \neq \emptyset$ for all (i,j). By its definition $P_{i,j} \cap P_{u,v} = \emptyset$ whenever $(i,j) \neq (u,v)$. That is $\{P_{i,j}\}$ is a set of mn nonempty sets which partition the set of cells. By the pigeonhole principle, we must have that the number of cells in $P_{i,j}$ is only 1 for all (i,j). Necessarily, for each pair (r,s) there is a unique pair (i,j) such that $T(E_{r,s}) = x_{r,s}E_{i,j}$ for some $x_{r,s} \in \mathbb{Z}_+$. That is, there is some permutation σ on $\{(i,j) \mid i = 1,2,\ldots,m; j = 1,2,\ldots,n\}$ such that for some scalars $x_{i,j} \in \mathbb{Z}_+$, $T(E_{i,j}) = x_{i,j}E_{\sigma(i,j)}$. We now only need to show that the $x_{i,j}$ are all 1. Since T is surjective and $T(E_{r,s}) \not\sqsubseteq E_{\sigma(i,j)}$ for $(r,s) \neq (i,j)$, there is some $\alpha \in \mathbb{Z}_+$ such that $T(\alpha E_{i,j}) = E_{\sigma(i,j)}$. But then, since T is linear, $T(\alpha E_{i,j}) = \alpha T(E_{i,j}) = \alpha x_{i,j}E_{\sigma(i,j)} = E_{\sigma(i,j)}$. That is, $\alpha x_{i,j} = 1$, or $x_{i,j} = 1$ in $\mathbb{Z}_+$. □

A *line* is a matrix of the form $R_i = \sum_{j=1}^{n} E_{i,j}$ or of the form $C_j = \sum_{i=1}^{m} E_{i,j}$. That is, a line is a matrix which includes all the ones in a row or column, and all other entries are 0.

Lemma 2.9. *Let $T : \mathcal{M}_{m,n}(\mathbb{Z}_+) \to \mathcal{M}_{m,n}(\mathbb{Z}_+)$ be a linear operator. If T is bijective on $\mathbb{E}$ and maps lines to lines, then T is a (P,Q)-operator.*

Proof. Let $\mathbb{L} = \{R_i \mid 1 \leq i \leq m\} \cup \{C_j \mid 1 \leq j \leq n\}$. Then, since T maps lines to lines and is bijective on $\mathbb{E}$, T is bijective on $\mathbb{L}$.

If $m \neq n$, the image of each R_i must be some R_k, and the image of each C_j must be some C_l since T is bijective on the cells.

If $m = n$, then either the image of every full row is a full row and hence the image of every full column is a full column, or the image of every full row is a full column and hence the image of every full column is a full row, since T is bijective on $\mathbb{L}$. If the image of every full row is a full column and the image of every full column is a full row, composing T with the transpose operator gives an operator that maps full rows to full rows and full columns to full columns. In these cases, letting σ be a permutation such that $T(R_i) = R_{\sigma(i)}$ and τ be a permutation such that $T(C_j) = C_{\tau(j)}$, we have that T is a (P,Q)-operator on $\mathbb{L}$ where P is the permutation matrix corresponding to σ and Q is the permutation matrix corresponding to τ. That is, $T(E_{i,j}) = PE_{i,j}Q$ or $m = n$, $T(E_{i,j}) = PE_{i,j}^tQ$ for all $E_{i,j} \in \mathbb{E}$.

Now, if $m = n$ and the image of one full row is a full row and the image of another full row is a full column, say without loss of generality that $T(R_1) = R_1$ and $T(R_2) = C_1$. Then, $\#(R_1 + R_2) = 2n$ while $\#(R_1 + C_1) = 2n - 1$, an impossibility since T is bijective. Therefore T is a (P,Q)-operator on $\mathbb{L}$.

Therefore, for any $A = [a_{i,j}] \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$, we have

$$\begin{aligned} T(A) &= T(\sum_{i=1}^{m}\sum_{j=1}^{n} a_{i,j}E_{i,j}) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{i,j}T(E_{i,j}) \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n} a_{i,j}PE_{i,j}Q = P(\sum_{i=1}^{m}\sum_{j=1}^{n} a_{i,j}E_{i,j})Q = PAQ. \end{aligned}$$

Similarly, when $m = n$, we have $T(A) = PA^tQ$. Hence T is a (P,Q)-operator on $\mathcal{M}_{m,n}(\mathbb{Z}_+)$. □

3. Isolation numbers and their preservers

Let $T : \mathcal{M}_{m,n}(\mathbb{Z}_+) \to \mathcal{M}_{m,n}(\mathbb{Z}_+)$ be a linear operator. Define $\overline{T} : \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ by $\overline{T}(\overline{A}) = \sum_{i=1}^{m}\sum_{j=1}^{n} \overline{T(a_{i,j}E_{i,j})}$ for any $A = [a_{i,j}] \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$. Then $\overline{T}$ is a well defined linear operator since it is a composition of well defined linear operators.

If $\overline{T}$ preserves Boolean ranks 1 and 2, then $\overline{T}$ is a (P, Q)-operator ([5]). Since the Boolean rank and isolation number of a matrix agree ([1]) when their Boolean ranks are 1 and 2, we have the following lemma.

Lemma 3.1. *Let $\overline{T} : \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ be a Boolean linear operator. Then the following are equivalent*:

(1) *$\overline{T}$ preserves the isolation number of matrices,*
(2) *$\overline{T}$ preserves isolation numbers* 1 *and* 2*,*
(3) *$\overline{T}$ is a (P, Q)-operator.*

From this lemma, we have the following theorem.

Theorem 3.2. *Let $T : \mathcal{M}_{m,n}(\mathbb{Z}_+) \to \mathcal{M}_{m,n}(\mathbb{Z}_+)$ be a linear operator and bijective on $\mathbb{E}$. Then the following are equivalent*:

(1) *T preserves the isolation number of matrices,*
(2) *T preserves isolation numbers* 1 *and* 2*,*
(3) *T is a (P, Q)-operator.*

Proof. That (1) implies (2) and (3) implies (1) is straight forward. We now show that (2) implies (3).

We assume that T preserves the isolation numbers 1 and 2. Since T is bijective on $\mathbb{E}$, we have $T(E_{i,j}) = \overline{T}(E_{i,j})$ for all $E_{i,j} \in \mathbb{E}$. Thus $\overline{T}$ preserves isolation numbers 1 and 2 on $\mathcal{M}_{m,n}(\mathbb{B})$. By Lemma 3.1, $\overline{T}$ is a (P, Q)-operator. For any $A = [a_{i,j}] \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$, we have

$$\begin{aligned}
T(A) = T(\sum_{i=1}^{m}\sum_{j=1}^{n} a_{i,j}E_{i,j}) &= \sum_{i=1}^{m}\sum_{j=1}^{n} a_{i,j}T(E_{i,j}) \\
= \sum_{i=1}^{m}\sum_{j=1}^{n} a_{i,j}\overline{T}(E_{i,j}) &= \sum_{i=1}^{m}\sum_{j=1}^{n} a_{i,j}(PE_{i,j}Q) \\
= \sum_{i=1}^{m}\sum_{j=1}^{n} P(a_{i,j}E_{i,j})Q &= P\sum_{i=1}^{m}\sum_{j=1}^{n}(a_{i,j}E_{i,j})Q = PAQ.
\end{aligned}$$

Similarly, when $m = n$, we have $T(A) = PA^tQ$. Therefore T is a (P, Q)-operator on $\mathcal{M}_{m,n}(\mathbb{Z}_+)$.

Proposition 3.3. *If $T : \mathcal{M}_{m,n}(\mathbb{Z}_+) \to \mathcal{M}_{m,n}(\mathbb{Z}_+)$ is a linear operator that preserves isolation numbers k and l with $k < l$, then T is nonsingular.*

Proof. Suppose that $T(A) = 0$ for some nonzero $A \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$. Then there exists some cell $E_{i,j} \sqsubseteq A$ such that $T(E_{i,j}) = 0$. Without loss of generality, assume that $T(E_{1,1}) = 0$. Since $\iota(E_{1,1} + E_{2,2} + \cdots + E_{l,l}) = l$, and T preserves isolation number l, we have $\iota(T(E_{2,2} + \cdots + E_{l,l})) = \iota(T(E_{1,1} + E_{2,2} + \cdots + E_{l,l})) = l$. Let $X = (T(E_{2,2} + \cdots + E_{l,l}))$. Then we can choose a set of l-isolated entries $S = \{x_{i(1),j(1)}, x_{i(2),j(2)}, \ldots, x_{i(l),j(l)}\}$ in X and hence there is some cell in $\{E_{2,2}, \ldots, E_{l,l}\}$ whose image under T dominates two isolated entries in S. Without loss of generality, we may assume that $T(E_{2,2})$ dominates two isolated entries $x_{i(1),j(1)}$ and $x_{i(2),j(2)}$. Further, there is some cell in $\{E_{2,2}, \ldots, E_{l,l}\}$ whose image under T dominates $x_{i(3),j(3)}$. Without loss of generality, we may assume that $T(E_{2,2} + E_{3,3})$ dominates three isolated entries $x_{i(1),j(1)}$, $x_{i(2),j(2)}$ and $x_{i(3),j(3)}$. Continuing this process, there are k cells in $\{E_{2,2}, \ldots, E_{l,l}\}$ the sum of whose image under T dominates isolated entries $x_{i(1),j(1)}, \ldots, x_{i(k+1),j(k+1)}$. Without loss of generality, we may assume that $T(E_{2,2} + \cdots + E_{k+1,k+1})$ dominates isolated entries $x_{i(1),j(1)}, x_{i(2),j(2)}, \ldots, x_{i(k+1),j(k+1)}$. Therefore $\iota(T(E_{2,2} + \cdots + E_{k+l,k+l})) \geq k+1$, a contradiction since $\iota(E_{2,2}+\cdots+E_{k+1,k+1}) = k$ and hence $\iota(T(E_{2,2}+\cdots+E_{k+1,k+1})) = k$. This contradiction establishes the lemma. □

The following lemma first appeared in [6, Lemma 3.3] and later in [2, Lemma 7]. We include the short proof here for completeness.

Lemma 3.4 ([6, Lemma 3.3]). *Let $T : \mathcal{M}_{m,n}(\mathbb{Z}_+) \to \mathcal{M}_{m,n}(\mathbb{Z}_+)$ be a linear operator. If an upper ideal $\mathbb{U}$ separates cells and T strongly preserves $\mathbb{U}$, then T is bijective on $\mathbb{E}$ and hence it is bijective on $\mathcal{M}_{m,n}(\mathbb{Z}_+)$.*

Proof. Since $\mathbb{E}$ is finite, there is a power of T, say T^r which is idempotent. Let $L = T^r$. Then L strongly preserves $\mathbb{U}$ since T does, and $L^2 = L$.

Suppose that $T(X) = O$. If a cell $E \sqsubseteq X$, then $T(E) = O$. Let F be any other cell. Since $\mathbb{U}$ separates cells, there is some $N \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$ which separates F from E. That is, $N \notin \mathbb{U}$, $N + F \notin \mathbb{U}$ but $N + E \in \mathbb{U}$. But then $T(N)+T(E) = T(N) \sqsubseteq T(N+F)$ and since $N+E \in \mathbb{U}$, $T(N) = T(N+E) \in \mathbb{U}$, contradicting that T strongly preserves $\mathbb{U}$ and $N \notin \mathbb{U}$. Thus, T and hence L is nonsingular.

Now, let F be any cell and suppose that $E \sqsubseteq L(F)$. If $E \neq F$, let N separate E from F. That is, $N \notin \mathbb{U}$, $N + E \in \mathbb{U}$ but $N + F \notin \mathbb{U}$. Then $L(N + F) = L(N) + L(F) = L(N) + L^2(F) \sqsupseteq L(N) + L(E)$, which is seen by the fact that L is idempotent and $E \sqsubseteq L(F)$. Thus, $L(N + F) \sqsupseteq L(N + E)$, and hence $L(N + F) \in \mathbb{U}$, contradicting that L strongly preserves $\mathbb{U}$. Thus, $L(E) = E$.

Now, suppose that $T(E) = T(F)$ for two distinct cells E and F. Then, $L(E) = T^r(E) = T^r(F) = L(F)$, and from above, $E = F$. That is T is injective on $\mathbb{E}$. Since $\mathbb{E}$ is finite, T is bijective on $\mathbb{E}$, and hence is bijective on $\mathcal{M}_{m,n}(\mathbb{Z}_+)$ by the linearity of T. □

Definition 3.5. A *two-claw* in a bipartite graph is a pair of edges incident with one vertex. In $\mathcal{M}_{m,n}(\mathbb{Z}_+)$, a *two-claw matrix* is a sum of two weighted cells, the entries are both in the same row or column. And a matrix A with all its $k = \#(A)$ nonzero entries in one row or column is called *k-claw*.

Lemma 3.6. *Let $T : \mathcal{M}_{m,n}(\mathbb{Z}_+) \to \mathcal{M}_{m,n}(\mathbb{Z}_+)$ be a linear operator. If T is bijective on $\mathbb{E}$ and T preserves two claws, then T is a (P, Q)-operator.*

Proof. If T preserves two-claws and is bijective on $\mathbb{E}$, then T preserves k-claws for all k. That is, T preserves lines. By Lemma 2.9, T is a (P, Q)-operator. $\square$

Theorem 3.7. *Let $T : \mathcal{M}_{m,n}(\mathbb{Z}_+) \to \mathcal{M}_{m,n}(\mathbb{Z}_+)$ be a linear operator on $\mathcal{M}_{m,n}(\mathbb{Z}_+)$ and let $\mathbb{U}$ be an upper ideal that separates cells. If T strongly preserves $\mathbb{U}$ and T preserves two claws, then T is a (P, Q)-operator.*

Proof. If T strongly preserves $\mathbb{U}$, then T is bijective on $\mathbb{E}$ by Lemma 3.4. Since T preserves two-claws, T is a (P, Q)-operator by Lemma 3.6. $\square$

Lemma 3.8. *Let $\mathbb{F}$ be a family of $(0, 1)$-matrices in $\mathcal{M}_{m,n}(\mathbb{Z}_+)$ such that $O \notin \mathbb{F}$ and $J \notin \mathbb{F}$. Then we have the following assertions*:

(1) *$\mathbb{U}(\mathbb{F})$ is an upper ideal.*
(2) *If $T : \mathcal{M}_{m,n}(\mathbb{Z}_+) \to \mathcal{M}_{m,n}(\mathbb{Z}_+)$ strongly preserves $\mathbb{F}$, then T strongly preserves $\mathbb{U}(\mathbb{F})$.*
(3) *If $\mathbb{F}$ is bigraphical, then $\mathbb{U}(\mathbb{F})$ is bigraphical.*

Proof. (1) It follows from the definition of $\mathbb{U}(\mathbb{F})$.

(2) If $A \notin \mathbb{U}(\mathbb{F})$, then there is some $N \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$ such that $A + N \in \mathbb{F}$. Then, $T(A) + T(N) = T(A + N) \in \mathbb{F}$ and hence $T(A) \notin \mathbb{U}(\mathbb{F})$.

For converse implication, let $L = T^r$ be idempotent (as in Lemma 3.4). Suppose that $T(A) \notin \mathbb{U}(\mathbb{F})$. Then, $T(A)+Y \in \mathbb{F}$ for some $Y \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$. Let $Z = T^{r-1}(Y)$. Then $T^{r-1}(T(A) + Y) = L(A) + Z \in \mathbb{F}$. But then, $L(A + Z) = L(A) + L(Z) = L^2(A) + L(Z) = L(L(A) + Z) \in \mathbb{F}$. Thus, $A + Z \in \mathbb{F}$ since L strongly preserve $\mathbb{F}$. That is, $A \notin \mathbb{U}(\mathbb{F})$.

(3) Suppose that $PAQ \notin \mathbb{U}(\mathbb{F})$ for some permutation matrices P, Q. Then there is some $PBQ \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$ such that $P(A + B)Q = PAQ + PBQ \in \mathbb{F}$. Since $\mathbb{F}$ is bigraphical, $A + B = P^t P(A + B)QQ^t \in \mathbb{F}$. Thus $A \notin \mathbb{U}(\mathbb{F})$. Hence $\mathbb{U}(\mathbb{F})$ is bigraphical. $\square$

Lemma 3.9. *Let $T : \mathcal{M}_{m,n}(\mathbb{Z}_+) \to \mathcal{M}_{m,n}(\mathbb{Z}_+)$ be a linear operator. If T is bijective on $\mathbb{E}$ and strongly preserves isolation number k, then T preserves two-claws.*

Proof. For $k = 1$ let $\iota(A) = 2$. Then we can write $A = A_1 + A_2$ with $\iota(A_1) = \iota(A_2) = 1$. Thus $\iota(T(A)) = \iota(T(A_1 + A_2)) \leq \iota(T(A_1)) + \iota(T(A_2)) = 2$. But $\iota(T(A)) \neq 1$ since T strongly preserves isolation number 1. Hence $\iota(T(A)) = 2$. That is, T preserves isolation number 2. Then T is a (P, Q)-operator by Theorem 3.2. Thus T preserves two-claws.

Thus, suppose that $k \geq 2$. Since T is bijective on $\mathbb{E}$, a cell goes to a cell under T. Suppose that $E_{i,j} + E_{i,k}$ is a sum of two cells in a line, that is a two-claw but $T(E_{i,j} + E_{i,k})$ is not a two-claw. Then $T(E_{i,j} + E_{i,k})$ is a matrix containing exactly two ones which are not collinear. Without loss of generality, we may assume that $T(E_{i,j}+E_{i,k})=E_{1,1}+E_{2,2}$. Let $E_3,\ldots,E_k$ be cells such that $T(E_r)=E_{r,r}, r=3,\ldots,k$. Since $E_{i,j}+E_{i,k}$ is a two-claw, $\iota(E_{i,j}+E_{i,k})=1$. Thus, $\iota(E_{i,j}+E_{i,k}+E_3+\cdots+E_k)<k$. But $T(E_{i,j}+E_{i,k}+E_3+\cdots+E_k)=E_{1,1}+E_{2,2}+E_{3,3}+\cdots+E_{k,k}$ has isolation number k, a contradiction since T strongly preserves isolation number k. Thus, T preserves two-claws. □

Our main theorem is:

Theorem 3.10. *Let $T:\mathcal{M}_{m,n}(\mathbb{Z}_+)\to\mathcal{M}_{m,n}(\mathbb{Z}_+)$ be a linear operator. Then, T strongly preserves isolation number k for any $2\le k\le\min\{m,n\}$ if and only if T is a (P,Q)-operator.*

Proof. If T is a (P,Q)-operator, then clearly, T strongly preserves isolation number k.

Now, suppose that T strongly preserves isolation number k with $k\ge 2$.

Let $\mathbb{F}_k$ be the set of all $(0,1)$-matrices in $\mathcal{M}_{m,n}(\mathbb{Z}_+)$ with isolation number k. So T strongly preserves $\mathbb{F}_k$. By Lemma 3.8(2), T strongly preserves $\mathbb{U}(\mathbb{F}_k)$. By Lemma 2.7 $\mathbb{U}(\mathbb{F}_k)$ separates cells. By Lemma 3.4, T is bijective on $\mathbb{E}$. By Lemma 3.9, T preserves two-claws. Therefore T is a (P,Q)-operator by Theorem 3.7. □

Proposition 3.11. *Let $T:\mathcal{M}_{m,n}(\mathbb{Z}_+)\to\mathcal{M}_{m,n}(\mathbb{Z}_+)$ be a linear operator and bijective on $\mathbb{E}$. Then, T strongly preserves isolation number 1 if and only if T is a (P,Q)-operator.*

Proof. If T is a (P,Q)-operator, then clearly, T strongly preserves isolation number 1.

Now, suppose that T strongly preserves isolation number 1. Let $\iota(A)=2$. Then we can write $A=A_1+A_2$ with $\iota(A_1)=\iota(A_2)=1$. Thus $\iota(T(A))=\iota(T(A_1+A_2))\le\iota(T(A_1))+\iota(T(A_2))=2$. But $\iota(T(A))\neq 1$ since T strongly preserves isolation number 1. Hence $\iota(T(A))=2$. That is, T preserves isolation number 2. Since T is bijective on $\mathbb{E}$, T is a (P,Q)-operator by Theorem 3.2. □

Thus we have obtained some characterizations of linear operators that preserve the isolation number of integer matrices.

For the further researches on the linear operators that preserve the isolation number of integer matrices, we suggest a problem:

Problem 3.12. *Let $T:\mathcal{M}_{m,n}(\mathbb{Z}_+)\to\mathcal{M}_{m,n}(\mathbb{Z}_+)$ be a linear operator. If T preserves any two isolation numbers h and k, then does T be a (P,Q)-operator?*

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